SOME NON-CLASSICAL METHODS IN EPISTEMIC LOGIC AND GAMES

by

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ABSTRACT

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In this dissertation, we consider some non-classical methods in epistemic logic and games. We first consider, dynamic epistemic logics in topological and geometric semantics, and then extend such ideas to the cases where inconsistencies are allowed. Then, as a case example, we discuss a well known paradox in game theory which is essentially a two-person Russell's paradox. Finally, we conclude with considering an alternative approach to games where strategies are considered as the primitives of the theory, and advancing some results.

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Last, but not least, my Lauren and her smile gave me even-more-than-I-need hope and strength, and made me finish this work easily: *Thank you, thank you, thank you.*

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Yet take no action, and the people nurture each other;

Make no laws, and the people deal fairly with each other;

Own no interest, and the people cooperate with each other;

Express no desire, and the people harmonize with each other.

Lao Tzu, *Tao Te Ching*

Composer: Z. Müren.

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Chapter 1

Public Announcement Logic in Geometric Frameworks

1.1 Introduction

Public announcement logic is a well-known and well-studied example of dynamic epistemic logics (Plaza, 1989; Gerbrandy, 1999; van Ditmarsch *et al.*, 2007). Dynamic epistemic logics are set out to formalize knowledge and belief changes and revisions in usually multi-agent settings by defining and introducing different ways of epistemic updates and interactions. The main contribution of public announcement logic (PAL, henceforth) to the field of knowledge representation and formal epistemology is mostly due to its succinctness and clarity in reflecting the intuition behind epistemic updates as it does not increase the expressiveness of the (modal) epistemic logic (Kooi, 2007). PAL updates the epistemic models by the announcements made by a truthful (external) agent. After the truthful announcement, the model is updated by eliminating the states that do not agree with the announcement. PAL has many applications in the fields of formal approaches to social interaction, dynamic logics, knowledge representation and belief revision (Balbiani *et al.*, 2008; Baltag & Moss, 2004; van Benthem, 2006; van Benthem *et al.*, 2005). Extensive applications of PAL to different fields and frameworks have made it a rather familiar framework to many researchers. Moreover, virtually almost all applications and implementations of PAL make use of Kripke models for knowledge representation. However, as it is very well known, Kripke models are not the only representational tool for modal and epistemic logics.

In this chapter, we consider PAL in two different geometrical frameworks: topological modal models and subset space models. Topological models are not new to modal logics, indeed they are the first models for modal logic (McKinsey & Tarski, 1944; McKinsey & Tarski, 1946). The past decades have witnessed a revival of academic interest towards the topological models for modal logics in many different frameworks (Aiello *et al.*, 2003; van Benthem & Bezhanishvili, 2007; van Benthem *et al.*, 2006; Bezhanishvili & Gehrke, 2005). However, to the best of our knowledge, topological models have not yet been applied to dynamic epistemic logics. Nevertheless, there have been some influential works on the notion of common knowledge in topological models which has motivated our current work here (van Benthem & Sarenac, 2004). In that work, it was shown that the three different definitions of common knowledge (infinite iteration, fixed point and shared environment) differ in topological models even though these definitions are equivalent in Kripke structures, based on Barwise's earlier investigation (Barwise, 1988). Nevertheless, the authors did not seem to take the next immediate step to discuss dynamic epistemologies in that framework. This is one of our goals in this chapter: to apply topological reasoning to dynamic epistemological cases, and present the completeness results. The second framework that we discuss, subset space logic, is a weaker yet expressive geometrical structure dispensing with the topological structure (Moss & Parikh, 1992; Parikh *et al.*, 2007). Subset space logic has been introduced to reason about the topological notion of *closeness* and the dynamic notion of *effort* in epistemic situations. In this chapter, we also define PAL in subset space logic with its axiomatization, and present the completeness of PAL in subset space logics (Başkent, 2007; Başkent, 2011b; Başkent, 2011a; Başkent, 2012b).

There are several reasons that motivate this work. First, topological models can distinguish some epistemic properties that Kripke models cannot (van Benthem & Sarenac, 2004). This is perhaps not surprising as the topological semantics of the necessity modality has Σ_2 complexity (in the form of an $\exists\forall$ statement), while Kripkean semantics offer Π_1 complexity (in the form of an $\forall$ statement) for the same modality, and furthermore topologies deal with infinite cases in a rather special way by their very own definition. Moreover, PAL update procedure is easily defined by using well-defined topological operations giving sufficient reasons to wonder what other different structures one may have in topological models.

The present chapter is organized as follows. First, we introduce the geometrical frameworks that we need: topological spaces and subset spaces which will also serve as our future reference in the remainder of the dissertation. Then, after a brief interlude on PAL, we give the axiomatizations of PAL in such spaces, and their completeness. The completeness proofs are rather immediate - which is usually the case in PAL systems. Then, we make some further observations on PAL in geometric models. Our observations will be about the stabilization of updated models, backward induction in games and persistency as some applications of topological PAL which make some difference.

1.2 Geometric Models

In this section, we briefly recall the geometric models for modal logics both for the current and following chapters. What we mean by *geometric* models is topological models and subset space models as they are inherently geometrical structures. We first start with topological models and their semantics, and then discuss subset space models. Our emphasis will be on the differences of geometric models from the Kripke models.

1.2.1 Topological Semantics for Modal Logic

Topological interpretations for modal logic historically precede the relational semantics (McKinsey & Tarski, 1944; Goldblatt, 2006). Moreover, as we will observe very soon, topological semantics is arithmetically more complex than relational semantics: the first is Σ_2 while the latter is Π_1. Now, let us start by introducing the definitions.

Definition 1.2.1. A topological space $\mathcal{S} = \langle S, \sigma \rangle$ is a structure with a set S and a collection σ of subsets of S satisfying the following conditions:

1. The empty set and S are in σ.

2. The union of any collection of sets in σ is also in σ.

3. The intersection of a finite collection of sets in σ is also in σ.

The collection σ is said to be a *topology* on S. The elements of S are called *points* and the elements of σ are called *opens*. The complements of open sets are called *closed* sets. Our main operator in topological spaces is called *interior* operator $\mathbb{I}$ which returns the interior of a given set. The interior of a set is defined as the largest open set contained in the given set.

A topological model $\mathcal{M}$ is a triple $\langle S, \sigma, v \rangle$ where $\mathcal{S} = \langle S, \sigma \rangle$ is a topological space, and v is a valuation function assigning subsets of S to propositional letters, i.e. $v : P \to \wp(S)$ for a set of propositional letters P. Here, note that the valuation function does not necessarily assign opens (or closed) as the valuations to propositions.

The basic modal language $\mathcal{L}$ has a countable set of propositional letters P, a truth constant $\top$, the usual Boolean operators $\neg$ and $\wedge$, and a modal operator $\Box$. The dual of $\Box$ is denoted by $\Diamond$ and defined as $\Box\varphi \equiv \neg\Diamond\neg\varphi$. When we are in topological models, we will use the symbol I for $\Box$ after the *interior* operator for intuitive reasons, and to prevent any future confusion. Likewise, we will use the symbol C for $\Diamond$. Since in this chapter, we are discussing several different but intuitively similar frameworks, we need to use different symbols for different models. However, in the following chapter, for easy read, we will use the standard notation for modalities.

The notation $\mathcal{M}, s \models \varphi$ will read *the point s in the model $\mathcal{M}$ makes the formula φ true*. We call the set of points that satisfy a given formula φ in model $\mathcal{M}$ *the extension* of φ, and denote as $(\varphi)^{\mathcal{M}}$. We will drop the superscript when the model we are in is obvious.

In topological models, the extensions of a Boolean formulas are obtained in the familiar sense. The extension of a modal formula in model $\mathcal{M}$, then, is given as follows $(\mathsf{I}\varphi)^{\mathcal{M}} = \mathbb{I}((\varphi)^{\mathcal{M}})$ - namely, the extension of $\mathsf{I}\varphi$ is the interior of the extension of φ. Now, based on this framework, the model theoretical semantics of modal logic in topological spaces is given as follows.

$$
\begin{array}{lll}
\mathcal{M}, s \models p & \text{iff} & s \in v(p) \text{ for } p \in P \\
\mathcal{M}, s \models \neg\varphi & \text{iff} & \mathcal{M}, s \not\models \varphi \\
\mathcal{M}, s \models \varphi \wedge \psi & \text{iff} & \mathcal{M}, s \models \varphi \text{ and } \mathcal{M}, s \models \psi \\
\mathcal{M}, s \models \mathsf{I}\varphi & \text{iff} & \exists U \in \sigma.(s \in U \wedge \forall t \in U,\ \mathcal{M}, t \models \varphi) \\
\mathcal{M}, s \models \mathsf{C}\varphi & \text{iff} & \forall U \in \sigma.(s \in U \rightarrow \exists t \in U,\ \mathcal{M}, t \models \varphi)
\end{array}
$$

A few words on the semantics are in order here. The necessity modality $\mathsf{I}\varphi$ says that there is an open set that contains the current state, and the formula φ is true everywhere in this set. Obviously, this is a rather complex statement, first, it requires us to determine the open set, and then check whether each point in this open set satisfies the given formula or not. On the other hand, the possibility modality $\mathsf{C}\varphi$ manifests the idea that for every open set that includes the current state, there is some point in the same set that satisfies φ. This is clearly reflected in the definition: in topological semantics, the definitions of modal satisfaction have the form $\exists\forall$ or $\forall\exists$. In Kripke models, as it is well-known, the form is either $\exists$ or $\forall$.

It has been shown by McKinsey and Tarski that the modal logic of topological spaces is S4 (McKinsey & Tarski, 1944). Moreover, logics of many other topological spaces have also been investigated (Aiello *et al.*, 2003; Cate *et al.*, 2009; Bezhanishvili *et al.*, 2005;

van Benthem *et al.*, 2006; van Benthem & Bezhanishvili, 2007). Furthermore, recently, the topological properties of paraconsistent systems have also been investigated since there exists a natural topological semantics for paraconsistent and paracomplete logics (Başkent, 2011c; Mortensen, 2000).

The proof theory of the topological models is as expected: we utilize modus ponens and necessitation. Modal logic S4 is long known to be sound and complete with respect to the well-known axiomatization of topological modal logic - which we reproduce here for the completeness of our arguments.

- Axioms of propositional logic
- $\mathsf{I}(\varphi \to \psi) \to (\mathsf{I}\varphi \to \mathsf{I}\psi)$
- $\mathsf{I}\varphi \to \varphi$
- $\mathsf{I}\varphi \to \mathsf{II}\varphi$

1.2.2 Subset Space Logic

Subset space logic (SSL, henceforth) was presented in early 1990s as a bimodal logic to formalize reasoning about sets and points with an underlying motivation from epistemic logic (Moss & Parikh, 1992). One of the modal operators in SSL is intended to quantify *over* the sets ($\Box$) whereas the other modal operator is intended to quantify *in* the current set (K). The underlying motivation to introduce these two modalities is to discuss the notion of *closeness*. In this context, K operator is intended for the knowledge operator (for one agent only, as SSL is originally presented for single-agent), and the $\Box$ modality is intended for the effort modality. Effort can correspond to various things: computation, observation, approximation - the procedures that can result in knowledge increase.

The language of subset space logic $\mathcal{L}_S$ has a countable set P of propositional letters, a truth constant $\top$, the usual Boolean operators $\neg$ and $\wedge$, and two modal operators K and $\Box$.

A subset space model is a triple $\mathcal{S} = \langle S, \sigma, v \rangle$ where S is a non-empty set, $\sigma \subseteq \wp(S)$ is a collection of subsets (*not* necessarily a topology), $v : P \to \wp(S)$ is a valuation function. Semantics of SSL, then is given inductively as follows.

$s, U \models p$	iff	$s \in v(p)$	
$s, U \models \neg\varphi$	iff	$s, U \not\models \varphi$	
$s, U \models \varphi \wedge \psi$	iff	$s, U \models \varphi$	and $s, U \models \psi$
$s, U \models \mathsf{K}\varphi$	iff	$t, U \models \varphi$	for all $t \in U$
$s, U \models \Box\varphi$	iff	$s, V \models \varphi$	for all $V \in \sigma$ such that $s \in V \subseteq U$

The duals of $\Box$ and K are $\Diamond$ and L respectively, and defined as usual. The tuple (s, U) is called a *neighborhood situation* if U is a neighborhood of s, i.e. if $s \in U \in \sigma$. The axioms of SSL reflect the fact that the K modality is S5-like whereas the $\Box$ modality is S4-like. Moreover, we will need an additional axiom to state the interaction between those two modalities: $\mathsf{K}\Box\varphi \to \Box\mathsf{K}\varphi$. Let us now give the complete set of axioms of SSL.

1. All the substitutional instances of the tautologies of the classical propositional logic

2. $(A \to \Box A) \wedge (\neg A \to \Box\neg A)$ for atomic sentence A

3. $\mathsf{K}(\varphi \to \psi) \to (\mathsf{K}\varphi \to \mathsf{K}\psi)$

4. $\mathsf{K}\varphi \to (\varphi \wedge \mathsf{K}\mathsf{K}\varphi)$

5. $\mathsf{L}\varphi \to \mathsf{K}\mathsf{L}\varphi$

6. $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$

7. $\Box\varphi \to (\varphi \wedge \Box\Box\varphi)$

8. $\mathsf{K}\Box\varphi \to \Box\mathsf{K}\varphi$

The rules of inference are as expected: modus ponens and necessitation for both modalities. Based on this framework, subset space logic is complete and decidable (Moss & Parikh, 1992; Georgatos, 1997).

Note that SSL is originally proposed as a single-agent system. There have been some attempts in the literature to suggest a multi-agent version of it, but to the best of our knowledge, there is no intuitive and clear presentation of a multi-agent version of SSL (Başkent, 2007; Başkent & Parikh, 2010). However, SSL is a quite powerful language and can be used to discuss some philosophical issues in methodology of science (Başkent, 2009; Başkent, 2012a).

1.2.3 Public Announcement Logic

Now, let us briefly present the basic notions of public announcement logic for the completeness of our arguments here. Public announcement logic is a way to represent changes and updates in knowledge. The way PAL updates the epistemic states of the knower is by "state-elimination". A truthful announcement φ is made, and consequently, the agents update their epistemic states by eliminating the possible states where φ is false (Plaza, 1989; Gerbrandy, 1999; van Ditmarsch *et al.*, 2007).

Public announcement logic is typically interpreted on multi-modal (or multi-agent) Kripke structures (Plaza, 1989). Notationwise, the formula $[\varphi]\psi$ is intended to mean that *after the public announcement of* φ*,* ψ *holds*. As usual, we put K_i for the epistemic modality for the agent i. Likewise, R_i denotes the epistemic accessibility relation for the agent i. The language of PAL will be that of multi-agent (multi-modal) epistemic logic with an additional public announcement operator $[*]$ where $*$ can be replaced with any well-formed formula in the language of basic epistemic logic. Note that we do not introduce different PAL modalities per agent.

In order to see the semantics for PAL, take a model $\mathcal{M} = \langle W, \{R_i\}_{i \in I}, V \rangle$ where i denotes the agents and varies over a finite set of agents I . For atomic propositions, the negation and the conjunction, the semantics is as usual. For modal operators, we have the following semantics.

$\mathcal{M}, w \models \mathsf{K}_i\varphi$ iff $\mathcal{M}, v \models \varphi$ for each v such that $(w, v) \in R_i$

$\mathcal{M}, w \models [\varphi]\psi$ iff $\mathcal{M}, w \models \varphi$ implies $\mathcal{M}|\varphi, w \models \psi$

Here, the updated model $\mathcal{M}|\varphi = \langle W', \{R'_i\}_{i \in I}, V' \rangle$ is defined by restricting $\mathcal{M}$ to those states where φ holds. Hence, $W' = W \cap (\varphi)^{\mathcal{M}}$; $R'_i = R_i \cap (W' \times W')$, and finally $V'(p) = V(p) \cap W'$.

The axiomatization of PAL is the axiomatization of S5_n with additional axioms for dynamic modality. Hence, we give the (standard) set of axioms for PAL as follows.

1. All the substitutional instances of the tautologies of the classical propositional logic
2. $\mathsf{K}_i(\varphi \to \psi) \to (\mathsf{K}_i\varphi \to \mathsf{K}_i\psi)$
3. $\mathsf{K}_i\varphi \to \varphi$
4. $\mathsf{K}_i\varphi \to \mathsf{K}_i\mathsf{K}_i\varphi$
5. $\neg\mathsf{K}_i\varphi \to \mathsf{K}_i\neg\mathsf{K}_i\varphi$
6. $[\varphi]p \leftrightarrow (\varphi \to p)$
7. $[\varphi]\neg\psi \leftrightarrow (\varphi \to \neg[\varphi]\psi)$
8. $[\varphi](\psi \wedge \chi) \leftrightarrow ([\varphi]\psi \wedge [\varphi]\chi)$
9. $[\varphi]\mathsf{K}_i\psi \leftrightarrow (\varphi \to \mathsf{K}_i[\varphi]\psi)$

The additional rule of inference which we will need for announcement modality is called the *announcement generalization* and is described as expected: From $\vdash \psi$, derive $\vdash [\varphi]\psi$.

PAL is complete and decidable. The completeness proof is quite straightforward. Once the soundness of the given axiomatization is proved, then it means that every complex formula in the language of PAL can be reduced to a formula in the basic language of (multi-agent) epistemic logic. Since S5 (or S5_n) epistemic logic is long known to be complete, we immediately deduce the completeness of PAL. The argument for decidability is also very similar as the translation from PAL to modal epistemic logic is trivially polynomial. Therefore, it can be claimed that the only advantage of PAL is its succinctness (Kooi, 2007).

Notice again that in this section, we have defined PAL in Kripke structures by following the literature. Use of Kripke structures make the intuition of PAL rather clear.

However, in the next section, we will see how PAL is defined in geometrical models. We will start with SSL and proceed to topological models with some further observations.

1.3 Subset Space PAL

Let us recall a classical example for subset space logic (Moss & Parikh, 1992). Assume that a policeman is measuring the speed of the passing cars where the speed limit is 50 mph. Suppose that the measurement device he uses has an error range of 5 mph. In one instance, he measures the speed of a car, and reads that the speed lies in the interval $[45, 55]$. Yet, the policeman does not exactly know the speed v. Therefore, he is at the neighborhood situation $(v, [45, 55])$. Then, let us assume that he hears an announcement, say, a message he receives via the police radio, saying that the speed of that particular car is faster than or equal to 48 mph. In other words, he learns that $v \in [48, \infty]$. Then, the policeman updates his situation to $(v, [48, 55])$ since the announcements are assumed to be truthful. Therefore, public announcement limits his possibilities leading to an improvement, and update in knowledge.

Formalization of SSL in public announcement setting follows this simple example. In SSL, we depend on neighborhood situations (which are tuples of the form (s, U) for $s \in U \in \sigma$) instead of the epistemic accessibility relations. Therefore, if we want to adopt public announcement logic to the context of subset space logic, we first need to focus on the fact that the public announcements shrink the observation sets for each agent.

Let us set a piece of notation. For a formula φ, recall that $(\varphi)^{\mathcal{S}}$ is the extension of φ in the model $\mathcal{S} = \langle S, \sigma, v\rangle$. In SSL, we define $(\varphi)^{\mathcal{S}} = \{(s, U) \in S \times \sigma : s \in U, (s, U) \models \varphi\}$.

Now, let us define the following projections $(\varphi)^{\mathcal{S}}_1 := \{s : (s, U) \in (\varphi)^{\mathcal{S}} \text{ for some } U \ni s\}$, and $(\varphi)^{\mathcal{S}}_2 := \{U : (s, U) \in (\varphi)^{\mathcal{S}} \text{ for some } s \in U\}$. We will drop the superscript when the model we are in is obvious.

Now, assume that we are in a subset space model $\mathcal{S} = \langle S, \sigma, v\rangle$. Then, after public announcement φ, we will move to the updated subset space model $\mathcal{S}_\varphi = \langle S|\varphi, \sigma_\varphi, v_\varphi\rangle$ where $S|\varphi = (\varphi)_1$, and σ_φ is the reduced collection of subsets after the public announcement φ, and v_φ is the reduct of v on $S|\varphi$. The crucial point is to construct σ_φ. As we need to eliminate the refutative states, we eliminate the points which do not satisfy φ for each observation set U in σ. We will disregard the empty set as no neighborhood situations can be formed with the empty set. Hence, $\sigma_\varphi = \{U_\varphi : U_\varphi = U \cap (\varphi)_2 \neq \emptyset, \text{ for each } U \in \sigma\}$.

But then, how would the neighborhood situations be affected by the public announcements? Consider the neighborhood situation (s, U) and the public announcement φ. Then the statement $s, U \models [\varphi]\psi$ will mean that after the public announcement of φ, ψ will hold in the neighborhood situation (s, U_φ). So, first we will remove the points in U which refute φ, and then ψ will hold in the updated set U_φ which was obtained from the original set U. Then the corresponding semantics can be suggested as follows:

$$s, U \models [\varphi]\psi \text{ iff } s, U \models \varphi \text{ implies } s, U_\varphi \models \psi$$

Before checking whether this semantics satisfies the axioms of PAL, let us give the language and semantics of the subset space PAL. The language of the subset space PAL is given as follows:

$$p \mid \bot \mid \neg\varphi \mid \varphi \wedge \psi \mid \Box\varphi \mid \mathsf{K}\varphi \mid [\varphi]\psi$$

Now, let us consider the soundness of the axioms of basic PAL that we discussed earlier in Section 1.2.3. We prove that those axioms are sound in SSL.

Theorem 1.3.1. *The axioms of the basic PAL are sound in subset space logic.*

Proof. As the atomic propositions do not depend on the neighborhood, the first axiom is satisfied by the subset space semantics of public announcement modality. To see this, assume $s, U \models [\varphi]p$. So, by the semantics $s, U \models \varphi$ implies $s, U_\varphi \models p$. So, $s \in v(p)$. So for any set V where $s \in V$, we have $s, V \models p$. Hence, $s, U \models \varphi$ implies $s, U \models p$, that is $s, U \models \varphi \to p$. Conversely, assume $s, U \models \varphi \to p$. So, $s, U \models \varphi$ implies $s \in v(p)$. As $s, U \models \varphi$, s will lie in U_φ, thus (s, U_φ) will be a neighborhood situation. Thus, $s, U_\varphi \models p$. Then, we conclude $s, U \models [\varphi]p$.

The axioms for negation and conjunction are also straightforward formula manipulations and hence skipped.

The important reduction axiom is the knowledge announcement axiom. Assume, $s, U \models [\varphi]\mathsf{K}\psi$. Suppose further that $s, U \models \varphi$. Then we have the following.

$$\begin{array}{lll} s, U \models [\varphi]\mathsf{K}\psi & \text{iff} & s, U_\varphi \models \mathsf{K}\psi \\ & \text{iff} & \text{for each } t_\varphi \in U_\varphi, \text{ we have } t_\varphi, U_\varphi \models \psi \\ & \text{iff} & \text{for each } t \in U,\ t, U \models \varphi \\ & & \text{implies } t, U \models [\varphi]\psi \\ & \text{iff} & s, U \models \mathsf{K}(\varphi \to [\varphi]\psi) \\ & \text{iff} & s, U \models \mathsf{K}[\varphi]\psi \end{array}$$

Thence, the above axioms are sound for the subset space semantics of public announcement logic. ■

Now, recall that SSL has an indispensable modal operator $\Box$. One can wonder whether we can have a reduction axiom for it as well. We start by considering the statement $[\varphi]\Box\psi \leftrightarrow (\varphi \to \Box[\varphi]\psi)$. Assume, $s, U \models [\varphi]\Box\psi$. Suppose further that $s, U \models \varphi$. Then, we deduce the following.

$$
\begin{aligned}
s, U \models [\varphi]\Box\psi \quad & \text{iff} \quad s, U_\varphi \models \Box\psi \\
& \text{iff} \quad \text{for each } V_\varphi \subseteq U_\varphi \text{ we have } s, V_\varphi \models \psi \\
& \text{iff} \quad \text{for each } V \subseteq U,\ s, V \models \varphi \\
& \qquad\ \ \text{implies } s, V \models [\varphi]\psi \\
& \text{iff} \quad s, U \models \Box(\varphi \to [\varphi]\psi) \\
& \text{iff} \quad s, U \models \Box[\varphi]\psi
\end{aligned}
$$

Now, it is easy to see that the following axiomatize the subset space PAL together with the axiomatization of SSL:

1. $[\varphi]p \leftrightarrow (\varphi \to p)$
2. $[\varphi]\neg\psi \leftrightarrow (\varphi \to \neg[\varphi]\psi)$
3. $[\varphi](\psi \wedge \chi) \leftrightarrow ([\varphi]\psi \wedge [\varphi]\chi)$
4. $[\varphi]\mathsf{K}\psi \leftrightarrow (\varphi \to \mathsf{K}[\varphi]\psi)$
5. $[\varphi]\Box\psi \leftrightarrow (\varphi \to \Box[\varphi]\psi)$

Referring to the above discussions, the completeness of subset space PAL follows easily.

Theorem 1.3.2. *PAL in subset space models is complete with respect to the axiom system given above.*

Proof. By the reduction axioms above, we can reduce each formula in the language of subset space PAL to a formula in the language of SSL. As SSL is known to be complete, so is subset space PAL. ∎

By the same idea, we can import the decidability result.

Theorem 1.3.3. *PAL in subset space models is decidable.*

1.4 Topological PAL

1.4.1 Single Agent Topological PAL

We can use similar ideas to give an account of PAL in topological spaces. Let $\mathcal{T} = \langle T, \tau, v \rangle$ be a topological model, and φ be a public announcement. We now need to construct the topological model $\mathcal{T}_\varphi$ which is the updated model after the announcement. Define $\mathcal{T}_\varphi = \langle T_\varphi, \tau_\varphi, v_\varphi \rangle$ where $T_\varphi = T \cap (\varphi)$, $\tau_\varphi = \{O \cap T_\varphi : O \in \tau\}$ and $v_\varphi = v \cap T_\varphi$. We now need to verify that τ_φ is a topology, indeed the *induced topology*. For the sake of the completeness of our arguments in this work, let us give the proof here.

Proposition 1.4.1. *If τ is a topology, then $\tau_\varphi = \{O \cap T_\varphi : O \in \tau\}$ is a topology as well.*

Proof. Clearly, the empty set is in τ_φ as τ is a topology. As τ is a topology on T, we have $T \in \tau$. Thus, $T \cap T_\varphi$, namely T_φ, is in τ_φ. Consider $\bigcup_i^\infty U_i$ where $U_i \in \tau_\varphi$. For each i, we have $U_i = O_i \cap T_\varphi$ for some $O_i \in \tau$. Thus, $\bigcup_i^\infty U_i = T_\varphi \cap \bigcup_i^\infty O_i$. Since τ is a topology, $\bigcup_i^\infty O_i \in \tau$. Thus, $T_\varphi \cap \bigcup_i^\infty O_i \in \tau_\varphi$ yielding the fact that $\bigcup_i^\infty U_i \in \tau_\varphi$. Similarly, consider $\bigcap_i^n U_i$ where $U_i \in \tau_\varphi$ for some $n < \omega$. Since $U_i = O_i \cap T_\varphi$ for some $O_i \in \tau$, we similarly observe that $\bigcap_i^n U_i = \bigcap_i^n (O_i \cap T_\varphi) = T_\varphi \cap \bigcap_i^n O_i$. Since τ is a topology, $\bigcap_i^n O_i \in \tau$, thus, $\bigcap_i^n U_i \in \tau_\varphi$. ∎

It is important to notice here that only modal formulas necessarily yield open or closed extensions. Extensions of Booleans may or may not be a topological set as they solely depend on the model and the valuation we are working with.

Now, when we restrict the carrier set of the topology to a subset of it, we still get a topology. Based on this simple observation, we can give a semantics for the public announcements in topological models.

$$\mathcal{T}, s \models [\varphi]\psi \text{ iff } \mathcal{T}, s \models \varphi \text{ implies } \mathcal{T}_\varphi, s \models \psi$$

In a similar fashion, we can expect that the reduction axioms work in topological spaces.

The reduction axioms for atoms and Booleans are quite straight-forward. So, consider the reduction axiom for the interior modality given as follows: $[\varphi]\mathsf{I}\psi \leftrightarrow (\varphi \rightarrow \mathsf{I}[\varphi]\psi)$.

Let $\mathcal{T}, s \models [\varphi]\mathsf{I}\psi$ which, by definition means $\mathcal{T}, s \models \varphi$ implies $\mathcal{T}_\varphi, s \models \mathsf{I}\psi$. If we spell out the topological interior modality, we get $\exists U_\varphi \ni s \in \tau_\varphi$ s.t. $\forall t \in U_\varphi, \mathcal{T}_\varphi, t \models \psi$. By definition, since $U_\varphi \in \tau_\varphi$, it means that there is an open $U \in \tau$ such that $U_\varphi = U \cap (\varphi)$. Under the assumption that $\mathcal{T}, s \models \varphi$, we observe that $\exists U \ni s \in \tau$ (as we just constructed it), such that after the announcement φ, the non-eliminated points in U (namely, the ones in U_φ) will satisfy ψ. Thus, we get $\mathcal{T}, s \models \varphi \rightarrow \mathsf{I}[\varphi]\psi$.

The other direction is very similar and hence we leave it to the reader. Therefore, the reduction axioms for PAL in topological spaces are given as follows.

1. $[\varphi]p \leftrightarrow (\varphi \rightarrow p)$
2. $[\varphi]\neg\psi \leftrightarrow (\varphi \rightarrow \neg[\varphi]\psi)$
3. $[\varphi](\psi \wedge \chi) \leftrightarrow ([\varphi]\psi \wedge [\varphi]\chi)$
4. $[\varphi]\mathsf{I}\psi \leftrightarrow (\varphi \rightarrow \mathsf{I}[\varphi]\psi)$

As a result, all the complex formulas involving the PAL operator can be reduced to simpler ones. This algorithm directly shows the completeness of PAL in topological spaces by reducing each formula in the language of topological PAL to the language of basic topological modal logic. Thus, we have the following result.

Theorem 1.4.2. *Public announcement logic in topological spaces is complete with respect to the axiomatization given.*

By the same idea, we can import the decidability result.

Theorem 1.4.3. *Public announcement logic in topological models is decidable.*

1.4.2 Product Topological PAL

There are variety of ways to merge given topological models to express the epistemic interaction between them: products, sums, fusions etc. (Gabbay *et al.*, 2003). In this section, we focus on one of these methods, namely product topologies, and discuss how public announcements are defined in such systems. Product topological frameworks for multi-agent epistemic logics have already been discussed widely in the literature (van Benthem *et al.*, 2006; van Benthem & Sarenac, 2004). Therefore, our treatment of the subject will be based on these works. After establishing the basic formalism, we will then introduce public announcement logic to product topologies. The idea is quite straight-forward. We are given two topologies (possibly with different spaces) with a modal (epistemic, doxastic etc) model on them. Then, by the standard techniques in the literature, we will merge them. After that, we will discuss how public announcements work in this unified structure. Our contribution here is to define public announcements in combined topological spaces, give their axiomatization and semantics, and show the completeness of our axiomatization.

We believe that product topologies reflect the *multi-agent-ness* of PAL quite well. Namely, given the epistemics of individual agents with corresponding topological spaces, PAL is meaningful once the unifying setting we are in is multi-agent. Therefore, the individual epistemics of the agents need to be combined, or put side by side. The definitions we are suggesting keep the given epistemics of the individual agents unchanged in a multi-dimensional framework, yet introduce ways to express their communication.

Assume that we have two agents 1 and 2 with topological spaces T_1 and T_2 to represent their epistemics in a shared formal language. Let us assume, for the sake of the example, both 1 and 2 are graduate students in a computer science department, and one day they receive an e-mail sent to all graduate students which says that there is *not* going to be an end-of-semester party. In their individual epistemics, this clearly creates an update. From topological semantics point of view, the opens in the topology represent set of the epistemic possible worlds per agent. In our example, if 1 did not know that there was going to be a party, this update may not create any change. If 2 thought that if there was going to be a party, he could have invited a friend over or if there was not going to be a party, he could have taken the same friend to a movie, this update will have an effect. In other words, at 1's current state there is no open set in which all states satisfy the proposition that there will be a party - namely, there is no set of epistemic possibilities satisfying the proposition. At 2's state, then the option of inviting the friend over to the party will be eliminated. Therefore, the open set about 2's current state will be updated without affecting 1's. In short, in their parallel lives, both 1 and 2 update their own epistemics, and they can express it in a common language. More importantly, in their joint epistemics (since they both are graduate students), there is also an update. Therefore, if x is the current state of 1, and 2 is the current state of y, and, if we want to discuss how they both update their space at the same time, we need to consider a space of tuples (x, y), and the open sets associated with those tuples. In that space, both 1 and 2 can communicate, and update their states with respect to their original epistemics given with T_1 and T_2 respectively. Therefore, their individual knowledge is still expressible in this higher space, and moreover, we can also express how announcement update their space.

Let $\mathcal{T} = \langle T, \tau \rangle$ and $\mathcal{T}' = \langle T', \tau' \rangle$ be two given topological spaces. Now, we introduce some definitions. Let $X \subseteq T \times T'$. We call X horizontally open (*h-open*) if for any $(x, y) \in X$, there is a $U \in \tau$ such that $x \in U$, and $U \times \{y\} \subseteq X$. In a similar fashion, we call X vertically open (*v-open*) if or any $(x, y) \in X$, there is a $U' \in \tau'$ such that $y \in U'$, and $\{x\} \times U' \subseteq X$. These notions can be seen as one dimensional projections of openness and closure that we will need soon.

Now, given two topological spaces $\mathcal{T} = \langle T, \tau \rangle$ and $\mathcal{T}' = \langle T', \tau' \rangle$, let us associate two modal operators I and I' respectively to these models. Then, we can obtain a product topology in a language with the two aforementioned modalities. The product model, then, is of the form $\langle T \times T', \tau, \tau' \rangle$. Therefore, we consider the cross product $\times$ as a way to represent model interaction among epistemic agents which gives us a model with two-dimensional space, and two topologies. The semantics of those modalities, then, are given as follows.

$(x, y) \models \mathsf{I}\varphi$ iff $\exists U \in \tau$, $x \in U$ and $\forall u \in U$, $(u, y) \models \varphi$

$(x, y) \models \mathsf{I}'\varphi$ iff $\exists U' \in \tau'$, $y \in U'$ and $\forall u' \in U'$, $(x, u') \models \varphi$

Here, given a tuple (x, y), the modality I ranges over the first component while the modality I' ranges over the second. In other words, we localize the product with respect to the given original topologies.

It has been shown that the fusion logic S4⊕S4 is complete with respect to products of arbitrary topological spaces (van Benthem & Sarenac, 2004). Then, the question for us is this: How would a state elimination based dynamic epistemic paradigm work in product topologies?

Now, step by step, we will present how to define public announcements in this framework. The difficulty lies in the fact that when we take the product of the given topological models, we increase the dimension of the space. Then, the intuition behind defining public announcements should follow the same idea: the announcement will update the product topology in all dimensions.

Let us now be a bit more precise. Before we start, note that here we focus on the product of two topologies representing the interaction between two agents with different spaces and topologies, but it can easily be generalized to n-agents. The language of product topological PAL is given as follows.

$$p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \mathsf{K}_1\varphi \mid \mathsf{K}_2\varphi \mid [\varphi]\varphi$$

For given two topological models $\mathcal{T} = \langle T, \tau, v \rangle$ and $\mathcal{T}' = \langle T', \tau', v \rangle$, the product topological model $M = \langle T \times T', \tau, \tau', v \rangle$ has the following semantics.

$M, (x, y) \models \mathsf{K}_1\varphi$ iff $\exists U \in \tau$, $x \in U$ and $\forall u \in U$, $(u, y) \models \varphi$

$M, (x, y) \models \mathsf{K}_2\varphi$ iff $\exists U' \in \tau'$, $y \in U'$ and $\forall u' \in U'$, $(x, u') \models \varphi$

$M, (x, y) \models [\varphi]\psi$ iff $M, (x, y) \models \varphi$ implies $M_\varphi, (x, y) \models \psi$

where $M_\varphi = \langle T_\varphi \times T'_\varphi, \tau_\varphi, \tau'_\varphi, v_\varphi \rangle$ is the updated model. We define all T_φ, T'_φ, τ_φ, τ'_φ, and v_φ as before. Therefore, the following axioms axiomatize the product topological PAL together with the axioms of S4$\oplus$S4.

1. $[\varphi]p \leftrightarrow (\varphi \to p)$
2. $[\varphi]\neg\psi \leftrightarrow (\varphi \to \neg[\varphi]\psi)$
3. $[\varphi](\psi \land \chi) \leftrightarrow ([\varphi]\psi \land [\varphi]\chi)$
4. $[\varphi]\mathsf{K}_i\psi \leftrightarrow (\varphi \to \mathsf{K}_i[\varphi]\psi)$

Theorem 1.4.4. *Product topological PAL is complete and decidable with respect to the given axiomatization.*

Proof. Proof of both completeness and decidability is by reduction, and similar to the ones presented before. Thus, we leave the details to the reader. ■

1.5 Applications

Now, we can briefly apply the previous discussions to some issues in public announcements, foundational game theory and subset spaces. The purpose of such applications is to give the reader a sense as to how topological frameworks might affect the aforementioned issues, and in general how dynamic epistemic situations can be represented topologically.

1.5.1 Announcement Stabilization

The Muddy Children game presents an interesting case for public announcement logic (Fagin *et al.*, 1995). The game can be described as follows. Assume a group of children played outside in the mud. Then, their father calls them in, and they come back in, and gather around the father in such a way that every children sees all the others, and the father sees them all. We also assume that there is no mirror in the room, so the children cannot see themselves. Since they were playing in the mud, some got dirty with mud on their forehead. Father then announces that "At least one of you has mud on his or her forehead." If no child steps forward saying that "Yes, I do have mud on my forehead" communicating the fact that she learned it from the announcement, the father keeps repeating the very same announcement. The puzzle pops up when we ask how many announcements suffice for a given number of children to figure out which children have mud on their forehead (van Ditmarsch *et al.*, 2007).

In that game, the model representing the epistemics of the group (see the Figure 1.1) gets updated after each children says that s/he does not know if she had mud on her forehead. The model keeps being updated until the announcement is negated, and then becomes common knowledge (van Benthem, 2007). Therefore, after each update, we get smaller and smaller models until the moment that the final model gets stabilized in the sense that the same announcement does not update it any longer.

As in most logical abstractions, the core idea of PAL suffers from some philosophical problems. Most importantly, in this framework, the minds of the agents are assumed to have a common language over which we put a logical structure. This is a strong assumption, and various objections can be raised against it which we will not discuss here. Second, the reduction axioms show that PAL is not more expressive than the basic (modal) epistemic logic. Then, naively, one can ask: what is the point of having PAL to discuss epistemic puzzles? The common response to this question deals with the succinctness of PAL (Kooi, 2007). Nevertheless, the aforementioned notion of model stabilization presents an interesting case within the framework of PAL, and shows how PAL approaches multi-agent knowledge interaction.

As van Benthem pointed out, this is closely related to several issues in modal and epistemic logics (van Benthem, 2007). First, PAL behaves like a fixed-point operator where the fixed point is the final model which is stabilized. Second, there seems to be a close relation between game theoretical strategy eliminations, and solution methods based on such

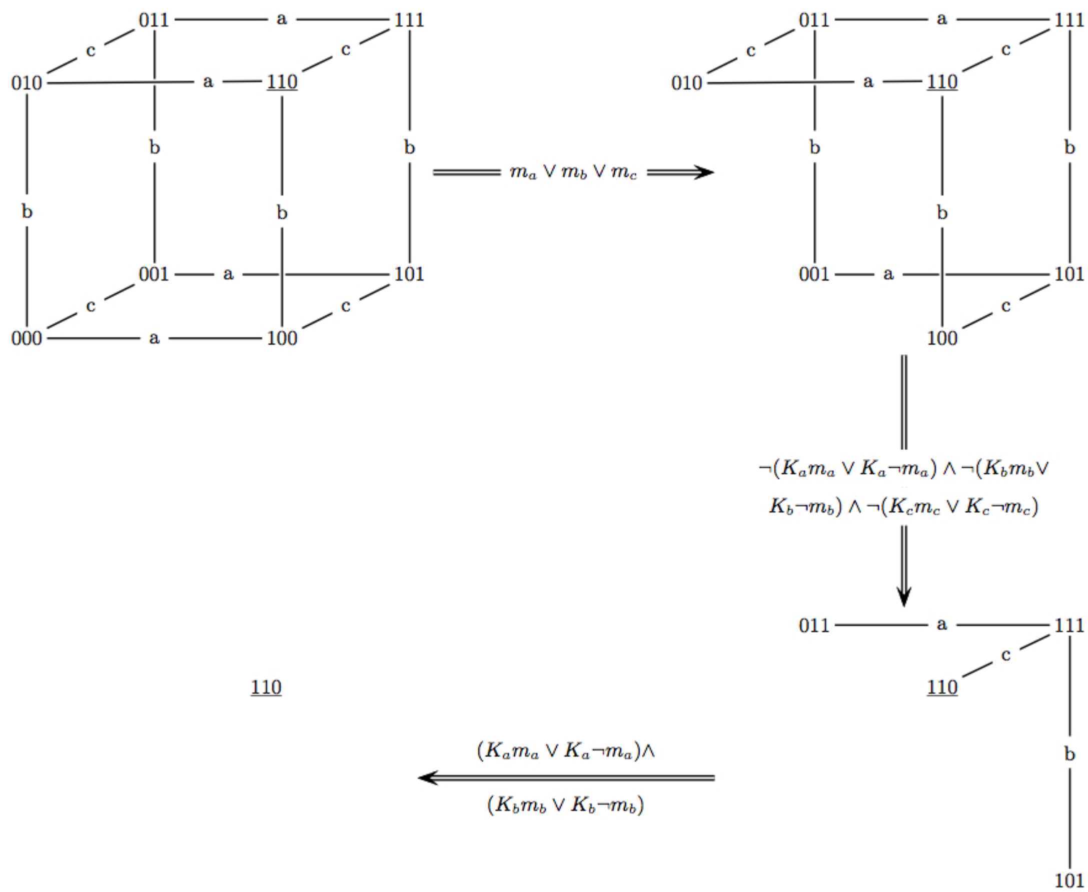

Figure 1.1: A model for muddy children played with 3 children a, b, c taken from (van Ditmarsch *et al.*, 2007). The state $n_a n_b n_c$ for $n_a, n_b, n_c \in \{0, 1\}$ represent that child i has mud on her forehead iff $n_i = 1$ for $i \in \{a, b, c\}$. The proposition m_i means that the child $i \in \{a, b, c\}$ has mud on her forehead. The current state is underlined.

approaches. Therefore, it is rather important to analyze announcement stabilization. Here, we will approach the issue from a topological angle.

For a model M and a formula φ, we define the announcement limit $\lim_\varphi M$ as the first model which is reached by successive announcements of φ that no longer changes after the last announcement is made. Announcement limits exist in both finite and infinite models (van Benthem & Gheerbrant, 2010). For instance, for any model M, $\lim_p M = M|p$ for propositional variable p. Therefore, the limit model is the first updated model when the announcement is a ground Boolean formula. In *muddy children*, the announcement shrinks the model step by step, round by round (van Benthem, 2007). However, sometimes in dialogue games it may take *too long* to solve such puzzles until the model gets stabilized as shown by Parikh (Parikh, 1991). Similarly, even Zermelo considered similar approaches in the early 20th century to understand as to how long it takes for the game to stabilize (Schwalbe & Walker, 2001).

Similar to the discussions of the aforementioned authors, we now analyze how the models stabilize in topological PAL. We know that topological models do present some differences in epistemic logical structures. For instance, in topological models, the stabilization of the fixed-point definition version of common knowledge may occur later than ordinal stage ω. In the fixed-point definition, the formula φ is said to be common knowledge $C_{1,2}\varphi$ among two-agents 1 and 2, and is represented with the (largest) fixed-point definition as follows: $C_{1,2}\varphi := \nu\varphi.\varphi \wedge \mathsf{K}_1\varphi \wedge \mathsf{K}_2\varphi$ where K_i, for $i = 1, 2$ is the familiar knowledge operator (Barwise, 1988). On the other hand, the infinite iteration definition of common knowledge is traditionally given as $C_{1,2}\varphi := \varphi \wedge \mathsf{K}_1\varphi \wedge \mathsf{K}_2\varphi \wedge \mathsf{K}_1\mathsf{K}_2\varphi \wedge \mathsf{K}_2\mathsf{K}_1\varphi \wedge \ldots$.

However, models stabilize in less than ω steps in Kripke models when the fixed point definition of common knowledge is used (as opposed to the infinite iteration definition) (van Benthem & Sarenac, 2004). This result, we underline, does not contradict with Parikh's earlier result as Parikh did not use the fixed point definition of common knowledge in his work (Parikh, 1991). Therefore, naively, we can say that topological models deal with infinite cases better, and has tools to express and represent such situations by their very definition. For instance, models with real topology are suitable tools to deal with such situations.

We also know that there are two possibilities for the limit models. Either it is empty or nonempty. If it is empty, it means that the negation of the announcement has become common knowledge, thus the announcement refuted itself. On the other hand, if the limit model is not empty, it means that the announcement has become common knowledge (van Benthem & Gheerbrant, 2010).

The case of announcement stabilization differs in topological models. We now observe the following.

Theorem 1.5.1. *For some formula φ and some topological model M, it may take more than ω stages to reach the limit model $\lim_\varphi M$.*

Proof. The proof is rather immediate for those familiar with the literature. So, we just mention the basic ideas here.

First, note that it was shown that in multi-agent topological models, stabilization of common knowledge with fixed-point definition may occur later than ω stage. However, in Kripke models it occurs before ω stage (Parikh, 1991; van Benthem & Sarenac, 2004).

Also note that it was also shown that if the limit model is not empty, the announcement has become common knowledge (van Benthem & Gheerbrant, 2010).

Therefore, combining these two observations, we conclude that in some topological models with non-empty limit models, the number of stages for the announcement to be common knowledge may take more than ω steps. ■

Even if the stabilization takes longer, we can still obtain stable models by taking intersections at the limit ordinals as a general rule (Parikh, 1991; Barwise, 1988; van Benthem & Gheerbrant, 2010). Therefore, we guarantee that the update procedure will terminate. Thus, the aforementioned result is self-evident.

Theorem 1.5.2. *Limit models exist for a given topological model.*

Then, it can be asked if there is a closure ordinal for announcement stabilization. This question can be answered directly. The recursive fixed point-definition of common knowledge is described as the *fixed-point of a descending approximation sequence defined over the set of ordinals* (van Benthem & Sarenac, 2004). This is well-defined as the sequence always terminates.

Yet another property of topological models is the fact that the topologies are *not* closed under arbitrary intersection due to the very definition. Then, one can ask the following question: "How does PAL work in infinite-conjunction announcements?" assuming our language allows such expression. The following example illustrates that point. Take the real closed interval $[-1, 1]$ with the usual Euclidean topology. For each $n \in \omega$, define the valuation for propositions as such $v(p_n) = [-1/n, 1/n]$. Therefore, p_1 holds in the entire space $[-1, 1]$, while p_2 holds in $[-1/2, 1/2]$. Consider now the announcements $\Box \bigwedge_{n\in\omega} p_n$ and $\bigwedge_{n\in\omega} \Box p_n$. The first formula is true in the interior $\mathbb{I}(\bigcap_{n\in\omega} p_n)$ which is equal to the empty set while the latter one is true in the intersection $\bigcap_{n\in\omega} \mathbb{I}(p_n)$ which is equal to the singleton $\{0\}$. Then, clearly these updates will yield the same models in Kripke models. But, in topological models, as the extensions of two formulas differ, updated models will clearly differ, too.

1.5.2 Backward Induction

The fact that limit models can be attained in more than ω steps can create some problems in games. Consider the backward induction solution where players trace back their moves to develop a winning strategy. Notice that the Aumann's backward induction solution assumes common knowledge of rationality (Aumann, 1995; Halpern, 2001) (Although according to Halpern, Stalnaker proved otherwise (Halpern, 2001; Stalnaker, 1998; Stalnaker, 1994; Stalnaker, 1996)). Even though there are several philosophical and epistemic issues about the centipede game and its relationship with rationality that the aforementioned authors mentioned, we will not pursue the philosophical direction here, and leave it to a future work (Artemov, 2009a; Artemov, 2009b).

The backward induction scheme can also be approached from a dynamic epistemic perspective. Recently, it has been shown that in any game tree model M taken as a PAL model, $\lim_{\mathsf{rational}} M$ is the actual subtree computed by the backward induction procedure where the announced proposition $\mathsf{rational}$ means that "at the current node, no player has chosen a strictly dominated move in the past coming here" - where strict domination is understood in the same way as in the classical game theory (van Benthem & Gheerbrant, 2010). Even

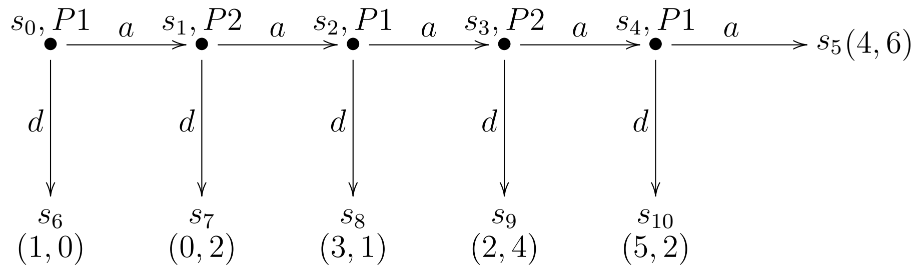

Figure 1.2: The Centipede game

if the mathematically precise definition of the proposition rational was not given by the authors, the announcement of node-rationality produces the same result as the backward induction procedure. Each backward step in the backward induction procedure can then be obtained by the public announcement of node rationality. This result is quite impressive in the sense that it establishes a closer connection between communication and rationality, and furthermore leads to several more intriguing discussions about rationality. Similarly, Fitting proposed an extension of propositional dynamic logic with epistemic modality to discuss rationality and, specifically the centipede game where rationality implies that the player makes the best move (Fitting, 2011). Similarly Parikh et al. discussed the centipede in a framework where players have a selection function (Parikh *et al.*, 2012). In this current work, we refrain ourselves from pursuing such line of thoughts for the time being.

However, there seems to be a problem in topological models. The admissibility of limit models can take more than ω steps in topological models as we have conjectured earlier. Therefore, the backward induction procedure can take ω steps or more.

Theorem 1.5.3. *In topological models of games, under the assumption of rationality, the backward induction procedure can take more than ω steps.*

Proof. Notice that each tree can easily be converted to a topology by taking the upward closed sets as opens. By the previous discussion, we know that backward induction solution can be attained by obtaining the limit models by publicly announcing the proposition rationality. Therefore, by Theorem 1.5.1, stabilization can take more than ω steps. Thus, the corresponding backward induction scheme can also take more than ω steps. ■

This is indeed a problem about the attainability and rationality in infinite games: how can a player continue playing the game after she has hit the limit ordinal ω-th step in the backward induction procedure? How can a rational player make an ω-many moves? In order not to diverge from our current focus, we leave this question open for further research.

1.5.3 Persistence

Let us now discuss a similar notion of stabilization in the framework of SSL, called *persistence* (Dabrowski *et al.*, 1996; Parikh *et al.*, 2007). Define *persistent* formula in a model M as the formula φ whose truth is independent from the subsets in M. In other words, φ is persistent if for all states s and subsets $V \subseteq U$, we have $s, U \models \varphi$ implies $s, V \models \varphi$. Clearly, Boolean formulas are persistent in every model.

The significance of persistent formulas is the fact that they are independent of the subsets they occupy which means that they are immune to the epistemics of the model. Now, since persistent formulas are independent from the epistemics of the model, intuitively, one may think that they should also be immune to the epistemic changes (revisions, updates) in the model. This is an interesting observation because now we have quite a powerful way to tell what can and cannot be updated by public announcements in SSL.

Theorem 1.5.4. *Let M be a model and φ be persistent in M. Then, for any formula χ and neighborhood situation (s, U), if $s, U \models \varphi$, then $s, U \models [\chi]\varphi$. In other words, true persistent formulas are immune to public announcements.*

Proof. Proof follows directly from the definitions and the fact that after the public announcement of χ, we always have $U_\chi \subseteq U$. ■

In other words, we can have some formulas in SSL framework that are immune to the announcements.

Note that the theorem is one directional. We showed that persistent formulas are immune to public announcements. This clearly does not imply that *only* persistent formulas are immune to announcements.

Moreover, for example, one can imagine a model where some subsets of the given neighborhood is not expressible in the language. Every formula that players can announce in this language may have a geometric extension, but not every geometric extension may correspond to a formula expressible in the language. For example, the standard topological approximation (by intersections) to a point may require infinitely many announcements, and propositional language is not strong enough to express infinite conjunctions (as announcements) (Başkent, 2007).

1.6 Conclusion and Future Work

In this chapter, we discussed PAL in two different geometrical systems. In subset space logic, we defined dynamic axioms for both epistemic and dynamic modalities and showed the corresponding completeness theorems. Moreover, we have applied the geometric ideas to model stabilization and persistent formulas. This gave us some hints about the connection between dynamic epistemic logics and rationality. We observed that in topological models, backward induction scheme loses its intuitiveness. However, there can be some mathematical solutions to this problem. For the backwards induction procedure that takes longer than ω, modal-mu calculus can also be considered with its natural game theoretical semantics to deal with infinite games. The mathematical reason for this is quite clear. As modal-mu calculus is the bisimulation invariant of the second-order logic (an extension of van Benthem's result that modal logic is the bisimulation invariant of the first-order logic), it has an impressive expressive power. Moreover, fixed-point logics, and their game theoretical semantics have sufficient tools to reason about games with infinite paths.

An interesting fact about topological models is that only modal formulas can definitely produce open or closed sets. However, one can stipulate that the extension of *any* modal formula can be open (or dually closed). If that is the case, one can obtain incomplete or inconsistent logics respectively (Başkent, 2011c; Mortensen, 2000). Moreover, some special algebras such as Heyting and co-Heyting algebras correspond to those logics. Therefore, the topological investigation of PAL can be carried out in these special topological spaces or algebras to discuss public announcements in paraconsistent and paracomplete logics.

Chapter 2

Paraconsistent Topologies and Homotopies

2.1 Motivation

Bisimulations and van Benthem's celebrated related theorem provide a direct insight how truth preserving operations work in modal logics. Apart from bisimulations, there are several other operations in basic modal logic that preserve the truth, and they are definable in some geometric logics (such as SSL) in a nice way (Blackburn *et al.*, 2001; Başkent, 2007).

However, there is a problem. Given a modal model, and several bisimilar copies of it, there is no method to *compare* or *measure* the differences between bisimilar models apart from the basic model theoretical methods (i.e. if they are submodels of each other, for instance). A rather negative slogan for this issue is the following: *Modal language cannot distinguish bisimilar models*.

Since the modal language *cannot count or measure*, several extensions of the language has been proposed to tackle this issue such as hybrid logics and majority logics (Blackburn, 2000; Fine, 1972). However, such extensions are language based, and introduce a non-natural and sometimes counter-intuitive operators to the language which we find *ad-hoc* and circular.

In this chapter, we focus on truth preserving operations rather than extending the modal language. In other words, we ask the following question: *Is there a truth preserving natural modal operation that can also distinguish the models it generates, and even compare them with respect to a parameter?* We argue that such an operation exists, and the answer to that question is positive.

However, to conceptualize our concerns and questions, we need to be careful in choosing the correct modal logical framework. Kripke models, in this respect, are criticized as they are overly simplistic and can overshadow some mathematical properties that can be apparent in some other modal models as we exemplified in the previous chapter. Therefore, in this section, we again concentrate on topological models as they can provide us with much stronger and richer structure of which we can take advantage. In this chapter, we utilize a rather elementary concept from topology. We first introduce homeomorphisms, and then homotopies to classical and non-classical modal logical frameworks, and show some invariance results. On the other hand, from an application oriented point of view, we also have some applications to illustrate how our constructions can be useful. Let us start with defining paraconsistency in topological semantics.

2.2 Topological Semantics and Paraconsistency

An easy and immediate semantics of paraconsistent/paracomplete logics can be given by using topologies. For this reason, it is helpful to remember some basic concepts of paraconsistency.

First, note that deductive explosion describes the situation where any formula can be deduced from an inconsistent set of formulas (or from the deductive closure of the set), e.g. for all formulas φ and ψ, we have $\{\varphi, \neg\varphi\} \vdash \psi$, where $\vdash$ denotes a logical consequence relation. In this respect, both "classical" and intuitionistic logics are known to be explosive.

Paraconsistent logic, on the other hand, is the umbrella term for logical systems where the logical consequence relation $\vdash$ is *not* explosive (Priest, 2002). A variety of philosophical and logical objections can be raised against paraconsistency, and almost all of these objections can be answered in a rigorous fashion. We will not here focus on the philosophical implications of paraconsistency (or dialetheism), and we refer the reader to the following for a comprehensive defense of paraconsistency with a variety of well-structured applications (Priest, 1998).

Use of topological semantics for paraconsistent logics is not new. To our knowledge, the earliest work discussing the connection between inconsistency and topology goes back to Goodman (Goodman, 1981)[1]. In his paper, Goodman discussed "pseudo-complements" in a lattice theoretical setting, and called the topological system he obtains "anti-intuitionistic logic". In a recent work, Priest discussed the dual of the intuitionistic negation operator, and considered that operator in topological framework (Priest, 2009). Similarly, Mortensen discussed topological separation principles from a paraconsistent and paracomplete point of view, and investigated the theories in such spaces (Mortensen, 2000). Similar approaches from a modal perspective was discussed by Béziau, too (Béziau, 2005).

Recall that while giving a topological semantics for modal logics, we associate the extensions of modal formulas with topological sets. In other words, the extensions of Booleans may or may not be open or closed. We can take one step further and suggest that extension of *any* propositional variable be an open set (Mortensen, 2000; Mints, 2000). In that setting, conjunction and disjunction works fine for finite intersections and unions. Nevertheless, the negation can be difficult as the complement of an open set is not generally an open set, thus may not be the extension of a formula in the language. For this reason, we need to use a new negation symbol $\dot{\sim}$ that returns the open complement (interior of the complement) of a given set. A similar idea can also be applied to closed sets where

[1] Thanks to Chris Mortensen for this remark. Even if Goodman's paper appeared in 1981, the work had been carried out around 1978. In his paper, Goodman indicated that the results were based on an early work that appeared in 1978 only as an abstract.

we assume that the extension of any propositional variable will be a closed set. In order to be able to avoid a similar problem with the negation, we stipulate yet another negation operator which returns the closed complement (closure of the complement) of a given set. In this setting, we use the symbol $\sim$ that returns the closed complement of a given set.

Now, for our current purposes here, let us take two separate topologies σ and τ of closed and open sets respectively. Then, it is easy to observe the following (Mortensen, 2000).

- In the topology of closed sets σ, any theory that includes the boundary points will be inconsistent.
- In the topology of open sets τ, any theory that includes the theory of the propositions that are true at the boundary is incomplete.

2.3 Homotopies

In this section, we introduce homotopies into the context of classical and paraconsistent (basic) modal logics. To the best of our knowledge, this is the first attempt to generalize this very basic notion of topology in classical and non-classical contexts.

Here, we will start off with the easier case and consider paraconsistent topological spaces. Second, we extend our results to classical case.

2.3.1 Paraconsistent Case

Topological modal logics with continuous functions were discussed in various early work (Artemov *et al.*, 1997; Kremer & Mints, 2005). A following theorem, which was stated and proved in variety of different work, would also work for paraconsistent logics (Kremer & Mints, 2005). Now, let us take two closed set topologies σ and σ' on a given set S and a homeomorphism $f : \langle S, \sigma \rangle \to \langle S, \sigma' \rangle$. Akin to a previous theorem of Kremer and Mints, we have a simple way to associate the respective valuations between two models M and

M' which respectively depend on σ and σ' so that we can have a truth preservation result. Therefore, define $V'(p) = f(V(p))$. Then, we have $M \models \varphi$ iff $M' \models \varphi$.

Theorem 2.3.1. *Let $M = \langle S, \sigma, V \rangle$ and $M' = \langle S, \sigma', V' \rangle$ be two paraconsistent topological models with a homeomorphism f from $\langle S, \sigma \rangle$ to $\langle S, \sigma' \rangle$. Define $V'(p) = f(V(p))$. Let $w \in S$ and $w' = f(w)$, then for all φ, we have $M, w \models \varphi$ iff $M', w' \models \varphi$ for all φ.*

Proof. The proof is by induction on the complexity of the formulas.

Let $M, w \models p$ for some propostional variable p. Then, $w \in V(p)$. Since we are in a paraconsistent topological model, $V(p)$ is a closed set and since f is a homeomorphism $f(V(p))$ is closed as well, and $f(w) \in f(V(p))$. Thus, $M', f(w) \models p$. Converse direction is similar and based on the fact that the inverse function is also continuous.

Negation $\sim$ is less immediate. Let $M, w \models \sim \varphi$. Therefore, w is in the closure of the complement of $V(\varphi)$. So, $w \in \mathsf{Clo}((V(\varphi))^c)$. Then, $f(w) \in f(\mathsf{Clo}(V(\varphi))^c)$. Moreover, since f is bicontinuous as f is a homeomorphism, we observe that $f(w) \in \mathsf{Clo}(f((V(\varphi))^c))$. Then, by the induction hypothesis, $f(w) \in \mathsf{Clo}((V'(\varphi))^c)$ yielding $M', f(w) \models \sim \varphi$. Converse direction is also similar.

We leave the conjunction case to the reader and proceed to the modal case. Assume $M, w \models \Diamond\varphi$. Thus, $w \in V(\Diamond\varphi)$. Thus, $w \in \mathsf{Clo}(V(\varphi))$. Then, $f(w) \in f(\mathsf{Clo}(V(\varphi)))$. Since f is a homomorphism, we have $f(w) \in \mathsf{Clo}(f(V(\varphi)))$. By the induction hypothesis, we then deduce that $f(w) \in \mathsf{Clo}(V'(\varphi))$ which in turn yields that $f(w) \in V'(\Diamond\varphi)$. Thus, we deduce $M', f(w) \models \Diamond\varphi$.

Converse direction is as expected and we leave it to the reader. ■

Note that the above theorem also works in paracomplete topological models, and we leave the details to the reader.

Theorem 2.3.2. *Let $M = \langle S, \tau, V \rangle$ and $M' = \langle S, \tau', V' \rangle$ be two paracomplete topological models with a homeomorphism f from $\langle S, \sigma \rangle$ to $\langle S, \sigma' \rangle$. Define $V'(p) = f(V(p))$. Let $w \in S$ and $w' = f(w)$, then for all φ, we have $M, w \models \varphi$ iff $M', w' \models \varphi$ for all φ.*

Now, assuming that f is a homeomorphism may seem a bit strong. We can then separate it into two chunks. One direction of the biconditional can be satisfied by continuity whereas the other direction is satisfied by the openness of f.

Corollary 2.3.3. *Let* $M = \langle S, \sigma, V\rangle$ *and* $M' = \langle S, \sigma', V'\rangle$ *be two paraconsistent topological models with a continuous* f *from* $\langle S, \sigma\rangle$ *to* $\langle S, \sigma'\rangle$. *Define* $V'(p) = f(V(p))$. *Then* $M, w \models \varphi$ *implies* $M', w' \models \varphi$ *for all* φ *where* $w' = f(w)$.

Corollary 2.3.4. *Let* $M = \langle S, \sigma, V\rangle$ *and* $M' = \langle S, \sigma', V'\rangle$ *be two paraconsistent topological models with an open* f *from* $\langle S, \sigma\rangle$ *to* $\langle S, \sigma'\rangle$. *Define* $V'(p) = f(V(p))$. *Then* $M', w' \models \varphi$ *implies* $M, w \models \varphi$ *for all* φ *where* $w' = f(w)$.

Proofs of both corollaries depend on the fact that Clo operator commutes with continuous functions in one direction, and it commutes with open functions in the other direction. Furthermore, similar corollaries can be given for paracomplete frameworks as the Int operator also commutes in one direction under similar assumptions, and we leave it to the reader as well.

Furthermore, any topological operator that commutes with continuous, open and homeomorphic functions will reflect the same idea and preserve the truth[2]. Therefore, these results can easily be generalized.

We can now take one step further to discuss homotopies in paraconsistent topological modal models.

Simply put, a *homotopy* is a description of how two continuous functions from a topological space to another can be deformed to each other. Homotopies are central in topology, and most known algebraic invariants of topological spaces are homotopy invariants (Dugundji, 1966). Moreover, a recent initiative by Voevodsky and Awodey considers homotopies within the framework of type theory to obtain *comprehensive, computational foundation for mathematics based on the homotopical interpretation of type theory*[3].

[2]Thanks to Chris Mortensen for pointing this out.
[3]See their webpage for the details `http://homotopytypetheory.org`

We can now state the formal definition of a homotopy.

Definition 2.3.5. Let S and S' be two topological spaces with continuous functions $f, g : S \to S'$. A homotopy between f and g is a continuous function $H : S \times [0,1] \to S'$ such that if $s \in S$, then $H(s,0) = f(s)$ and $H(s,1) = g(s)$.

In other words, a homotopy between f and f' is a family of continuous functions $H_t : S \to S'$ such that for $t \in [0,1]$ we have $H_0 = f$ and $H_1 = g$ and the map $t \mapsto H_t$ is continuous from $[0,1]$ to the space of all continuous functions from S to S'. Notice that homotopy relation is an equivalence relation. Thus, if f and f' are homotopic, we denote it with $f \approx f'$. We will now use homotopies to obtain a generalization of Theorem 2.3.1.

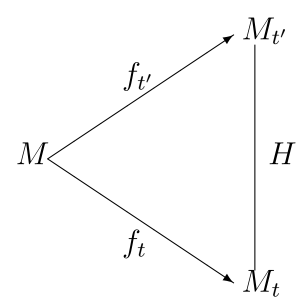

Figure 2.1: Homotopic Models

Definition 2.3.6. Given a model $M = \langle S, \sigma, V \rangle$, we call the family of models $\{M_t = \langle S_t \subseteq S, \sigma_t, V_t \rangle\}_{t \in [0,1]}$ generated by M and homotopic functions homotopic models. In the generation, we put $V_t = f_t(V)$.

Theorem 2.3.7. *Given two topological paraconsistent models $M = \langle S, \sigma, V \rangle$ and $M' = \langle S', \sigma', V' \rangle$ with two continuous functions $f, f' : S \to S'$ both of which respect the valuation: $V' = f(V) = f'(V)$. If there is a homotopy H between f and f', then homotopic models satisfy the same formulas.*

Proof. To make the proof a bit more readable, note that the object whose names have a prime$'$, are in the range of the functions.

To make the proof go easily, we will assume that continuous functions are onto. If not, we can easily rearrange the range in such a way that it will be.

Let $M_t = \langle S'_t, \sigma_t, V_t \rangle$ and $M_{t'} = \langle S'_{t'}, \sigma_{t'}, V_{t'} \rangle$ be homotopic models with continuous $f_t, f_{t'} : S \to S'$. Observe that $M_t = \langle S'_t, \sigma_t, f_t(V) \rangle$, or equally $M_t = \langle S'_t, \sigma_t, H_t(V) \rangle$ for homotopy H.

Take a point s_t such that $f_t(s)$ for some $s \in S$.

Then, take $\langle S'_t, \sigma_t, H_t(V) \rangle, f_t(s) \models \varphi$ for arbitrary φ. Since, H is continuous on t by some h, and by Corollary 2.3.3, we observe $M_{t'} = \langle S_{t'}, \sigma_{t'}, H_{t'}(V) \rangle, s_{t'} \models \varphi$. In this case $s_{t'}$ exists as H_t is continuos on t and $s_{t'} = h(f_t(s)) = h(s_t)$. Therefore, $M_t \models \varphi$ implies $M_{t'} \models \varphi$.

Notice that in this case, we did not need to present a proof on the complexity of φ. The reason for that is the fact that the extension of each formula is a closed set (since we are in a paraconsistent setting). ■

Corollary 2.3.8. *$M \models \varphi$ implies $M_t \models \varphi$, but not the other way around.*

Corollary 2.3.9. *In Theorem 2.3.7, if we take the cases for $t = 0$ and $t = 1$, we obtain Corollary 2.3.3.*

Notice that we have discussed the truth in the image sets that are obtained under the functions $f, f', f_t, \ldots$. Nevertheless, the converse can also be true, once the continuous functions have continuous inverses: this is exactly what is guaranteed by homeomorphisms. The corresponding notion at the level of homotopies is an *isotopy*. An isotopy is a continuous transformation between homeomorphic functions. Thus, we have the following.

Theorem 2.3.10. *Given two topological paraconsistent models $M = \langle S, \sigma, V \rangle$ and $M' = \langle S', \sigma', V' \rangle$ with two homeomorphisms $f, f' : S \to S'$ both of which respect the valuation: $V' = f(V) = f'(V)$. If there is an isotopy H between f and f', then, for all φ, we have the following*

$$M_t \models \varphi \textit{ iff } M \models \varphi \textit{ iff } M' \models \varphi$$

Proof. Immediate, thus left to the reader. ■

What makes the non-classical case easy is the fact that the extension of each formula is an open or a closed set. Furthermore, a similar theorem can be stated for paracomplete cases with a similar proof.

Theorem 2.3.11. *Given two topological paracomplete models $M = \langle S, \sigma, V \rangle$ and $M' = \langle S', \sigma', V' \rangle$ with two continuous functions $f, f' : S \to S'$ both of which respect the valuation: $V' = f(V) = f'(V)$. If there is a homotopy H between f and f', then homotopic models satisfy the same formulas.*

2.3.2 Classical Case

The reason why homotopies work nicely in the paraconsistent case is immediate: because we stipulate that the extension of propositions to be open (or dually closed) sets. This is a strong assumption, and makes the proof relatively easy.

Can we then have results in classical case similar to those we had in non-classical case? This is the problem we are going to address in this section.

Let $N = \langle T, \eta, V \rangle$ and $N' = \langle T', \eta', V' \rangle$ be classical topological modal models. Define a homotopy $H : T \times [0, 1] \to T'$. Therefore, as before, for each $t \in [0, 1]$, we obtain models $N_t = \langle T_t, \eta_t, V_t \rangle$.

Theorem 2.3.12. *Given two classical topological modal models $N = \langle T, \eta, V \rangle$ and $N' = \langle T', \eta', V' \rangle$ with two continuous functions $f, f' : T \to T'$ both of which respect the valuation: $V' = f(V) = f(V')$. If there is a homotopy H between f and f', then N and N' satisfy the same formulas.*

Proof. Let two classical topological modal models $N = \langle T, \eta, V \rangle$ and $N' = \langle T', \eta', V' \rangle$ with two continuous functions $g, g' : T \to T'$ both of which respect the valuation: $V' = f(V) = f(V')$ be given. Let H be a homotopy $H : T \times [0, 1] \to T'$. We will show that homotopic models satisfy the same formula.

Take two homotopic models N_t and $N_{t'}$ for $t, t' \in [0,1]$. Let $f_t(w_t) \in T_t$ be a point in N_t where $f_t : T \to T_t$. Consider $N_t, f_t(w_t) \models \varphi$. We will show that for some $f_{t'}(w_{t'}) \in T_{t'}$, we will have $N_{t'}, f_{t'}(w_{t'}) \models \varphi$.

Proof is by induction on the complexity of φ. First, let $\varphi = p$ for a propositional variable p. Then, let $N_t, f_t(w_t) \models p$. By using H, we can rewrite as $N_t, H(w_t, t) \models p$. Therefore, $H(w_t, t) \in V_t(p)$. Since H is continuous on t, we have a continuous function from $i : t \to t'$. Now, since $V_t = f(V)$ we observe $H(w_t, t) \in f_t(V(p))$ which is equivalent to say $H(w_t, t) \in H(V(p), t)$. We can compose both sides with i to get $H(w_{t'}, t') \in H(V(p), t')$ for some $w_{t'} \in T$. In short, we obtain $N_{t'}, f_{t'}(w_{t'}) \models p$.

Note that since H is a homotopy, the function $i : t \to t'$ exists and is continuous. Similarly, another function $j : t' \to t$ exists and is continuous, and j is needed to prove the other direction. The cases for Boolean φ is similar and thus left to the reader.

Let us now consider the modal case $\varphi = \Box\psi$ for some ψ. Now, let $N_t, f_t(w_t) \models \Box\psi$. Thus, $f_t(w_t) \in \mathsf{Int}([\psi])$ where $[\psi]$ denotes the extension of ψ. Since f_t is continuous the inverse image of an open set is open, thus $f_t^{-1}(\mathsf{Int}([\psi]))$ is open. For the previously constructed i, we observe $i^{-1} \circ (f_t^{-1}(\mathsf{Int}([\psi])))$ is also open as f_t and i are both continuous. Thus, the inverse image of $\mathsf{Int}([\psi])$ is open under $f_{t'}$. Therefore, by the similar reasoning, $H(w_{t'}, t') \in \mathsf{Int}([\psi])$ for some $w_{t'}$ in the neighborhood. Thus, $N_{t'}, f_{t'}(w_{t'}) \models \Box\psi$.

The reverse direction of the proof from $N_{t'}$ to N_t is similar, and this concludes the proof. ■

In conclusion, under a suitable valuation, isotopic models are truth invariant both in classical and non-classical cases that we have investigated.

2.4 A Modal Logical Application

Consider the following two bisimilar Kripke models M and M'. Assume that $\{w, w'\}$ and $\{u, u', y'\}$ and $\{v, v', x'\}$ do satisfy the same propositional letters. Then, it is easy to see that w and w' are bisimilar, and therefore satisfy the same modal formulas.

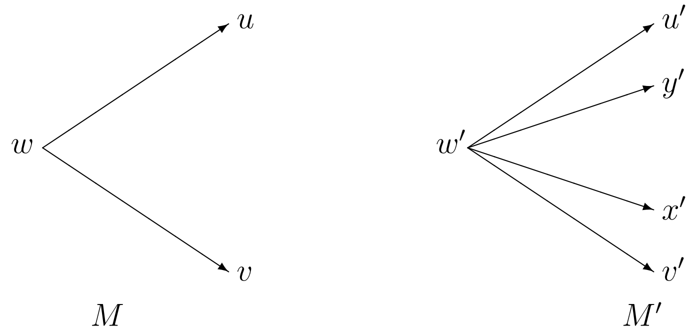

Figure 2.2: Two bisimilar Models

We can still pose a conceptual question about the relation between M and M'. Even if these two models are bisimilar, they are different models. Moreover, it is plausible to *contract* M' to M in a validity preserving fashion. Therefore, we may need to transform one model to a bisimilar model of it. Furthermore, given a model, we may need to *measure* the level of change from the fixed model to another model which is bisimilar to the given one.

Especially, in epistemic logic, such concerns do make sense. Given an epistemic situation, we can model it *upto bisimulations*. In other words, from an agent's perspective, bisimilar models are indistinguishable. But, from a model theoretical (say, global) perspective, they are distinguishable. Therefore, there can be several modal logical ways to model the given epistemic situation. We will now define how these models are related and different from each other by using the constructions we have presented earlier.

Before proceeding further, let us give the definition of a related concept, called *topological bisimulations* (Aiello & van Benthem, 2002).

Definition 2.4.1. Let $M = \langle S, \sigma, v\rangle$ and $M' = \langle S', \sigma', v'\rangle$ be two topological models. A topo-bisimulation is a nonempty relation $\rightleftarrows \subseteq S \times S'$ such that if $s \rightleftarrows s'$, then we have the following:

1. BASE CONDITION

   $s \in v(p)$ if and only if $s \in v'(p)$ for any propositional variable p.

2. FORTH CONDITION

   $s \in U \in \sigma$ implies that there exists $U' \in \sigma'$ such that $s' \in U'$ and for all $t' \in U'$ there exists $t \in U$ with $t \rightleftarrows t'$

3. BACK CONDITION $s' \in U' \in \sigma'$ implies that there exists $U \in \sigma$ such that $s \in U$ and for all $t \in U$ there exists $t' \in U'$ with $t \rightleftarrows t'$

We can take one step further and define a homoemorphism that respect bisimulations. Let $M = \langle S, \sigma, v \rangle$ and $M' = \langle S', \sigma', v' \rangle$ be two topo-bisimilar models. If there is a homoemorphism f from $\langle S, \sigma \rangle$ to $\langle S, \sigma \rangle$ that respect the valuation, we call M and M homeo-topo-bisimilar models. We give the precise definition as follows.

Definition 2.4.2. Let $M = \langle S, \sigma, v \rangle$ and $M' = \langle S', \sigma', v' \rangle$ be two topological models. A homeo-topo-bisimulation is a nonempty relation $\rightleftarrows_f \subseteq S \times S'$ based on a homeomorphism f from S into S' such that if $s \rightleftarrows_f s'$, then we have the following:

1. BASE CONDITION

   $s \in v(p)$ if and only if $s \in v'(p)$ for any propositional variable p.

2. FORTH CONDITION

   $s \in U \in \sigma$ implies that there exists $f(U) \in \sigma'$ such that $s' \in f(U)$ and for all $t' \in f(U)$ there exists $t \in U$ with $t \rightleftarrows_f t'$

3. BACK CONDITION $s' \in f(U) \in \sigma'$ implies that there exists $U \in \sigma$ such that $s \in U$ and for all $t \in U$ there exists $t' \in f(U)$ with $t \rightleftarrows_f t'$

Based on this definition, we immediately observe the following.

Theorem 2.4.3. *Homeo-topo-bisimulation preserve truth.*

Proof. The proof is a simple application of a proof by induction on the complexity of the formulas, and thus left to the reader. ■

Notice that we can define more than one homeomorphism between topo-bisimilar models.

Now, we can discuss the homotopy of homeo-topo-bisimilar models. What we aim to derive is the following. Given a topological model (either classical, intuitionistic or paraconsistent), we will construct two homeomorphic image of it respecting homeo-topo-bisimulation where these two homeomorphisms are homotopic. Then, by using homotopy, we will measure the level of change of the intermediate homeomorphic models with respect to these two functions.

Let M be a given topological model. Construct M_f and M_g as the homeomorphic image of M respecting the valuation where f and g are homeomorphism. For simplicity, assume that $M \rightleftarrows_f M_f$ and $M \rightleftarrows_g M_g$. Now, if f and g are homotopic, then we have functions h_x for x continuous on $[0, 1]$ with $h_0 = f$ and $h_1 = g$.

Therefore, given $x \in [0, 1]$, the model M_x will be obtained by applying h_x to M respecting the valuation. Hence, $M_0 = M_f$ and $M_1 = M_g$. Therefore, given M, the distance of any homeo-topo-bisimilar model M_x to M will be x, and it will be the measure of non-modal change in the model. In other words, even if $M \rightleftarrows_{h(x)} M_x$, we will say M and M_x are x-different than each other.

Consider the topological models for public announcement logic. In order to have a nice framework, assume that announcements update the model in a continuous way. Assume that we index announcements with $x \in [0, 1]$. A simple example for this situation might be a game where players are trying to approximate to an interval, and the announcements give an upper bound for the interval (not necessarily the lowest upper bound). In other words, announcements shrink the interval, make it closer and closer to the designated interval that the players do not know. Therefore, different announcements indexed with x_1 and x_2 give different approximations (with a measure). Therefore, the x-difference between updated models may correspond to the measure theoretical distance to the designated interval (from God's point of view as the agents may not know it).

The procedure we described offers a well-defined method of indexing the homeo-topo-bisimilar models. But, indexing is not random. It is continuously on the closed unit interval.

Note that invariance results are usually used to prove undefinability results in modal logic (Blackburn *et al.*, 2001). For example, in order to show irreflexivity is not modally definable, one needs to come up with two bisimilar models - one is irreflexive, the other is not.

Homeo-topo-bisimulations can also be used to show some topological properties are not modally definable. For instance, in this respect, dimensions of spaces is not modally definable in topological modal logic. Similarly, as trefoil and circle are homeomorphic, knots are also not definable.

2.5 Conclusion

This chapter elementarily steps into an amazing field of logic and mathematics: algebraic and geometric topology. The topological notions we have discussed here have very immediate and natural topological interpretations. In this regard, what we have achieved here is simply the baby steps of providing the connections between geometric topology and logic. Clearly, a chapter in a dissertation is not a fair treatment of the subject which begs for much closer attention.

Chapter 3

A Paradox in Game Theory

In this chapter, we put topological semantics and paraconsistency into work by considering a well-known paradox from epistemic game theory. We will consider a well-known game theoretical paradox, and analyze it non-classical frameworks.

3.1 Introduction

3.1.1 Motivation

The Brandenburger-Keisler paradox is a two-person self-referential paradox in epistemic game theory (Brandenburger & Keisler, 2006). Due to its considerable impact on various branches of game theory and logic, it has gained increasing interest in the literature.

In short, for players Ann and Bob, the Brandenburger-Keisler paradox ('BK paradox', henceforth) arises when we consider the following statement "Ann believes that Bob assumes that Ann believes that Bob's assumption is wrong" and ask the question if "Ann believes that Bob's assumption is wrong."

The notion of "assumption" here is quite strong. What they mean by assumption is "strongest belief" is a belief that implies all the other beliefs. In due course, we will mention the formal definition of the aforementioned modality to make it precise.

There can be considered two main reasons why the Brandenburger-Keisler argument turns out to be a paradox. First, the limitations of set theory present some restrictions on the mathematical model which is used to describe self-referantiality and circularity in the formal language. Second, Boolean logic comes with its own Aristotelian meta-logical assumptions about consistency. Namely, Aristotelian principle about consistency, *principium contradictionis*, maintains that contradictions are impossible. In this paper, we will consider some alternatives to such assumptions, and investigate their impact on the BK paradox.

The BK paradox is based on ZFC set theory. ZFC set theory comes with its own *restrictions*, one of which is the *axiom of foundation*. It can be deduced from this axiom that no set can be an element of itself. In non-well-founded set theory, on the other hand, the axiom of foundation is replaced by the *anti-foundation axiom* which leads to, among many other things, generation of sets which are members of themselves (Mirimanoff, 1917; Aczel, 1988). Therefore, we claim that switching to non-well-founded set theory suggests a new approach to the paradox, and game theory in general. The power of non-well-founded set theory comes from its genuine methods to deal with circularity (Barwise & Moss, 1996; Moss, 2009).

Second, what makes the BK paradox a *paradox* is the *principium contradictionis*. Paraconsistent logics challenge this assumption (Priest, 1998; Priest, 2006). Therefore, we also investigate the BK paradox in paraconsistent systems. This line of research, as we shall see, is rather fruitful. The reason for this is the following. The BK paradox is essentially a self-referential paradox, and similar to any other paradox of the same kind, it can be analyzed from a category theoretical or algebraic point of view (Yanofsky, 2003; Abramsky & Zvesper, 2010). Moreover, paraconsistent logics also present an algebraic and category theoretical structure which makes this approach possible. In this work, we make the connection between self-referantiality and paraconsistency clearer, and see whether we can *solve* the paradox if we embrace a paraconsistent framework.

What is the significance of adopting non-classical frameworks then? There are many situations where circularity and inconsistency are integral parts of the game. First, a game when some players can *reset* the game can be thought of a situation where the phenomenon of circularity appears. Namely, if the players are allowed to "restart" the game, some game states can be described by using those states themselves. Second, inconsistencies occur in games quite often as well. Situations where information or knowledge sets of some players become inconsistent after receiving some information in a dialogue without a consequent belief revision are such examples where inconsistencies occur (Lebbink *et al.*, 2004; Rahman & Carnielli, 2000).

In this chapter, we show that adopting the non-well-founded set theory makes a significant change in the structure of the paradox. We achieve this by constructing counter-models for the BK argument. Second, by paraconsistent logic, we show that, even when we allow non-trivial inconsistencies the BK argument *can* be satisfied in certain situations leading to the paradox. We also use topological products to give a weak-completeness result generalizing some of the results in the original paper.

3.1.2 Related Literature

The Brandenburger - Keisler paradox was presented in its final form in a relatively recent paper in 2006 (Brandenburger & Keisler, 2006). This paper was followed by some further results within the same domain (Brandenburger *et al.*, 2008).

A general framework for self-referential paradoxes was discussed earlier by Yanofsky in 2003 (Yanofsky, 2003). In his paper, Yanofsky used Lawvere's category theoretical arguments in well-known mathematical arguments such as Cantor's diagonalization, Russell's paradox, and Gödel's Incompleteness theorems. Lawvere, on the other hand, discussed self-referential paradoxes in cartesian closed categories in his early paper which appeared in 1969 (Lawvere, 1969). Most recently, Abramsky and Zvesper used Lawvere's arguments to analyze the BK paradox in a category theoretical framework (Abramsky & Zvesper, 2010).

Pacuit approached the paradox from a modal logical perspective and presented a detailed investigation of the paradox in neighborhood models and in hybrid systems (Pacuit, 2007). Neighborhood models are used to represent modal logics weaker than $\mathbf{K}$, and can be considered as weak versions of topological semantics (Chellas, 1980). This argument later was extended to assumption-incompleteness in modal logics (Zvesper & Pacuit, 2010).

To the best of our knowledge, the idea of using non-well-founded sets as Harsanyi type spaces was first suggested by Lismont, and extended later by Heifetz (Lismont, 1992; Heifetz, 1996). Heifetz motivated his approach by "making the types an explicit part of the states' structure", and hence obtained a circularity that enabled him to use non-well-founded sets.

Mariotti et al., on the other hand, used compact belief models to represent interactive belief structures in a topological framework with further topological restrictions (Mariotti *et al.*, 2005).

Paraconsistent games in the form of dialogical games were largely discussed by Rahman and his co-authors (Rahman & Carnielli, 2000). Co-Heyting algebras are used in "region based theories of space" within the field of *mereotopology* (Stell & Worboys, 1997). Mereotopology discusses the qualitative topological relations between the wholes, parts, contacts and boundaries and so on.

The organization of the current chapter is as follows. First, we recall the BK paradox stated in basic modal language. Then, we consider the matter from non-well-founded set theoretical point of view. Next, we introduce category theoretical and topological paraconsistent frameworks to deal with the paradox, and analyze the behavior of the paradox in such frameworks. Finally, we conclude with several research directions for future work.

3.1.3 The Paradox

The BK paradox can be considered as a game theoretical two-person version of Russell's paradox where players interact in a self-referential fashion. Let us call the players Ann and Bob with associated type spaces U^a and U^b respectively. Now, consider the following statement which we call the *BK sentence*:

> *Ann believes that Bob assumes that Ann believes that Bob's assumption is wrong.*

A Russell-like paradox arises if one asks the question whether *Ann believes that Bob's assumption is wrong*. In both cases, we get a contradiction, hence the paradox. Thus, the BK sentence is impossible. The key here is the assumption modality. Let us now set the framework, and then define the assumption modality in that framework.

Brandenburger and Keisler use belief sets to represent the players' beliefs. The model (U^a, U^b, R^a, R^b) that they consider is called a *belief model* where $R^a \subseteq U^a \times U^b$ and $R^b \subseteq U^b \times U^a$. The expression $R^a(x, y)$ represents that in state x, Ann believes that the state y is possible for Bob, and similarly for $R^b(y, x)$. We will put $R^a(x) = \{y : R^a(x, y)\}$, and similarly for $R^b(y)$. At a state x, we say Ann believes $P \subseteq U^b$ if $R^a(x) \subseteq P$. Now, a modal logical semantics for the interactive belief structures can be given. We use two different modalities $\Box$ and $\heartsuit$ which stand for the belief and assumption operators respectively with the following semantics.

$$x \models \Box^{ab}\varphi \quad \text{iff} \quad \forall y \in U^b . R^a(x, y) \text{ implies } y \models \varphi$$

$$x \models \heartsuit^{ab}\varphi \quad \text{iff} \quad \forall y \in U^b . R^a(x, y) \text{ iff } y \models \varphi$$

Those operators can also be given a modal definition (Brandenburger & Keisler, 2006). An interactive belief frame is a structure (W, P, U^a, U^b) with a binary relation $P \subseteq W \times W$, and disjoint sets U^a and U^b such that (U^a, U^b, P^a, P^b) is a belief model with $U^a \cup U^b = W$, $P^a = P \cap U^a \times U^b$, and $P^b = P \cap U^b \times U^a$. Now, for a given valuation function which assigns propositional variables to subsets of W, the semantics of the belief and assumption modalities are given as follows.

$$x \models \Box^{ab}\varphi \quad \text{iff} \quad w \models \mathbf{U}^a \wedge \forall y(P(x,y) \wedge y \models \mathbf{U}^b \text{ implies } y \models \varphi)$$
$$x \models \heartsuit^{ab}\varphi \quad \text{iff} \quad w \models \mathbf{U}^a \wedge \forall y(P(x,y) \wedge y \models \mathbf{U}^b \text{ iff } y \models \varphi)$$

A belief structure (U^a, U^b, R^a, R^b) is called *assumption complete* with respect to a set of predicates Π on U^a and U^b if for every predicate $P \in \Pi$ on U^b, there is a state $x \in U^a$ such that x assumes P, and for every predicate $Q \in \Pi$ on U^a, there is a state $y \in U^b$ such that y assumes Q. Since we assume that the type spaces for the players are disjoint, when we say that x assumes P, we mean that the agent whose type space has x assumes the predicate P.

We will use special propositions $\mathbf{U}^a$ and $\mathbf{U}^b$ with the following meaning: $w \models \mathbf{U}^a$ if $w \in U^a$, and similarly for $\mathbf{U}^b$. Namely, $\mathbf{U}^a$ is true at each state for player Ann, and $\mathbf{U}^b$ for player Bob.

Brandenburger and Keisler showed that no belief model is complete for its first-order language. Therefore, "not every description of belief can be represented" with belief structures (Brandenburger & Keisler, 2006). The incompleteness of the belief structures is due to the *holes* in the model. A model, then, has a hole at φ if either $\mathbf{U}^b \wedge \varphi$ is satisfiable but $\heartsuit^{ab}\varphi$ is not, or $\mathbf{U}^a \wedge \varphi$ is satisfiable but $\heartsuit^{ba}\varphi$ is not. A big hole is then defined by using the belief modality $\Box$ instead of the assumption modality $\heartsuit$.

In the original paper, the authors make use of two lemmas before identifying the holes in the system. These lemmas are important for us as we will challenge them in the next section. First, let us define a special propositional symbol $\mathbf{D}$ with the following valuation $D = \{w \in W : (\forall z \in W)[P(w,z) \to \neg P(z,w)]\}$.

Lemma 3.1.1 ((Brandenburger & Keisler, 2006))**.**

1. *If* $\heartsuit^{ab}\mathbf{U}^b$ *is satisfiable, then* $\Box^{ab}\Box^{ba}\Box^{ab}\heartsuit^{ba}\mathbf{U}^a \to \mathbf{D}$ *is valid.*

2. $\neg\Box^{ab}\heartsuit^{ba}(\mathbf{U}^a \wedge \mathbf{D})$ *is valid.*

Based on these lemmas, authors observe that there are no complete belief models. Here, we give the theorem in two forms.

Theorem 3.1.2 ((Brandenburger & Keisler, 2006))**.**

- *First-Order Version: Every belief model M has either a hole at U^a, a hole at U^b, a big hole at one of the formulas*

  (i) $\forall x.P^b(y, x)$

  (ii) x *believes* $\forall x.P^b(y, x)$

  (iii) y *believes [*x *believes* $\forall x.P^b(y, x)$*]*

  a hole at the formula

  (iv) $D(x)$

  or a big hole at the formula

  (v) y *assumes* $D(x)$

  Thus, there are no belief model which is complete for a language $\mathcal{L}$ which contains the tautologically true formulas and formulas (i)-(v).

- *Modal Version: There is either a hole at $\mathbf{U}^a$, a hole at $\mathbf{U}^b$, a big hole at one of the formulas*

$$\heartsuit^{ba}\mathbf{U}^a, \qquad \Box^{ab}\heartsuit^{ba}\mathbf{U}^a, \qquad \Box^{ba}\Box^{ab}\heartsuit^{ba}\mathbf{U}^a$$

  a hole at the formula $\mathbf{U}^a \wedge \mathbf{D}$, or a big hole at the formula $\heartsuit^{ba}(\mathbf{U}^a \wedge \mathbf{D})$. Thus, there are no complete interactive frame for the set of all modal formulas built from $\mathbf{U}^a$, $\mathbf{U}^b$, and $\mathbf{D}$.

3.2 Non-well-founded Set Theoretical Approach

3.2.1 Introduction

Non-well-founded set theory is a theory of sets where the axiom of foundation is replaced by the *anti-foundation axiom* which is due to Mirimanoff (Mirimanoff, 1917). Decades later, the axiom was re-formulated by Aczel within the domain of graph theory which motivated our approach here (Aczel, 1988). In non-well-founded (NWF, henceforth) set theory, we can have true statements such as '$x \in x$', and such statements present interesting properties in game theory. NWF theories, in this respect, are natural candidates to represent circularity (Barwise & Moss, 1996).

To the best of our knowledge, Lismont introduced non-well-founded type spaces to show the existence of universal belief spaces (Lismont, 1992). Then, Heifetz used NWF sets to represent type spaces and obtained rather sophisticated results (Heifetz, 1996). He mapped a given belief space to its NWF version, and then proved that in the NWF version, epimorphisms become equalites.

However, Harsanyi noted earlier that circularity might be needed to express infinite hierarchy of beliefs.

> It seems to me that the basic reason why the theory of games with incomplete information has made so little progress so far lies in the fact that these games give rise, or at least appear to give rise, to an infinite regress in reciprocal expectations on the part of the players. In such a game player 1's strategy choice will depend on what he expects (or believes) to be player 2's payoff function U_2, as the latter will be an important determinant of player 2's behavior in the game. But his strategy choice will also depend on what he expects to be player 2's first-order expectation about his own payoff function U_1. Indeed player 1's strategy choice will also depend on what he expects to be player 2's second-order expectation - that is, on what player 1 thinks that player 2 thinks that player 1 thinks about player 2's payoff function U_2... and so on *ad infinitum*.
>
> (Harsanyi, 1967)

Note that Harsanyi's concern for infinite regress or circularity is related to the epistemics of the game. However, some other ontological concerns can be raised as well about the type spaces, and the way we define the states in the type spaces. In this respect, Heifetz motivated his approach, which is related to our perspective here, by arguing that NWF type spaces can be used "once states of nature and types would no longer be associated with states of the world, but constitute *their very definition*." [ibid, (his emphasis)].

This is, indeed, a prolific approach to Harsanyi type spaces to represent uncertainty. Here is Heifetz on the very same issue.

> Nevertheless, one may continue to argue that a state of the world should indeed be a circular, self-referantial object: A state represents a situation of human uncertainty, in which a player considers what other players may think in other situations, and in particular about what they may think there about the current situation. According to such a view, one would seek a formulation where states of the world are indeed self-referring mathematical entities.
>
> (Heifetz, 1996, p. 204).

Notice that BK paradox is a situation where the aforementioned belief interaction among the players plays a central role. Therefore, in our opinion, it is worthwhile to pursue what NWF type spaces might provide in such situations. On the other hand, NWF set theory is not immune to the problems that the classical set theory suffers from. For example, note that Russell's paradox is not solved in NWF setting, and moreover the subset relation stays the same in NWF theory (Moss, 2009). The reason is quite straight-forward. As Heifetz also noted "Russell's paradox applies to the collection of all sets which do not contain themselves, not to the collection of sets which *do* contain themselves" (Heifetz, 1996, (his emphasis)). Therefore, we do not expect the BK paradox to disappear in NWF setting. Yet, NWF set theory will give us many other tools in game theory. We will also go back to this issue when we use category theoretical tools.

3.2.2 Non-well-founded Belief Models

Let us start with defining belief models in NWF theory. What we call a non-well-founded model is a tuple $M = (W, V)$ where W is a non-empty non-well-founded set (*hyperset*, for short), and V is a valuation assigning propositional variables to the elements of W. However, notice that our definition does not rule out well-founded sets, we merely treat them as a special case.

Now, we give the semantics of (basic) modal logic in non-well-founded setting (Gerbrandy, 1999). We use the symbol $\models^+$ in NWF models to distinguish it.

$$
\begin{array}{lll}
M, w \models^+ \Diamond\varphi & \text{iff} & \exists v \in w. \text{ such that } M, v \models^+ \varphi \\
M, w \models^+ \Box\varphi & \text{iff} & \forall v \in w.\ v \in w \text{ implies } M, v \models^+ \varphi
\end{array}
$$

Notice that NWF definition is more general than ordinary Kripke models since it allows $v = w$ that produce $w \in w$ which is not possible in ordinary Kripke models which are defined with well-founded sets.

Based on this definition, we can now give a non-standard semantics for the belief and assumption modalities $\Box^{ij}$ and $\heartsuit^{ij}$ respectively for $i, j \in \{a, b\}$.

$$
\begin{array}{lll}
M, w \models^+ \Box^{ij}\varphi & \text{iff} & M, w \models^+ \mathbf{U}^i \wedge \\
 & & \forall v (v \in w \wedge M, v \models^+ \mathbf{U}^j \rightarrow M, v \models^+ \varphi) \\
M, w \models^+ \heartsuit^{ij}\varphi & \text{iff} & M, w \models^+ \mathbf{U}^i \wedge \\
 & & \forall v (v \in w \wedge M, v \models^+ \mathbf{U}^j \leftrightarrow M, v \models^+ \varphi)
\end{array}
$$

Several comments on the NWF semantics are in order here. First, notice that this definition of NWF semantics for belief and assumption modalities depend on the earlier modal definition of those operators given (Brandenburger & Keisler, 2006; Gerbrandy, 1999). Second, belief or assumption of φ at a state w is defined in terms of the truth of φ at the states that constitutes w, including possibly w itself. Therefore, these definitions address the philosophical and foundational points that Heifetz made about the uncertainty in type spaces. We call a belief state $w \in W$ *Quine state* if $w = \{w\}$. We call a belief state $w \in W$ an *urelement* if it is not the empty set, and it can be a member of a set but cannot have members. Finally, we call a set A *transitive* if $a \in A$ and $b \in a$, then $b \in A$.

For example consider the model $W = \{w, v\}$ with $V(p) = \{w\}$ and $V(q) = \{v\}$ with a language with two propositional variables for simplicity. Let us assume that both w and v are Quine states. What does it mean to say that the player a assumes p at w in NFW belief models? Since w is a Quine state, the only state it can access is itself. Therefore, the statement $w \models^{+} \heartsuit^{ab} p$ forces $w \in U^a \cap U^b$, which is impossible since the type spaces of a and b are assumed to be disjoint. On the other hand, for $w \in U^a$, we have $w \models^{+} \heartsuit^{ab} q$. Notice that $w \models^{+} \mathbf{U}^a$. Moreover, since w is the only member of w, and $w \not\models^{+} \mathbf{U}^b$, together with the assumption that $w \not\models^{+} q$, we observe that the bicondition is satisfied. Therefore, Quine states can only assume false statements. Moreover, they believe in any statements.

Theorem 3.2.1. *Let $M = (W, V)$ be a NWF belief model with disjoint type spaces U^a and U^b respectively for two players a and b. If $w \in U^i$ be a Quine state or an urelement belief state for $i \in \{a, b\}$, then i assumes φ at w if and only if $M, w \not\models^{+} \varphi$. Moreover, i believes in any formula ψ at w.*

Proof. Let us first start with considering the Quine states.

Without loss of generality, let $w \in U^a$ where w is a Quine state. Suppose $w \models^{+} \heartsuit^{ab} \varphi$. Since $w \in U^a$, $w \models^{+} \mathbf{U}^a$. Since, $w \in w$, and U^a and U^b are disjoint, we have $w \not\models^{+} \mathbf{U}^b$. Therefore, since $w \models^{+} \heartsuit^{ab} \varphi$, we conclude $w \not\models^{+} \varphi$.

For the converse direction, suppose that $w \not\models^{+} \varphi$. Since $w \in w$, and $w \not\models^{+} \mathbf{U}^b$ the biconditional is satisfied. Moreover, since $w \in U^a$, $w \models +\mathbf{U}^a$. Therefore, $w \models^{+} \heartsuit^{ab} \varphi$.

The proof for the belief operator is immediate for Quine states.

Now, let us consider urelements. Without loss of generality, let $w \in U^a$ be an urelement. Then, $w \models^{+} \mathbf{U}^a$. For the left-to-right direction, note that since there is no $v \in w$, the conditional is vacuously satisfied. For the right-to-left direction, suppose $w \not\models^{+} \varphi$. Since, $w \notin w$ and $w \not\models^{+} \mathbf{U}^b$, the biconditional is satisfied again.

The proof for the belief operator is also immediate for urelement belief states, and thus left to the reader.

This concludes the proof. ■

Notice that the proof heavily depends on the fact that the type spaces for the players are assumed to be disjoint. Let us now see how belief models change once we allow the intersection of NWF type spaces.

Theorem 3.2.2. *Let $M = (W, V)$ be a NWF belief model for two players a and b where U^a and U^b is not necessarily disjoint. For a Quine state w, if $w \models^+ \heartsuit^{ij}\top$ for $i, j \in \{a, b\}$, then $w \in U^a \cap U^b$. In other words, Quine states with true assumptions belong to the both players.*

Proof. Without loss of generality, assume that Quine state w is in U^a. Suppose $w \models^+ \heartsuit^{ab}\top$. Then, we observe $w \models^+ \mathbf{U}^a$. Left-to-right direction of the biconditional is immediate. Now, consider the right-to-left direction. Since, $\top$ is satisfied at any state, and w is non-empty (i.e. $w \in w$), we conclude that $w \models^+ \mathbf{U}^b$. Therefore, $w \in U^b$. Thus, $w \in U^a \cap U^b$. ■

Game theoretical implications of Theorem 3.2.2 is worth mentioning. From the standard belief structure's point of view, Quine states correspond to the states which are reflexive. In other words, at such a state w, player i considers w possible for player j. Thus, such a state w is forced to be in the intersection of the type spaces.

On the other hand, intersecting type spaces do not seem to create a problem for belief models. To overcome this issue, one can introduce a *turn* function from the space of the belief model to the set of players assigning states to players. The functional definition of this construction necessitates that every state should be assigned to a unique player. Therefore, the game can determine whose turn it is at Quine atoms. Additionally, urelements, since they cannot have elements, are end states in games. At such states, players do not consider any states possible for the other player.

Now, based on the NWF semantics we gave earlier, it is not difficult to see that the following formulas discussed in the original paper are still valid as before if we maintain the assumption of the disjointness of type spaces.

$$\Box^{ab}\mathbf{U}^b \leftrightarrow \mathbf{U}^a,\ \ \Box^{ba}\mathbf{U}^a \leftrightarrow \mathbf{U}^b,\ \ \Box^{ab}\mathbf{U}^a \leftrightarrow \bot,\ \ \Box^{ba}\mathbf{U}^b \leftrightarrow \bot$$

Furthermore, the following formulas are not valid as before.

$$\Box^{ab}\mathbf{U}^b \to \mathbf{U}^b,\ \ \Box^{ab}\mathbf{U}^b \to \Box^{ba}\Box^{ab}\mathbf{U}^b,\ \ \Box^{ab}\mathbf{U}^b \to \Box^{ab}\Box^{ab}\mathbf{U}^b$$

However, for the sake of the completeness of our arguments, let us, for the moment, allow that type spaces may not be disjoint.

Consider a NWF belief model (W, V) where $w = \{w\}$ with $U^a = U^b = W$. In such a model $\Box^{ab}\mathbf{U}^a \leftrightarrow \bot$ fails, but $\Box^{ab}\mathbf{U}^a \leftrightarrow \top$ is satisfied. Similar observations can be made for $\Box^{ba}\mathbf{U}^b \leftrightarrow \bot$ and $\Box^{ba}\mathbf{U}^b \leftrightarrow \top$. Similarly, all $\Box^{ab}\mathbf{U}^b \to \mathbf{U}^b$, $\Box^{ab}\mathbf{U}^b \to \Box^{ba}\Box^{ab}\mathbf{U}^b$, and $\Box^{ab}\mathbf{U}^b \to \Box^{ab}\Box^{ab}\mathbf{U}^b$ are satisfied in the aforementioned NWF model.

Thus, following Heifetz's arguments, we maintain that NWF models with self-referring states are better candidates to formalize uncertainty in games. Therefore, based on the above observations, it is now conceivable to imagine a NWF belief model in which previously constructed standard holes do not exist. Our aim now is to construct a NWF belief model in which the Lemma 3.1.1 fails. For this purpose of us, however, we still maintain the assumption that type space be disjoint.

As a first step, we redefine the diagonal set in the NWF setting. Recall that, in the standard case, diagonal set D is defined with respect to the accessibility relation P which we defined earlier. In NWF case, we will use membership relation for that purpose.

Definition 3.2.3. Define $D^+ := \{w \in W : \forall v \in W.(v \in w \to w \notin v)\}$.

We define the propositional variable $\mathbf{D}^+$ as the propositional variable with the valuation set D^+.

Now, we observe how the NWF models make a difference in the context of the BK paradox. Notice that BK argument relies on two lemmas which we have mentioned earlier in Lemma 3.1.1. Now, we present counter-models to Lemma 3.1.1 in NWF theory.

Proposition 3.2.4. *In a NWF belief structure, if* $\heartsuit^{ab}\mathbf{U}^b$ *is satisfiable, then the formula* $\Box^{ab}\Box^{ba}\Box^{ab}\heartsuit^{ba}\mathbf{U}^a \wedge \neg\mathbf{D}^+$ *is also satisfiable.*

Proof. Let $W = \{w, v\}$ with $w = \{v\}$, $v = \{w\}$ where $U^a = \{w\}$ and $U^b = \{v\}$. To maintain the disjointness of the types, assume that neither w nor v is transitive.

Then, $w \models^+ \heartsuit^{ab}\mathbf{U}^b$ since all states in b's type space is assumed by a at w. Similarly, $v \models^+ \heartsuit^{ba}\mathbf{U}^a$ as all states in a's type space is assumed by b at v. Then, $w \models^+ \Box^{ab}\heartsuit^{ba}\mathbf{U}^a$. Continuing this way, we conclude, $w \models^+ \Box^{ab}\Box^{ba}\Box^{ab}\heartsuit^{ba}\mathbf{U}^a$.

However, by design, $w \not\models^+ \mathbf{D}^+$ since $v \in w$ and $w \in v$. Thus, the formula $\Box^{ab}\Box^{ba}\Box^{ab}\heartsuit^{ba}\mathbf{U}^a \wedge \neg\mathbf{D}^+$ is satisfiable as well. ■

Proposition 3.2.5. *The formula* $\Box^{ab}\heartsuit^{ba}(\mathbf{U}^a \wedge \mathbf{D}^+)$ *is satisfiable in some NWF belief structures.*

Proof. Take a non-transitive model (W, V) with $W = \{w, v, u, t\}$ where $w = \{v, w\}$, $v = \{u\}$, and $u = \{t\}$ where $u \notin t$. Let $U^a = \{w, u\}$, and $U^b = \{v, t\}$. Now, observe that the formula $\mathbf{U}^a \wedge \mathbf{D}^+$ is satisfiable only at u (as $w \in w$, w does not satisfy $\mathbf{U}^a \wedge \mathbf{D}^+$). Now, $v \models^+ \heartsuit^{ba}(\mathbf{U}^a \wedge \mathbf{D}^+)$. Finally, $w \models^+ \Box^{ab}\heartsuit^{ba}\mathbf{U}^a \wedge \mathbf{D}^+$. Note that even if $w \in w$ as $w \notin U^b$, by definition of the box modality, w satisfies $\Box^{ab}\heartsuit^{ba}\mathbf{U}^a \wedge \mathbf{D}^+$. ■

Therefore, the Lemma 3.1.1 is refuted in NWF belief models. Notice that Lemma 3.1.1 is central in Brandenburger and Keisler's proof of the incompleteness of belief structures. Thus, we ask whether the failure of Lemma 3.1.1 would mean that there can be complete NWF belief structures. The answer to this question requires some category theoretical tools that we will introduce in the following section. For now, we construct a counter-model for Theorem 3.1.2 in NWF setting.

Consider the following counter-model. Let $W = \{w, v, u, t, r, s, x, y, z\}$ with $w = \{v, w\}$, $v = \{u\}$, $u = \{t\}$ ($u \notin t$), $r = \{v, t, s, y\}$, $s = \{w, u, r, x, z\}$, $x = \{x, s\}$, $y = \{y, x\}$, $z = \{z, y\}$ where $U^a = \{w, u, r, x, z\}$ and $U^b = \{v, t, s, y\}$. Now, observe the following.

- $\mathbf{U} \wedge \mathbf{D}^+$ is satisfied only at u, since we have $w \in w$, $r \in s \wedge s \in r$, $x \in x$ and $z \in z$
- No hole at $\mathbf{U}^a$ as $s \models^+ \heartsuit^{ba}\mathbf{U}^a$
- No hole at $\mathbf{U}^b$ as $r \models^+ \heartsuit^{ab}\mathbf{U}^b$
- No big hole at $\heartsuit^{ba}\mathbf{U}^a$ as $x \models^+ \square^{ab}\heartsuit^{ba}\mathbf{U}^a$
- No big hole at $\square^{ab}\heartsuit^{ba}\mathbf{U}^a$ as $y \models^+ \square^{ba}\square^{ab}\heartsuit^{ba}\mathbf{U}^a$
- No big hole at $\square^{ba}\square^{ab}\heartsuit^{ba}\mathbf{U}^a$ as $z \models^+ \square^{ab}\square^{ba}\square^{ab}\heartsuit^{ba}\mathbf{U}^a$
- No hole at $\mathbf{U}^a \wedge \mathbf{D}^+$ as $v \models^+ \heartsuit^{ba}(\mathbf{U}^a \wedge \mathbf{D}^+)$
- No big hole at $\heartsuit^{ba}(\mathbf{U}^a \wedge \mathbf{D}^+)$ as $w \models^+ \square^{ab}\heartsuit^{ba}(\mathbf{U}^a \wedge \mathbf{D}^+)$

The crucial point in the semantical evaluation of big holes is the fact that the antecedent of the conditional in the definition of the box modality is not satisfied if some elements of the current states are not in the desired type space. Therefore, the entire statement of the semantics of the box modality is still satisfied if the current state has some elements from the same type space. This helped us to construct the counter-model.

This counter-model shows that Theorem 3.1.2 with its stated form does not hold in NWF belief structures. Yet, we have to be careful here. Our counter model does not establish the fact that NWF belief models are complete. It does establish the fact that they do not have the same holes as the standard belief models. We will get back to this question later on, and give an answer from category theoretical point of view.

Similarly, NWF theory should not be taken as such a revolutionary approach to epistemic game theory replacing the classical (Kripke) models. Gerbrandy noted:

> (...)[T]here are many 'more' Kripke models than there are possibilities of knowledge structures: each possibly corresponds [to] a whole class of bisimilar, but structurally different, models. In other words, a semantics for modal logic in the form of Kripke models has a finer structure than a semantics in terms of non-well-founded sets.
>
> (Gerbrandy, 1999)

Now, one can ask whether there exists a BK-sentence in NWF framework that can create a self-referential paradox. In order to answer this question, we will need some arguments from category theory.

3.2.3 Games Played on Non-well-founded Sets

Finally, note that the BK paradox is about the belief of the players. However, we can use NWF sets not only to represent the epistemics of games but also to represent games themselves in extensive form. As we have emphasized earlier, Aczel's graph theoretical approach identifies directed graphs and sets, and this will be our approach here as well.

Aczel's Anti-Foundation Axiom states that for each connected rooted directed graph, there corresponds a unique set (Aczel, 1988). Therefore, given any rooted directed graph, we can construct a game with the set that corresponds to the given graph up to the order of players.

Therefore, now, we can use *any* directed graphs, not necessarily only trees, to represent games in extensive form. Clearly, the given graph may have loops that can create infinite regress, thus creating infinite games. For instance, the situations in which some players may *reset* the game, or make a move that can take the game backward are examples of such games where cyclic representation in extensive form is needed. The crucial point here is the fact that such a set/game space exists, and is unique.

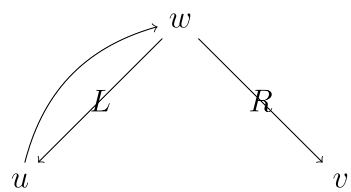

Figure 3.1: An example for a NWF game.

Theorem 3.2.6. *For every (labeled) rooted directed connected graph, there corresponds to a unique two-player NWF belief structure up to the permutation of type spaces, and the order of players.*

Proof. Follows directly from the Aczel's Anti-Foundation axiom. ■

Let us illustrate the theorem with a simple example.

Example 3.2.7. Consider the following labeled, connected directed graph. We can now construct the two-player NWF belief structure of this game is as follows. Put $W = \{w, v, u\}$ where $w = \{u, v\}$ and $u = \{w\}$. Assume that $U^a = \{w\}$, $U^b = \{u, v\}$ (or any other combination of type spaces). Therefore, this graph corresponds to the game where Bob can *reset* the game if Alice plays L at w.

In conclusion, allowing NWF sets in extensive game representations, we can express a much larger class of games.

3.3 Paraconsistent Approach

Paraconsistent logics can be captured by using several rather strong algebraic, topological and category theoretical structures. In this chapter, we approach paraconsistency from such directions, and analyze the BK paradox within paraconsistent logics interpreted in such systems.

3.3.1 Algebraic and Category Theoretical Approach

A recent work on the BK paradox shows the general pattern of such paradoxical cases, and gives some positive results such as fixed-point theorems (Abramsky & Zvesper, 2010). In this section, we instantiate the fixed-point results of the aforementioned work to some other mathematical structures that can represent paraconsistent logics. The surprising result, as we shall see, is the fact that even if we endorse a paraconsistent logical structure to accommodate paradoxes, there shall still be BK-sentences in such structures.

First, note that endorsing a paraconsistent logic does not mean that *all* contradictions are true. It means that *some* contradictions do not entail a trivial theory, and moreover absurdity ($\bot$) always lead to trivial theories. Thus, in paraconsistent logics, there are some contradictions which are not absurd.

Note that the semantical issue behind the failure of *ex contradictione quodlibet* (i.e. explosion) in paraconsistent systems is the fact that in such logical systems, the extension of the conjunction of some formulas and their negations may not be the empty set. Therefore, semantically, in paraconsistent logics, there exist some states in which a formula and its negation is true.

There are variety of different logical and algebraic structures to represent paraconsistent logics (Priest, 2002). Co-Heyting algebras are natural algebraic candidates to represent paraconsistency. We will resort to Co-Heyting algebras because of their algebraic and category theoretical properties which will be helpful later on.

Definition 3.3.1. Let L be a bounded distributive lattice. If there is a binary operation $\Rightarrow: L \times L \to L$ such that for all $x, y, z \in L$,

$$x \leq (y \Rightarrow z) \text{ iff } (x \wedge y) \leq z,$$

then we call $(L, \Rightarrow)$ a Heyting algebra.

Dually, if we have a binary operation $\setminus : L \times L \to L$ such that

$$(y \setminus z) \leq x \text{ iff } y \leq (x \vee z),$$

then we call $(L, \setminus)$ a co-Heyting algebra. We call $\Rightarrow$ implication, $\setminus$ subtraction.

An immediate example of a co-Heyting algebra is the closed subsets of a given topological space and subtopoi of a given topos (Lawvere, 1991; Mortensen, 2000; Başkent, 2011c; Priest, 2002). Therefore, we can now use a closed set topology to represent paraconsistent belief sets within the BK-paradox. Such belief sets, then, would constitute a Co-Heyting algebra. However, we need to be careful about defining the negation in such systems.

Both operations $\Rightarrow$ and $\setminus$ give rise to two different negations. The *intuitionistic negation* $\dot{\neg}$ is defined as $\dot{\neg}\varphi \equiv \varphi \to \mathbf{0}$ and *paraconsistent negation* $\sim$ is defined as $\sim\varphi \equiv \mathbf{1} \setminus \varphi$ where $\mathbf{0}$ and $\mathbf{1}$ are the bottom and the top elements of the lattice respectively. Therefore, $\dot{\neg}\varphi$ is the largest element disjoint from φ, and $\sim\varphi$ is the smallest element whose join with φ gives the top element $\mathbf{1}$ (Rasiowa, 1974; Reyes & Zolgaghari, 1996). In a Boolean algebra, both intuitionistic and paraconsistent negations coincide, and give the usual Boolean negation where we interpret $\varphi \Rightarrow \psi$ as $\neg\varphi \vee \psi$, and $\varphi \setminus \psi$ as $\varphi \wedge \neg\psi$ with the usual Boolean negation $\neg$. What makes closed set topologies a paraconsistent structure is the fact that theories that are true at boundary points include formulas and their negation (Mortensen, 2000; Başkent, 2011c). Because, a formula φ and its paraconsistent negation $\sim\varphi$ intersect at the boundary of their extensions. We will discuss the paraconsistent negation in the following sections as well.

On the other hand, the algebraic structures such as Co-Heyting algebras, we have mentioned can be approached from a category theory point of view. Before discussing Lawvere's argument, we need to define *weakly point surjective* maps. An arrow $f : A \times A \to B$ is called *weakly point surjective* if for every $p : A \to B$, there is an $x : \mathbf{1} \to A$ such that for all $y : \mathbf{1} \to A$ where $\mathbf{1}$ is the terminal object, we have

$$p \circ y = f \circ \langle x, y \rangle : \mathbf{1} \to B$$

In this case, we say, p is represented by x. Moreover, a category is cartesian closed (CCC for short, henceforth), if it has a terminal object, and admits products and exponentiation. A set X is said to have the fixed-point property for a function f, if there is an element $x \in X$ such that $f(x) = x$. Category theoretically, an object X is said to have the fixed-point property if and only if for every endomorphism $f : X \to X$, there is $x : \mathbf{1} \to X$ with $xf = x$ (Lawvere, 1969).

Theorem 3.3.2 ((Lawvere, 1969))**.** *In any cartesian closed category, if there exists an object A and a weakly point-surjective morphism $g : A \to Y^A$, then Y has the fixed-point property for g.*

It was observed that CCC condition can be relaxed, and Lawvere's Theorem works for categories that have only finite products (Abramsky & Zvesper, 2010)[1]. These authors showed how to reduce Lawvere's Lemma to the BK paradox, and how to reduce the BK paradox to Lawvere's Lemma[2]. Now, our goal is to take one step further, and investigate some other cartesian closed categories which represent the non-classical frameworks that we have investigated in this paper. Therefore, by Lawvere and Abramsky & Zvesper results, we will be able to show the existence of fixed-points in our framework, which will give the BK paradox in those frameworks.

In their paper, Abramsky & Zvezper, first define the BK sentence by using relations between type spaces instead of maps, and then, represent the Lawvere's fixed-point lemma in a relational framework. Then, they conclude that if the relational representation of the BK sentence satisfies some conditions then they have a fixed-point, which in turn creates

[1]This point was already made by Lawvere and Schanuel in *Conceptual Mathematics*. Thanks to Noson Yonofsky for pointing this out.

[2]In order to be able to avoid the technicalities of categorical logic, we do not give the detalis of their construction and refer the reader to (Abramsky & Zvesper, 2010)

the BK sentence. Their approach makes use of the standard (classical) BK paradox, and utilizes regular logic in their formalization. In our approach, we directly use Lawvere's Lemma (Theorem 3.3.2 here) to deduce our results.

Now, we observe the category theoretical properties of co-Heyting algebras and the category of hypersets. Recall that the category of Heyting algebras is a CCC. A canonical example of a Heyting algebra is the set of opens of a topological space (Awodey, 2006). The objects of such a category will be the open sets. The unique morphisms in that category exists from O to O' if $O \subseteq O'$. What about co-Heyting algebras? It is easy to prove the following. We give the proof for the completeness of our arguments here.

Proposition 3.3.3. *Co-Heyting algebras are Cartesian closed categories.*

Proof. Let $(L, \setminus)$ be a co-Heyting algebra. First, observe that the element $\mathbf{0}$ is the terminal element (dual of the $\mathbf{1}$ in Heyting algebra). Second, for x, y in L, the product exists and defined as $x \wedge y$ as $x \wedge y \leq x$, and $x \wedge y \leq y$. Moreover, $x \leq y$ and $x \leq z$ imply that $x \leq y \wedge z$. Third, the exponent x^y is defined as $x \wedge \neg y$. Notice that we also write this as $x \setminus y$. The evaluation of subtraction is $x \leq (x \setminus y) \vee y$. Unraveling the definition, we observe that the definition is sound: ∎

Example 3.3.4. As we have mentioned, the co-Heyting algebra of the closed sets of a topology is a well-known example of a CCC. Similar to the arguments that show that open set topologies are CCC, we can observe that closed set topologies are CCC as well.

Given two objects C_1, C_2, we define the unique arrow from C_1 to C_2, if $C_1 \supseteq C_2$. The product is the union of C_1 and C_2 as the finite union of closed sets exists in a topology. The exponent $C_1^{C_2}$ is then defined as $\mathsf{Clo}(\overline{C_1} \cap C_2)$ where $\overline{C_1}$ is the complement of C_1.

Now, we have the following corollary for Theorem 3.3.2.

Corollary 3.3.5. *In a co-Heyting algebra, if there is an object A and a weakly point-surjective morphism $g : A \to Y^A$, then Y has the fixed-point property. Therefore, there exists a co-Heyting algebraic model with an impossible BK sentence.*

I find this result surprising. Namely, even if we allow nontrivial inconsistencies and represent them as a co-Heyting algebra, we will still have fixed-points. This is our first step to establish the possibility of having the BK paradox in paraconsistent setting.

Corollary 3.3.6. *There exists a paraconsistent belief model in which the BK paradox remains unsatisfiable.*

Proof. The procedure is relatively straight-forward. Take a paraconsistent belief model M where belief sets constitute a closed set topology. Such a topology is a co-Heyting algebra, and therefore a CCC. Therefore, Lawvere's Lemma (Theorem 3.3.2) applies. Since Lawvere's Lemma is reducible to the BK sentence by Abramsky & Zvesper's result, we observe that M possesses a fixed-point that creates the BK sentence (under the paraconsistent negation) which is an unsatisfiable sentence in the model M. ■

This simple result shows that even if we allow some contradictions, there will exist a BK sentence in the model which is not satisfiable. This is perhaps not surprising. As Mariotti *et al.* pointed out earlier, interactive belief models can produce situations which are not expressible in the language (Mariotti *et al.*, 2005)[3].

In our earlier discussion, we have presented some counter-models for the classical BK sentence. However, we haven't concluded that NWF models are complete. We need Lawvere's Lemma to show that NWF belief models cannot be complete. Consider the category **AFA** of hypersets with total maps between them[4]. Category **AFA** admits a final object $\mathbf{1} = \{\emptyset\}$. Moreover, it also admits exponentiation and products in the usual sense, making it a CCC. Thus, Lawvere's Lemma applies.

Corollary 3.3.7. *There exists an impossible BK sentence in non-well-founded interactive belief structures.*

[3]In their work, however, Mariotti et al. makes the same point by resorting Cantor's diagonal arguments. Therefore, either using CCC or Cantor's diagonlization, a pattern for such self-referential paradoxes is visible (Yanofsky, 2003).

[4]Thanks to Florian Lengyel for pointing this out.

3.3.2 Topological Approach

Now, we make it explicit how paraconsistent topological belief models are constructed. In our construction, we will make use of relational representation of belief models which in turn produces belief and assumption modalities. We will then interpret those modalities over paraconsistent topological models. We now step by step construct the BK argument in paraconsistent topological setting. We will call such a belief model a *paraconsistent topological belief model*.

For the agents a and b, we have a corresponding non-empty type space A and B, and define closed set topologies τ_A and τ_B on A and B respectively. Furthermore, in order to establish connection between τ_A and τ_B to represent belief interaction among the players, we introduce additional constructions $t_A \subseteq A \times B$, and $t_B \subseteq B \times A$. We then call the structure $F = (A, B, \tau_A, \tau_B, t_A, t_B)$ a paraconsistent topological belief model. In this setting, the set A represents the possible epistemic states of the player a in which she holds beliefs about player b, or about b's beliefs etc, and vice versa for the set B and the player a. Moreover, the topologies represent those beliefs. For instance, for player a at the state $x \in A$, t_A returns a closed set in $Y \in \tau_B \subseteq \wp(B)$. In this case, we write $t_A(x, Y)$ which means that at state x, player a believes that the states y in $Y \in \tau_B$ are possible for the player b, i.e. $t_A(x, y)$ for all $y \in Y$. Moreover, a state $x \in A$ *believes* $\varphi \subseteq B$ if $\{y : t_A(x, y)\} \subseteq \varphi$. Furthermore, a state $x \in A$ *assumes* φ if $\{y : t_A(x, y)\} = \varphi$. Notice that in this definition, we identify logical formulas with their extensions.

The modal language which we use has two modalities representing the beliefs of each agent. Akin to some earlier modal semantics for the paradox, we give a topological semantics for the BK argument in paraconsistent topological belief models. Let us first give the formal language which we use. The language for the topological belief models is given as follows.

$$\varphi := p \mid \sim\varphi \mid \varphi \wedge \varphi \mid \Box_a \mid \Box_b \mid \boxplus_a \mid \boxplus_b$$

where p is a propositional variable, $\sim$ is the paraconsistent topological negation symbol which we have defined earlier, and $\Box_i$ and $\boxplus_i$ are the belief and assumption operators for player i, respectively. We use different notations for modal operators to distinguish them from the original syntax in which they were defined.

We have discussed the semantics of the negation already. For $x \in A$, $y \in B$, the semantics of the modalities are given as follows with a modal valuation attached to F.

$$\begin{aligned}
x \models \Box_a \varphi &\quad \text{iff} \quad \exists Y \in \tau_B \text{ with } t_A(x, Y) \to \forall y \in Y. y \models \varphi \\
x \models \boxplus_a \varphi &\quad \text{iff} \quad \exists Y \in \tau_B \text{ with } t_A(x, Y) \leftrightarrow \forall y \in Y. y \models \varphi \\
y \models \Box_b \varphi &\quad \text{iff} \quad \exists X \in \tau_A \text{ with } t_B(y, X) \to \forall x \in X. x \models \varphi \\
y \models \boxplus_b \varphi &\quad \text{iff} \quad \exists X \in \tau_A \text{ with } t_B(y, X) \leftrightarrow \forall x \in X. x \models \varphi
\end{aligned}$$

We define the dual modalities $\Diamond_a$ and $\Diamond_b$ as usual.

Now, we have sufficient tools to represent the BK sentence in our paraconsistent topological belief structure with respect to a state x_0:

$$x_0 \models \Box_a \boxplus_b \varphi \wedge \Diamond_a \top$$

Let us analyze this formula in our system. Notice that the second conjunct guarantees that for the given $x_0 \in A$, there exists a corresponding set $Y \in \tau_B$ with $t_A(x, Y)$. On the other hand, the first conjunct deserves closer attention:

$$\begin{aligned}
x_0 \models \Box_a \boxplus_b \varphi \quad &\text{iff} \quad \exists Y \in \tau_B \text{ with } t_A(x_0, Y) \Rightarrow \forall y \in Y.\ y \models \boxplus_b \varphi \\
&\text{iff} \quad \exists Y \in \tau_B \text{ with } t_A(x_0, Y) \Rightarrow \\
&\qquad [\forall y \in Y, \exists X \in \tau_A \text{ with } t_B(y, X) \Leftrightarrow \forall x \in X.\ x \models \varphi]
\end{aligned}$$

Notice that in our framework, some special x can satisfy falsehood $\perp$ to give $x_0 \models \Box_a \boxplus_b \perp \wedge \Diamond_a \top$ for some x_0. Let the extension of p be X_0. Pick $x_0 \in \partial X_0$ where $\partial(\cdot)$ operator denotes the boundary of a set $\partial(\cdot) = \mathsf{Clo}(\cdot) - \mathsf{Int}(\cdot)$. By the assumptions of our framework X_0 is closed. Moreover, by simple topology ∂X_0 is closed as well. By the second conjunct of the formula in question, we know that some $Y \in \tau_B$ exists such that

$t_A(x_0, Y)$. Now, for all y in Y, we make an additional supposition and associate y with ∂X_0 giving $t_B(y, \partial X_0)$. We know that for all $x \in \partial X_0$, we have $x \models p$ as $\partial X_0 \subseteq X_0$ where X_0 is the extension of p. Moreover, $x \models \sim p$ for all $x \in \partial X_0$ as $\partial X_0 \subseteq (\sim X_0)$, too. Thus, we conclude that $x_0 \models \Box_a \boxplus_b \bot \wedge \Diamond_a \top$ for some *carefully selected* x_0.

In this construction, we have several suppositions. First, we picked the actual state from the boundary of the extension of some proposition (ground or modal). Second, we associate the epistemic accessibility of the second player to the same boundary set. Namely, a's beliefs about b includes her current state.

Now, the BK paradox appears when one substitutes φ with the following diagonal formula (whose extension is a closed set by definition of the closed set topology), hence breaking the aforementioned circularity:

$$D(x) = \forall y.[t_A(x, y) \rightarrow \ \sim t_B(y, x)]$$

The BK impossibility theorem asserts that, under the seriality condition, there is no such x_0 satisfying the following.

$$\begin{aligned} x_0 \models \Box_a \boxplus_b D(x) \quad \text{iff} \quad & \exists Y \in \tau_B \text{ with } t_A(x_0, Y) \Rightarrow \\ & [\forall y \in Y, \exists X \in \tau_A \text{ with } t_B(y, X) \Leftrightarrow \\ & \forall x \in X.\ x \models \forall y'.(t_A(x, y') \rightarrow \ \sim t_B(y', x))] \end{aligned}$$

Motivated by our earlier discussion, let us analyze the logical statement in question. Let X_0 satisfy the statement $t_A(x, y')$ for all $y' \in Y$ and $x \in X_0$ for some Y. Then, $\partial X_0 \subseteq X_0$ will satisfy the same formula. Similarly, let $\sim X_0$ satisfy $\sim t_B(y', x)$ for all $y' \in Y$ and $x \in X_0$. Then, by the similar argument, $\partial(\sim X_0)$ satisfy the same formula. Since $\partial(X_0) = \partial(\sim X_0)$, we observe that any $x_0 \in \partial X_0$ satisfy $t_A(x, y')$ and $\sim t_B(y', x)$ with the aforementioned quantification. Thus, such an x_0 satisfies $\Box_a \boxplus_b D(x)$. Therefore, the states at the boundary of some closed set *satisfy* the BK sentence in paraconsistent topological belief structures. Thus, this is a counter-model for the BK sentence in the paraconsistent topological belief models.

Theorem 3.3.8. *The BK sentence is satisfiable in some paraconsistent topological belief models.*

Proof. See the above discussion for the proof which gives a model that satisfies the BK sentence. ■

This brings us little more closure to the open question to precisely characterize the models which satisfy the BK sentence.

3.3.3 Product Topologies

In the previous section, we introduced t_A and t_B to represent the belief interaction between the players. Topological models, moreover, provide us with some further tools to represent such an interaction (Gabbay *et al.*, 2003).

In this section, we use product topologies to represent belief interaction among the players. Novelty of this approach is not only to economize on the notation and the model, but also to present a more natural way to represent the belief interaction. For our purposes here, we will only consider two-player games, and our results can easily be generalized to n-player. Here, we make use of the constructions presented in some recent works (van Benthem *et al.*, 2006; van Benthem & Sarenac, 2004).

Definition 3.3.9. Let a, b be two players with corresponding type space A and B. Let τ_A and τ_B be the (paraconsistent) closed set topologies of respective type spaces. The product topological paraconsistent belief structure for two agents is given as $(A \times B, \tau_A \times \tau_B)$.

In this framework, we assume that the topologies are *full* on their sets - namely $\bigcup \tau_A = A$, and likewise for B. In other words, we do not want any non-expressibility results just because the given topologies do not cover such states. If the topologies are not full, we can reduce the given space to a subset of it on which the topologies are full.

In this setting, if player a believes proposition $P \subseteq B$ at state $x \in A$, we stipulate that there is a closed set $X \in \tau_A$ such that $x \in X$ and a closed set $Y \in \tau_B$ with $Y \subseteq P$, all implying $X \times Y \in \tau_A \times \tau_B$. Player a assumes P if $Y = P$, and likewise for player b. Similar to the previous section, we make use of paraconsistent topological structures with closed sets and paraconsistent negation.

Borrowing some standard definitions from topology, we recall that given a set $S \subseteq A \times B$, we say that S is *horizontally closed* if for any $(x, y) \in S$, there exists a closed set X with $x \in X \in \tau_A$ and $X \times \{y\} \subseteq S$. Similarly, S is *vertically closed* if for any $(x, y) \in S$, there exists a closed set Y with $y \in Y \in \tau_B$, and $\{x\} \times Y \subseteq S$ (van Benthem *et al.*, 2006; van Benthem & Sarenac, 2004). In this framework, we say player a at $x \in A$ is said to believe a set $Y \subseteq B$ if $\{x\} \times Y$ is vertically closed.

Now, we define assumption-complete structures in product topologies. For a given language $\mathcal{L}$ for our belief model, let $\mathcal{L}^a$ and $\mathcal{L}^b$ be the families of all subsets of A and B respectively. Then, we observe that by assumption-completeness, we require every non-empty set $Y \in \mathcal{L}^b$ is assumed by some $x \in A$, and similarly, every non-empty set $X \in \mathcal{L}^a$ is assumed by some $y \in B$.

We can now characterize assumption-complete paraconsistent topological belief models. Given type spaces A and B, we construct the coarsest topologies on respective type spaces τ_A and τ_B where each subset of A and B are in τ_A and τ_B. Therefore, it is easy to see that $A \times B$ is vertically and horizontally closed for any $S \subseteq A \times B$. Moreover, under these conditions, our belief structure in question is assumption-complete.

We can relax some of these conditions. Assume that now τ_A and τ_B are not the coarsest topologies on A and B respectively. Therefore, we define, we *weak assumption-completeness* for a topological belief structure if every set $S \in A \times B$ is both horizontally and vertically closed. In other words, weak assumption-complete models focus only on the formula that are available in the given structure. There can be some formulas expressible in $\mathcal{L}$, but not available in $\mathcal{L}^a$ or in τ_A for some reasons. Epistemic game theory, indeed, is full of such cases where players may or may not be allowed to send some particular signals, and some information may be unavailable to certain players. The following theorem follows directly from the definitions.

Theorem 3.3.10. *Let $M = (A \times B, \tau_A \times \tau_B)$ be a product topological paraconsistent belief model. If M is horizontally and vertically closed, then it is weak assumption-complete.*

3.4 Conclusion and Future Work

In this chapter, we presented two non-classical frameworks to formalize beliefs in games: non-well-founded sets and paraconsistent logic.

Paraconsistency presents a rather interesting framework for game theory (Rahman & Carnielli, 2000). Therefore, a possible next step in this direction would be to define *inconsistent games*. Our work so far has lead us to consider paraconsistent information sets in games where the information sets of players may contain contradictions without leading to triviality so that we can still make meaningful inferences[5]. Then, the question is the following: "What does it mean to make an inconsistent move?" This is a rather encrypted question. First, it should be made clear what this question means. An inconsistent move would require a dynamic logic where one can define a negation of a move, and define some understanding of move inconsistency. Dialogical games where actions are in the form of utterances seem to provide some answers to this question (Rahman & Carnielli, 2000).

Moreover, from a logical perspective, paraconsistency has its dual intuitionistic form where the information/belief sets of the agents maybe *incomplete* or *paracomplete*. Therefore, for some player i, there can be a formula φ for which neither φ nor its negation is believed by any player. This is an intriguing aspect for a game where players simply do not know what move to make. The concept of *empty move* (after the empty move, the clock ticks, and time passes but the game board does not change) may seem meaningful for such issues.

Our work raises the question of paraconsistent games. In this work, we focused on

[5]Thanks to Graham Priest for drawing my attention to this point.

the paraconsistency, but one can very well start with perfect information games with non-classical probabilities. In such cases, sum of the probabilities of playing a and not-a may be higher than 1 (Williams, 2011)[6].

[6]Thanks to Brian Weatherson for pointing this out.

Chapter 4

On Game Strategies

Another non-classical approach to games would be to change the primitives of the theory. In this section, we follow a recent trend in game theory by treating strategies as primitives, and give them a dynamic twist.

4.1 Introduction

In game theory, a *strategy* for a player is defined as "a set of rules that describe exactly how (...) [a] player should choose, depending on how the [other] players have chosen at earlier moves" (Hodkinson *et al.*, 2000). Notice that this definition of strategies is *static*. In other words, once you have a strategy with our without probabilities, the traditional framework of game theory does not allow changes in strategies - but instead suggest to embed such possible changes within the strategy itself. Moreover, according to the same definition, strategies are presumably constructed *before* the game is actually played, and does not seem to address the epistemic of the games adequately.

For example, consider the game of chess. According to Zermelo's well-known theorem, chess is determined (Schwalbe & Walker, 2001). Then, why would you play chess if you know you will lose (or won't win) the game? Clearly, if we have logical omniscience for the players (which we don't), then it is pointless for the player who is going to lose, to

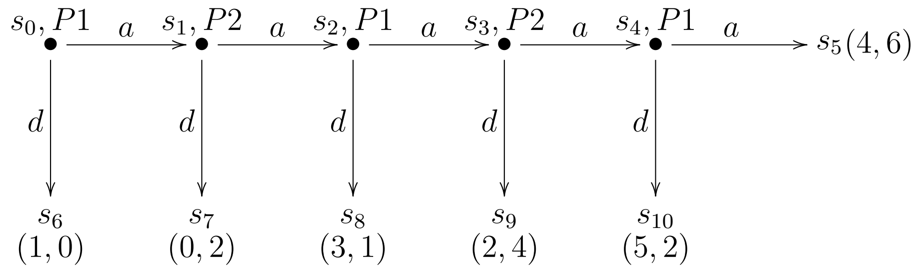

Figure 4.1: The Centipede game

even start playing the game as she knows the outcome already. If we are not logicalally omniscient, and have only a limited amount of computational power and memory (which we do), then chess is only a perfect information game for God. Therefore, there seems to be a problem. The static, pre-determined notion of strategies falls short of analyzing perfect information games. Because, we, people, do not strategize as such even in perfect information games - largely because we are not logically omniscient, and we have limited memory, and computational and deductive power.

While people play games, they observe, learn, recollect and update their strategies *during* the game as well as adopting deontological strategies and goals before the game. Players update and revise their strategies, for instance, when their opponent makes an *unexpected* or *irrational* move. Similarly, sometimes external factors may force the players not to make certain moves. For instance, assume that you are playing a video game by using a gamepad or a keyboard, and in the middle of the game, one of the buttons on the gamepad breaks. Hence, from that moment on, you will not be able to make certain moves in the game that are controlled by that button on the gamepad. This is most certainly not part of your strategy. Therefore, you will need to revise your strategy in such a way that some moves will be excluded from your strategy from that moment on. However, for your opponent, that is not the case as she can still make all the moves available to her.

Moreover, in some cases, assumptions about the game or the players may fail as well. For instance, consider the centipede game between two players $P1$ and $P2$ (Figure 4.1). Under the assumption of common knowledge of rationality, the usual backward induction scheme produces the solution that $P1$ needs to make a d move at s_0 (Aumann, 1995). What happens then, if $P1$ is prohibited or prevented from making a d move at s_0 and onwards right after the beginning of the game (or similarly, if the key on the gamepad that is used to make a d move is broken)? It means that from a behavioral perspective, the assumption of common rationality is violated, and $P2$ may need to update her strategy *during the game* based on what she has observed. There can be many reasons why $P1$ may make such a move. The move d might have been prohibited for $P1$ right after the game has started, or it may be a taboo for that specific player to make that move, or it may have been simply forbidden or restricted by an external factor (nature, God etc.). Such reasons are common when we think how societies are organized. One can very well think of such restrictions as taboos, intervention of the society or simply illegal moves (even though they are available) (Paul *et al.*, 2009).

In this chapter, we introduce what we call *move updates* where some moves become unavailable during the game. The motivation is to make the strategies dynamic, and describe a way as to how they can be revised.

Clearly, there can be considered many other forms of updates, revisions and restrictions of strategies (or of other elements of games) that can happen during the game play. For instance, some states may become unavailable in the midst of the game for some players. Moreover, manipulation games or sabotage games where a third party or God/nature affects the outcome of the game are also examples of such games where dynamic strategy analysis is much needed (van Benthem, 2005).

Our goal here is to present a formal framework for move based strategy restrictions by extending *strategy logic* (henceforth, SL) which was introduced by Ramanujam and Simon (Ramanujam & Simon, 2008b). The rest of the chapter is structured as follows. Section 4.2 provides a short review of SL. In Section 4.3, the framework of SL is extended with strategic move restrictions, and completeness of the resulting logic is shown. Then, we investigate some decision theoretical problems and give a complexity bound for the model checking problem in SL - which was an open problem so far. Finally, we conclude by placing our extended strategy logic in the context of related work, followed by a conclusion and ideas for future research.

4.2 Strategy logic

In this section, we give a short overview of strategy logic (Ghosh *et al.*, 2010b; Ramanujam & Simon, 2008b). The focus is on games played between two players given by the set $N = \{1, 2\}$, and a single set of moves Σ for both. Let $\mathbf{T} = (S, \Rightarrow, s_0)$ be a tree rooted at $s_0 \in S$, on the set of vertices S. A partial function $\Rightarrow: S \times \Sigma \to S$ specifies the labeled edges of such a tree where labels represent the moves at the states. The extensive form game tree, then, is a pair $T = (\mathbf{T}, \lambda)$ where $\mathbf{T}$ is a tree as defined before, and $\lambda : S \to N$ specifies whose turn it is at each state. A strategy μ^i for a player $i \in N$ is a function $\mu^i : S^i \to \Sigma$ where $S^i = \{s \in S : \lambda(s) = i\}$. For player i and strategy μ^i, the strategy tree $T_\mu = (S_\mu, \Rightarrow_\mu, s_0, \lambda_\mu)$ is the least subtree of T satisfying the following two natural conditions:

1. $s_0 \in S_\mu$;
2. For any $s \in S_\mu$, if $\lambda(s) = i$, then there exists a unique $s' \in S_\mu$ and action a such that $s \overset{a}{\Rightarrow}_\mu s'$. Otherwise, if $\lambda(s) \neq i$, then for all s' with $s \overset{a}{\Rightarrow} s'$ for some a, we have $s \overset{a}{\Rightarrow}_\mu s'$.

In other words, in the strategy tree, the root is included, for the states that belong to the strategizing player, a unique move is assigned to the player, and for the other player, all possible moves are considered. Notice that, in SL, strategies return unique moves. Nevertheless, we would still have a tree even if the strategies are set-valued.

The most basic constructions in SL are strategy specifications. First, for a given countable set X, a set of basic formulas $BF(X)$ is defined as follows, for $a \in \Sigma$:

$$BF(X) := x \in X \mid \neg\varphi \mid \varphi \wedge \varphi \mid \langle a \rangle \varphi$$

Let P^i be a countable set of atomic observables for player i, with $P = P^1 \cup P^2$. The syntax of strategy specifications is given as follows for $\varphi \in BF(P^i)$:

$$Strat^i(P^i) := [\varphi \to a]^i \mid \sigma_1 + \sigma_2 \mid \sigma_1 \cdot \sigma_2$$

The specification $[\varphi \to a]^i$ at player i's position stands for "play a whenever φ holds". The specification $\sigma_1 + \sigma_2$ means that the strategy of the player conforms to the specification σ_1 or σ_2; and $\sigma_1 \cdot \sigma_2$ means that the strategy of the player conforms to the specifications σ_1 and σ_2.

Let $M = (T, V)$ where $T = (S, \Rightarrow, s_0, \lambda)$ is an extensive form game tree as defined before, and $V : S \to 2^P$ is a valuation function for the set of propositional variables P. The truth of a formula $\varphi \in BF(P)$ is given as usual for the propositional, Boolean and modal formulas.

The notion "strategy μ *conforms* to specification σ for player i at state s" (notation $\mu, s \models_i \sigma$) is defined as follows, where $\mathbf{out}_\mu(s)$ denotes the unique outgoing edge at s with respect to μ.

$\mu, s \models_i [\varphi \to a]^i$	iff	$M, s \models \varphi$ implies $\mathbf{out}_\mu(s) = a$
$\mu, s \models_i \sigma_1 + \sigma_2$	iff	$\mu, s \models_i \sigma_1$ or $\mu, s \models_i \sigma_2$
$\mu, s \models_i \sigma_1 \cdot \sigma_2$	iff	$\mu, s \models_i \sigma_1$ and $\mu, s \models_i \sigma_2$

Now, based on the strategy specifications, the syntax of the strategy logic SL is given as follows:

$$p \mid \neg\varphi \mid \varphi_1 \wedge \varphi_2 \mid \langle a \rangle \varphi \mid (\sigma)_i : a \mid \sigma \leadsto_i \psi$$

for $p \in P$, $a \in \Sigma$, $\sigma \in Strat^i(P^i)$, and $\psi \in BF(P^i)$. We read $(\sigma)_i : a$ as "at the current state the strategy specification σ for player i suggests that the move a can be played". Subsequently, we read $\sigma \leadsto_i \psi$ as "following strategy σ player i can ensure ψ". The other connectives and modalities are defined as usual.

We now define the set of available moves at a state s as $moves(s) := \{a \in \Sigma : \exists s' \in S \text{ with } s \stackrel{a}{\Rightarrow} s'\}$. Then, based on $moves$, we inductively construct the set of enabled moves at state s in strategy σ as follows.

- $[\psi \to a]^i(s) \; = \begin{cases} \{a\} & : \lambda(s) = i; M, s \models \psi, a \in moves(s) \\ \emptyset & : \lambda(s) = i; M, s \models \psi, a \notin moves(s) \\ \Sigma & : \text{otherwise} \end{cases}$
- $(\sigma_1 + \sigma_2)(s) = \sigma_1(s) \cup \sigma_2(s)$
- $(\sigma_1 \cdot \sigma_2)(s) \; = \sigma_1(s) \cap \sigma_2(s)$

The truth definition for the strategy formulas are as follows:

$$
\begin{array}{lll}
M, s \models \langle a \rangle \varphi & \text{iff} & \exists s \text{ such that } s \stackrel{a}{\Rightarrow} s' \text{ and } M, s' \models \varphi \\
M, s \models (\sigma)_i : a & \text{iff} & a \in \sigma(s) \\
M, s \models \sigma \leadsto_i \psi & \text{iff} & \forall s' \text{ such that } s \Rightarrow_\sigma^* s' \text{ in } T_s|\sigma, \\
& & \text{we have } M, s' \models \psi \wedge (\mathbf{turn}_i \to \mathbf{enabled}_\sigma)
\end{array}
$$

where $\sigma(s)$ is as before, and $\Rightarrow_\sigma^*$ denotes the reflexive transitive closure of $\Rightarrow_\sigma$. Furthermore, T_s is the tree that consists of the unique path from the root (s_0) to s and the subtree rooted at s, and $T_s|\sigma$ is the least subtree of T_s that contains a unique path from s_0 to s and from s onwards, for each player i node, all the moves enabled by σ, and for each node of the opponent player, all possible moves . The proposition $\mathbf{turn}_i$ denotes that it is i's turn to play. Finally, define $\mathbf{enabled}_\sigma = \bigvee_{a \in \Sigma}(\langle a \rangle \top \wedge (\sigma)_i : a)$.

Now, we give the axioms of strategy logic.

- All the substitutional instances of the tautologies of propositional calculus

- $[a](\varphi \to \psi) \to ([a]\varphi \to [a]\psi)$
- $\langle a\rangle\varphi \to [a]\varphi$
- $\langle a\rangle\top \to ([\psi \to a]^i : a)_i$ for all $a \in \Sigma$
- $[\mathbf{turn}_i \wedge \psi \wedge ([\psi \to a]^i)_i : a)] \to \langle a\rangle\top$
- $\mathbf{turn}_i \wedge ([\psi \to a]^i)_i : c \leftrightarrow \neg\psi$ for all $a \neq c$
- $(\sigma + \sigma')_i : a \leftrightarrow (\sigma : a)_i \vee (\sigma' : a)_i$
- $(\sigma \cdot \sigma')_i : a \leftrightarrow (\sigma : a)_i \wedge (\sigma' : a)_i$
- $\sigma \leadsto_i \psi \to [\psi \wedge \mathbf{inv}_i^\sigma(a,\psi) \wedge \mathbf{inv}_{-i}^\sigma(\psi) \wedge \mathbf{enabled}_\sigma]$

Here, $\mathbf{inv}_i^\sigma(a,\psi) = (\mathbf{turn}_i \wedge (\sigma)_i : a) \to [a](\sigma \leadsto_i \psi)$ which expresses the fact that after an a move by i which conforms to σ, the statement $\sigma \leadsto_i \psi$ continues to hold, and $\mathbf{inv}_{-i}^\sigma(\psi) = \mathbf{turn}_i \to \odot(\sigma \leadsto_i \psi)$ states that after any move of $-i$, $\sigma \leadsto_i \psi$ continues to hold. Here, $\bigcirc\varphi \equiv \bigvee_{a\in\Sigma}\langle a\rangle\varphi$ and $\odot\varphi \equiv \neg\bigcirc\neg\varphi$.

Now, we discuss the inference rules that SL employs: modus ponens and generalization for $[a]$ for each $a \in \Sigma$. The induction rule is a bit more complex: From the formulas $\varphi \wedge (\mathbf{turn}_i \wedge (\sigma)_i : a) \to [a]\varphi, \varphi \wedge \mathbf{turn}_{-i} \to \odot\varphi$, and $\varphi \to \psi \wedge \mathbf{enabled}_\sigma$ derive $\varphi \to \sigma \leadsto_i \psi$. The axiom system of SL is sound and complete with respect to the given semantics (Ramanujam & Simon, 2008b).

This summarizes the strategy logic developed by the Chennai school of logic. Notice that payoffs and epistemics of the agents are not included in SL, and one can consider such future work directions as the extension of the basic framework of SL. In this work, we focus on giving a dynamic touch to SL.

4.3 Restricted strategy logic

4.3.1 Basics

Let us now extend SL to *restricted strategy logic*, henceforth RSL, by allowing move restrictions during the game[1]. Recall that our motivation can be illustrated with the example of a gamepad/keyboard which gets broken during the game play disallowing the player to make certain moves from that moment on.

We denote the move restriction by $[\sigma!a]^i$ for a strategy specification σ, and move a for player i. Informally, after the move restriction of σ by a, player i will not be able to make an a move. We incorporate restrictions in RSL at the level of strategy specifications. In SL, recall that strategies are functions. Therefore, they only produce one move per state. However, our dynamic take in strategies cover more general cases where strategies can offer a *set* of moves to the player. From some perspective, this may very well be considered a non-deterministic strategy. Thus, in RSL, we define strategy μ^i as $\mu^i : S^i \to 2^\Sigma$. By $\mathbf{outr}_{\mu^i}(s)$ we will denote the set of moves returned by μ^i at s. Then, the extended syntax of strategy specifications for player i is given as follows.

$$Strat^i(P^i) := [\psi \to a]^i \mid \sigma + \sigma \mid \sigma \cdot \sigma \mid [\sigma!a]^i$$

Notice that the restrictions affect only the player who gets a move restriction. In other words, if a is prohibited to player i, it does not mean that some other player j cannot make an a move. In other words, if my gamepad/keyboard is broken, it doesn't mean that yours is broken as well.

Once a move is restricted at a state, we will prune the strategy tree removing the prohibited move from that state on. Therefore, given $\mu^i : S^i \to 2^\Sigma$, we define the updated strategy relation $\mu^i!a : S^i \to 2^{\Sigma - \{a\}}$. We are now ready to define confirmation of restricted

[1] I owe the name RSL to Sujata Ghosh.

specifications to strategies. Note that we skip the cases for $\cdot$ and $+$ as they are exactly the same.

$$\mu, s \models_i [\varphi \to a]^i \quad \text{iff} \quad M, s \models \varphi \text{ implies } a \in \mathbf{outr}_\mu(s)$$
$$\mu^i, s \models_i [\sigma!a]^i \quad \text{iff} \quad a \notin \mathbf{outr}_{\mu^i}(s) \text{ and } \mu!a, s \models_i \sigma$$

In the sequel, we omit the superscript that indicates the agents, thus, we write S for S^i, and $\sigma!a$ for $[\sigma!a]^i$ when it is obvious. Given a strategy μ and its strategy tree $T_\mu = (S_\mu, \Rightarrow_\mu, s_0, \lambda_\mu)$, we define the restricted strategy structure $T_{\mu!a}$ with respect to an action a. Once we removed the restricted moves, the updated structure may not be a tree (it may be a forest). For this reason, we take $(\mu!a, s)$ as the connected component of $T_{\mu!a}$ that includes s. Therefore, for a fixed strategy, restrictions may yield different restricted strategy trees at different states. This is perfectly fine for our intuition, because the state of the game where the restriction is made is important. In other words, it matters where my gamepad is broken during the game play. Now, for player i and strategy μ^i and move a, the restricted strategy tree $T_{\mu!a} = (S_{\mu!a}, \Rightarrow_{\mu!a}, s', \lambda_{\mu!a})$ is the least subtree of T satisfying the following two conditions:

1. $s' \in S_{\mu!a}$;
2. For any $s \in S_{\mu!a}$, if $\lambda(s) = i$, then for each action $b \neq a \in \mu!a(s)$, there exists a unique $t \in S_{\mu!a}$ such that $s \overset{b}{\Rightarrow}_{\mu!a} t$. Otherwise, if $\lambda(s) \neq i$, then for all t with $s \overset{b}{\Rightarrow} t$ for $b \neq a$, we have $s \overset{b}{\Rightarrow}_{\mu!a} t$.

Note that we introduce such conditions so that an RSL model can easily be considered as a submodel of a SL model with some additional assumptions. We can now make some further observations. By an abuse of the notation, we will use $\leftrightarrow$ to denote the equivalence of strategy specifications with respect to the conformation relation.

Proposition 4.3.1. *For any strategy μ, state s, specification σ, and formula ψ, $\mu, s \not\models [[\psi \to a]!a]$.*

Proof. Immediate by the definitions. ■

Proposition 4.3.2. *For strategy specifications σ and σ', and move a, $(\sigma \cdot \sigma')!a \leftrightarrow (\sigma!a) \cdot (\sigma'!a)$ and $(\sigma + \sigma')!a \leftrightarrow (\sigma!a) + (\sigma'!a)$.*

Proof. Consider the first case.

$$\begin{aligned}
\mu, s \models (\sigma \cdot \sigma')!a \quad & \text{iff} \quad a \notin \mathbf{outr}_\mu(s) \text{ and } \mu!a, s \models \sigma \cdot \sigma' \\
& \text{iff} \quad a \notin \mathbf{outr}_\mu(s) \text{ and } \mu!a, s \models \sigma \text{ and } \mu!a, s \models \sigma' \\
& \text{iff} \quad a \notin \mathbf{outr}_\mu(s) \text{ and } \mu!a, s \models \sigma), \text{ and} \\
& \qquad\;\; a \notin \mathbf{outr}_\mu(s) \text{ and } \mu!a, s \models \sigma') \\
& \text{iff} \quad \mu, s \models \sigma!a \text{ and } \mu, s \models \sigma'!a
\end{aligned}$$

The remaining cases are similar. ■

Restrictions stabilize immediately. For $n \geq 1$, we use notation $\sigma!^n a$ to denote $n \geq 1$-consecutive restrictions of σ by move a. Similarly, we put $\mu!^n a$ for the corresponding strategy tree μ.

Proposition 4.3.3. *For arbitrary strategy specification σ, move a and state s, $(\sigma!a)!a \leftrightarrow \sigma!a$. Moreover, we have $\sigma!^n a \leftrightarrow \sigma!a$.*

Proof. We start with considering the cases where the restrictions are applied consecutively twice. We use Proposition 2.

$$\begin{aligned}
\mu, s \models (\sigma!a)!a \quad & \text{iff} \quad a \notin \mathbf{outr}_\mu(s) \text{ and } \mu!a, s \models \sigma!a \\
& \text{iff} \quad a \notin \mathbf{outr}_\mu(s) \text{ and } (\mathbf{out}_\mu(s) \neq a \text{ and } (\mu!a)!a, s \models \sigma) \\
& \text{iff} \quad a \notin \mathbf{outr}_\mu(s) \text{ and } \mu!a, s \models \sigma \\
& \text{iff} \quad \mu, s \models \sigma!a
\end{aligned}$$

Now, we generalize to the case for n. The proof is by induction on n. The case for $n = 1$ is trivial and the case for $n = 2$ was presented above. Assume now that $n \geq 2$ and the claim holds for every integer less than or equal to n. We will now show it for $n + 1$.

$$
\begin{array}{llll}
\mu, s \models \sigma!^n a & \text{iff} & & \\
\mu, s \models (\sigma!^{n-1}a)!a & \text{iff} & a \notin \mathbf{outr}_\mu(s) \text{ and } \mu!a, s \models \sigma!^{n-1}a & \\
 & \text{iff} & a \notin \mathbf{outr}_\mu(s) \text{ and } \mu!a, s \models \sigma!^a & \text{induction hyp.} \\
 & \text{iff} & \mu, s \models \sigma!a & \text{by definition}
\end{array}
$$

■

Since restrictions are local and operate by elimination, the order of the restrictions does not matter.

Proposition 4.3.4. *For any moves a and b, $(\sigma!a)!b \leftrightarrow (\sigma!b)!a$.*

Proof. First, notice that for a strategy μ and moves a, b, we have $(\mu!a)!b = (\mu!b)!a$ by definition. Now, let us consider strategy specification σ with moves a and b.

$$
\begin{array}{lll}
\mu, s \models (\sigma!a)!b & \text{iff} & b \notin \mathbf{outr}_\mu(s) \text{ and } \mu!b, s \models \sigma!a \\
 & \text{iff} & b \notin \mathbf{outr}_\mu(s) \text{ and } (a \notin \mathbf{outr}_\mu(s) \text{ and } (\mu!b)!a, s \models \sigma) \\
 & \text{iff} & b \notin \mathbf{outr}_\mu(s) \text{ and } (a \notin \mathbf{outr}_\mu(s) \text{ and } (\mu!a)!b, s \models \sigma) \\
 & \text{iff} & a \notin \mathbf{outr}_\mu(s) \text{ and } (b \notin \mathbf{outr}_\mu(s) \text{ and } (\mu!a)!b, s \models \sigma) \\
 & \text{iff} & a \notin \mathbf{outr}_\mu(s) \text{ and } \mu!a, s \models \sigma!b \\
 & \text{iff} & \mu, s \models (\sigma!b)!a
\end{array}
$$

■

4.3.2 A Case Study: The Centipede Game

Let us consider the centipede game (see Figure 4.1), and see how RSL can formalize it when a restricted strategy specification can change the game after an unexpected (or even irrational) move. Let us call the players $P1$ and $P2$. The set of actions in the centipede game is $\Sigma = \{d, a\}$ where d, a mean that the player moves down or across, respectively. Utilities for individual players are indicated by a tuple (x, y) where x is the utility for $P1$,

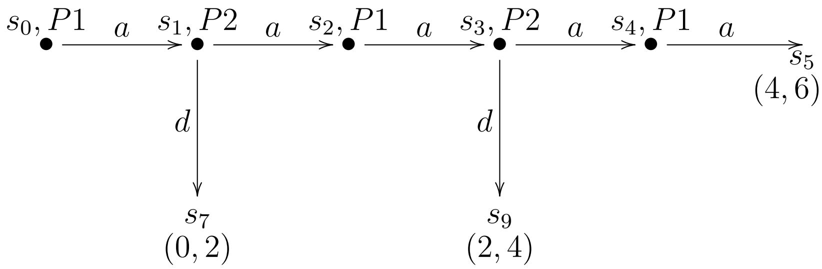

Figure 4.2: Restricted centipede game-tree

and y is the utility for $P2$. For the sake of generality, we will not impose any further conditions on the strategies.

In a recent work, Artemov approached the centipede game from a rationality and epistemology based point of view (Artemov, 2009b). Now, similar to his approach, we will use symbols $\mathbf{r}_1$ and $\mathbf{r}_2$ to denote the propositions "$P1$ is rational" and "$P2$ is rational", respectively. Let us now construct rational strategies μ and ν for $P1$ and $P2$ respectively following the backward induction scheme. At s_4, $P1$ makes a d move, if she is rational. Therefore, we have $\mu, s_4 \models [\mathbf{r}_1 \to d]^{P1}$. However, at s_3, $P2$ would be aware of $P1$'s possible move at s_4 and the fact that $P1$ is rational as well, thus makes a d move if she herself is rational. So, we have $\nu, s_3 \models [(\mathbf{r}_2 \wedge (\langle a \rangle \mathbf{r}_1 \vee \langle d \rangle \mathbf{r}_1)) \to d]^{P2}$. Following the same strategy, we obtain the following.

$\mu, s_2 \models [\mathbf{r}_1 \wedge (\langle a \rangle(\mathbf{r}_2 \wedge (\langle a \rangle \mathbf{r}_1 \vee \langle d \rangle \mathbf{r}_1) \vee \langle d \rangle(\mathbf{r}_2 \wedge (\langle a \rangle \mathbf{r}_1 \vee \langle d \rangle \mathbf{r}_1))) \to d]^{P1}$

$\nu, s_1 \models [\mathbf{r}_2 \wedge (\langle a \rangle(\mathbf{r}_1 \wedge (\langle a \rangle(\mathbf{r}_2 \wedge (\langle a \rangle \mathbf{r}_1 \vee \langle d \rangle \mathbf{r}_1) \vee \langle d \rangle(\mathbf{r}_2 \wedge (\langle a \rangle \mathbf{r}_1 \vee \langle d \rangle \mathbf{r}_1))))\langle d \rangle(\mathbf{r}_1 \wedge (\langle a \rangle(\mathbf{r}_2 \wedge (\langle a \rangle \mathbf{r}_1 \vee \langle d \rangle \mathbf{r}_1) \vee \langle d \rangle(\mathbf{r}_2 \wedge (\langle a \rangle \mathbf{r}_1 \vee \langle d \rangle \mathbf{r}_1))))) \to d]^{P2}$

$\mu, s_0 \models [(\mathbf{r}_1 \wedge (\langle a \rangle(\mathbf{r}_2 \wedge (\langle a \rangle(\mathbf{r}_1 \wedge (\langle a \rangle(\mathbf{r}_2 \wedge (\langle a \rangle \mathbf{r}_1 \vee \langle d \rangle \mathbf{r}_1) \vee \langle d \rangle(\mathbf{r}_2 \wedge (\langle a \rangle \mathbf{r}_1 \vee \langle d \rangle \mathbf{r}_1))))\langle d \rangle(\mathbf{r}_1 \wedge (\langle a \rangle(\mathbf{r}_2 \wedge (\langle a \rangle \mathbf{r}_1 \vee \langle d \rangle \mathbf{r}_1) \vee \langle d \rangle(\mathbf{r}_2 \wedge (\langle a \rangle \mathbf{r}_1 \vee \langle d \rangle \mathbf{r}_1)))))) \vee \langle d \rangle(\mathbf{r}_2 \wedge (\langle a \rangle(\mathbf{r}_1 \wedge (\langle a \rangle(\mathbf{r}_2 \wedge (\langle a \rangle \mathbf{r}_1 \vee \langle d \rangle \mathbf{r}_1) \vee \langle d \rangle(\mathbf{r}_2 \wedge (\langle a \rangle \mathbf{r}_1 \vee \langle d \rangle \mathbf{r}_1))))\langle d \rangle(\mathbf{r}_1 \wedge (\langle a \rangle(\mathbf{r}_2 \wedge (\langle a \rangle \mathbf{r}_1 \vee \langle d \rangle \mathbf{r}_1) \vee \langle d \rangle(\mathbf{r}_2 \wedge (\langle a \rangle \mathbf{r}_1 \vee \langle d \rangle \mathbf{r}_1))))))) \to d]^{P1}$

Let $\Diamond \varphi := \langle a \rangle \varphi \vee \langle d \rangle \varphi$. Then, we have the following statements.

$\mu, s_4 \models [\mathbf{r}_1 \to d]^{P1}$

$\nu, s_3 \models [\mathbf{r}_2 \wedge \Diamond \mathbf{r}_1 \to d]^{P2}$

$\mu, s_2 \models [\mathbf{r}_1 \wedge \Diamond(\mathbf{r}_2 \wedge \Diamond\mathbf{r}_1) \rightarrow d]^{P1}$

$\nu, s_1 \models [\mathbf{r}_2 \wedge \Diamond(\mathbf{r}_1 \wedge \Diamond(\mathbf{r}_2 \wedge \Diamond\mathbf{r}_1)) \rightarrow d]^{P2}$

$\mu, s_0 \models [\mathbf{r}_1 \wedge \Diamond(\mathbf{r}_2 \wedge \Diamond(\mathbf{r}_1 \wedge \Diamond(\mathbf{r}_2 \wedge \Diamond\mathbf{r}_1))) \rightarrow d]^{P1}$

Therefore, backward inductively, under the assumption of common rationality, we observe that $P1$ should make a d move at s_0. Furthermore, this can be generalized to many other games.

Theorem 4.3.5. *Assuming common rationality in RSL framework, backward induction scheme produces a unique solution in games with ordinal pay-offs.*

The argument for the proof of the theorem is quite straight-forward. Even if the strategies may be set valued, and hence return a set of moves per state, the assumption of common rationality forces the player to choose the move which returns the highest pay-off. Since this fact is known among players, by induction, we can show that the solution is unique.

Let us follow the backward induction scheme again with the updated game tree given above. Notice that the specification $[\mathbf{r}_1 \rightarrow d]^{P1}$ does not conform with $\mu!d$ (as $\mu!d, s \not\models [[\mathbf{r}_1 \rightarrow d]^{P1}!d]^{P1}$ for all s)[2]. Therefore, after the move restriction rational move for $P1$ is not admissible for her from that point on. Thus, $\mu!d$ conforms to those specifications that implies an a move. Thus, $\mu!d, s_4 \models [\top \rightarrow a]^{P1} \cdot \neg[\mathbf{r}_1 \rightarrow d]^{P1}$. Therefore, at s_3, being rational, $P2$ choses the move with the highest pay-off, and makes an a move. Thus, $\nu, s_3 \models \mathbf{r}_2 \wedge (\Diamond\neg[\mathbf{r}_1 \rightarrow d]) \rightarrow a]^{P2}$. Similarly, at s_1, we have $\nu, s_1 \models \mathbf{r}_2 \wedge (\Diamond\neg[\mathbf{r}_1 \rightarrow d] \wedge \Diamond(\mathbf{r}_2 \wedge (\Diamond\neg[\mathbf{r}_1 \rightarrow d]))) \rightarrow a]^{P2}$. Finally, at the root, we have $\mu!d, s_0 \models [\top \rightarrow a]^{P1}$. Thus, in the restricted centipede game, clearly, $P1$ has to make an a move.

Even if it is not the thesis of this paper, this example also shows that we can view rationality as a *set of restrictions*. Namely, some restrictions force the players to play irrationally, while some restrictions allow them to play rationally. RSL, in this respect,

[2]We slightly abuse the formal language here. For a specification σ, we put $\mu, s \models \neg\sigma$ if it is not the case that $\mu, s \models \sigma$.

gives a framework where both rational and irrational strategies can be analyzed.

4.3.3 Axiomatization, Completeness, and Complexity

Before discussing the axiomatization of RSL, we note that the set of available moves $moves$ is defined as previously. Then, we define the set of enabled moves for the move restriction operator as $([\sigma!a]^i)(s) = \sigma(s) - \{a\}$. Namely, if the move a is not allowed any more, it should not be enabled.

We now give the syntax of RSL, which is the same as that of SL.

$$p \mid (\sigma)_i : a \mid \neg\varphi \mid \varphi_1 \wedge \varphi_2 \mid \langle a \rangle \varphi \mid \sigma \leadsto_i \psi$$

The semantics and the truth definitions of the formulas are defined as earlier with the exception of strategy specifications for restrictions (cf. Section 2). The axiom system of RSL consists of the axioms and rules of SL together with the following additional axiom for the added specification construct.

- $(\sigma!a)_i : c \leftrightarrow \mathbf{turn}_i \wedge \neg((\sigma)_i : a) \wedge (\sigma)_i : c$

The soundness of the given axiom is straightforward and hence skipped. The new specification we introduced does not bring along an extra derivation rule since the new operator is at the level of specifications, not the formulas.

Now, we observe that restricted moves are not enabled. We state it as a theorem with an immediate proof.

Theorem 4.3.6. $(\sigma!a)_i : a \leftrightarrow \bot$.

Proof. Immediate. ■

As expected, the system RSL is complete.

Theorem 4.3.7. *RSL is complete with respect to the given semantics.*

Proof. The completeness of RSL is by reduction to SL. Since we have one additional strategy specification, we describe an immediate reduction for that.

First notice that an RSL model is a submodel of a suitable SL model. The only problem is that, in RSL, strategies are constructed as relations whereas in SL, the strategies are functions. Therefore, an RSL strategy can be thought of as a union of several SL strategies. Therefore, an RSL game-tree is a tree model for an SL model with the suitable union of strategies. Moreover, in RSL, we also obtain a game-tree model with relational strategies.

Now, the reduction of RSL to SL should be rather immediate. Given a formula of the form $(\sigma)_i : c$ where $\sigma = \sigma'!a$ for some a, we observe that (by soundness), it is equivalent to a formula in the language of SL: $\mathbf{turn}_i \wedge \neg((\sigma)_i : a) \wedge (\sigma)_i : c$. Similarly, consider the given formula of the form $\sigma \leadsto_i \psi$ where $\sigma = \sigma'!a$ for some a. Notice also that, in SL, $\sigma \leadsto_i \psi$ is axiomatically reduced to a formula that uses formulas of the form $(\sigma)_i : a$ which we have covered just before. Therefore, the two different types of formulas that may include restricted strategy specifications are now reduced to a formula in the language of SL. Since SL is complete already (Ghosh *et al.*, 2010b), and our translation is truth preserving due to soundness, we conclude that RSL is complete. ■

However, the decidability of RSL and SL is not immediate. To the best of our knowledge, SL has not yet been proved to be decidable (or undecidable). Here, we show an upper bound for the model checking problem for both logics.

Now, we discuss a reduction of SL to (multi-modal) Computational Tree Logic CTL*. We refer the readers who are not familiar with CTL* to (Emerson, 1990; Emerson & Halpern, 1986). First, we discuss how to construct a CTL* model based on a given SL/RSL model, then describe a translation from SL to CTL*.

Let us take a strategy model $M = (T, V)$ where $T = (S, \Rightarrow, s_0, \lambda)$. Notice that the function $\Rightarrow$ was defined from $S \times \Sigma$ to S. We can *redefine* it by *currying*. Given $\Rightarrow: S \times \Sigma \to S$, we can define the transition $\overset{a}{\Rightarrow}: S \to S$ for each move a. Therefore, we can curry $\Rightarrow$ to get $\Rightarrow = \bigcup_{a \in \Sigma} \overset{a}{\Rightarrow}$. We can think of M as a pointed multi-modal CTL* tree model $M^* = (S, s_0, \{\stackrel{a}{\Rightarrow}\}_{a\in\Sigma}, V)$ that has next time modalities for each action. In this case, corresponding to each binary relation $\stackrel{a}{\Rightarrow}$, we will have a dynamic next-time modality X_a which quantifies over the sub-path on the same branch (note that state formulas are also path formulas (Emerson & Halpern, 1986)). For the reflexive-transitive closure of $\stackrel{a}{\Rightarrow}$, we will use $\Box$ for all accessible future times in the same branch. Before giving the translation of SL specifications and formulas into CTL*, let us introduce some special propositions. We label the states that are returned by a strategy μ with the proposition $\mathbf{strategy}_\mu$, stipulating that $\mathbf{strategy}_\mu$ holds only at those points. Notice that given two points in the domain of μ, there is a unique path between these two (which satisfies $\mathbf{strategy}_\mu$). Moreover, we use $\mathbf{turn}_i$ as a proposition that denotes that it is i's turn to play, i.e. $s \models \mathbf{turn}_i$ iff $s \in S^i$.

We now give two translations. First, $\mathbf{tr}$ translates strategy specifications to CTL* formulas while the second $\mathbf{Tr}$ translates SL formulas to CTL* formulas. Given a strategy μ, conformation to μ is translated as follows.

- $\mathbf{tr}([\psi \to a]^i) = \mathbf{Tr}(\psi) \to \mathsf{E}(\mathbf{strategy}_\mu \wedge \mathsf{X}_a \top)$
- $\mathbf{tr}(\sigma_1 + \sigma_2) = \mathbf{tr}(\sigma_1) \vee \mathbf{tr}(\sigma_2)$
- $\mathbf{tr}(\sigma_1 \cdot \sigma_2) = \mathbf{tr}(\sigma_1) \wedge \mathbf{tr}(\sigma_2)$

Notice that in the first case, the translation makes use of the formula translation $\mathbf{Tr}$ for formula ψ as defined below. Assuming the correctness of $\mathbf{Tr}$ (which we show in Theorem 4.3.9), correctness of the translation $\mathbf{tr}$ is straightforward.

Theorem 4.3.8. *Let μ be a strategy and σ a strategy specification in SL, and let μ^* be the corresponding (sub)tree in a CTL* model. Then, $\mu, s \models \sigma$ iff $\mu^*, s \models \mathbf{tr}(\sigma)$.*

Theorem 4.3.8. Let $\mu : S \to \Sigma$ be a strategy and let σ be a strategy specification in SL for a fixed player i. For simplicity, we omit the superscripts that indicate the player. First, we describe how to obtain a multi-modal CTL* tree μ^*, and then show that the translation is truth-preserving. As a reminder, in a strategy tree, we include the root, and for the states that belong to the strategizing player, we assign a unique move to the player. In addition, we include all other moves that do not belong to the strategizing player. Therefore, a strategy tree can be thought of a as branching multi-modal CTL* model. We put $w \models \mathbf{turn}_i$ if $w \in S^i$ and at each $w \in S^i$, we have one outgoing edge. For $v \notin S^i$, we have all admissible moves $\mu(v)$ at v. Then, the CTL* model is constructed as follows. Take S as the set of states of the CTL* model μ^*. Next, for all moves $a \in \Sigma$, we have a corresponding accessibility relation R_a in the CTL* model similar to the Kripke semantics for propositional dynamic logic. Finally, we keep the valuation the same as in the SL model. Now, let us consider the case $\sigma = [\psi \to a]^i$:

$$\begin{aligned} \mu, s \models \sigma \quad & \text{iff} \quad \mu, s \models [\psi \to a]^i \\ & \text{iff} \quad \mu, s \models \psi \text{ implies } \mathbf{out}_\mu = a \qquad \text{(by definition)} \\ & \text{iff} \quad \mu^*, s \models \mathbf{Tr}(\psi) \text{ implies } \mathsf{E}(\mathbf{strategy}_\mu \wedge \mathsf{X}_a \top) \end{aligned}$$

Here we make use of the formula translation $\mathbf{Tr}$ whose correctness will be shown next. Furthermore, we obtain the last line by the induction hypothesis, and by the earlier observation we have made about $\mathbf{out}_\mu = a$. The remaining cases for the translation $\mathbf{tr}$ are straightforward inductions on specifications $\sigma_1 + \sigma_2$ and $\sigma_1 \cdot \sigma_2$, and hence left to the reader. Unconventionally, here we make use of Theorem 4.3.9 in the proof of the above statement just because specifications preceed the formulas in SL. ■

Here follows the translation $\mathbf{Tr}$ of formulas from SL to CTL* skipping the Boolean cases. Note that the translation is very similar to the Kripke semantics for Propositional Dynamic Logic where for each action a, a relation R_a and a modality $\langle a \rangle$ associated with R_a are introduced.

- $\mathbf{Tr}(\langle a \rangle \varphi) = \mathsf{X}_a \mathbf{Tr}(\varphi)$
- $\mathbf{Tr}((\sigma)_i : c) = \mathsf{X}_c \top$ for $c \in \sigma(s)$
- $\mathbf{Tr}((\sigma)_i : c) = \bot$ if $c \notin \sigma(s)$

- $\mathbf{Tr}(\sigma \rightsquigarrow_i \psi) = \mathsf{E}\Box(\mathbf{strategy}_\sigma \wedge (\mathbf{Tr}(\psi) \wedge (\mathbf{turn}_i \rightarrow \mathbf{enabled}_\sigma)))$

Now, the atom $\mathbf{enabled}_\sigma$ is true at a state s in CTL* if and only if for at least one $a \in \Sigma$ we have $\mathsf{X}_a\top$ and $a \in \sigma(s)$. Finally, the CTL* correspondence of $\sigma(s)$ to the set of enabled moves at s by the strategy specification σ is defined exactly as before with one small arrangement in the definition of admissible moves at a given state s, namely $moves(s) = \{a : s \models_{CTL*} \mathsf{X}_a\top\}$. As an example, consider the translation of the proposition $\mathbf{out}_\mu = a$. Recall that it means that a is the unique outgoing edge according to strategy μ at the state where the formula is interpreted. Therefore, there is a branch that is followed by strategy μ, and at that branch, at the current state, a is an admissible move. This corresponds to the translation $\mathsf{E}(\mathbf{strategy}_\mu \wedge \mathsf{X}_a\top)$. The following theorem summarizes our efforts here.

Theorem 4.3.9. *Let M be a SL model and let M^* be its CTL* correspondent. Then, $M, s \models \varphi$ iff $M^*, s \models \mathbf{Tr}(\varphi)$ for any state $s \in S$.*

Proof. Take a SL model M and the corresponding multi-modal CTL* model M^*. We then have the following:

$$
\begin{array}{lll}
M, s \models \langle a \rangle \varphi & \text{iff} \quad \exists s' \text{ s.t. } s \stackrel{a}{\Rightarrow} s' \text{ and } M, s' \models \varphi & \text{(by definition)} \\
& \qquad \exists s' \text{ s.t. } s \stackrel{a}{\Rightarrow} s' \text{ and } M^*, s' \models \mathbf{Tr}(\varphi) & \text{(by induction)} \\
& \qquad M^*, s \models \mathsf{X}_a \mathbf{Tr}(\varphi) & \text{(by definition)}
\end{array}
$$

Similarly, consider the SL formula $\sigma \rightsquigarrow_i \psi$:

$M, s \models \sigma \rightsquigarrow_i \psi$ iff for all s' such that $s \Rightarrow^*_\sigma s'$ in $T_s|\sigma$, we have,

$M, s' \models \psi \wedge (\mathbf{turn}_i \rightarrow \mathbf{enabled}_\sigma)$

(by definition)

iff for all s' such that $s \Rightarrow^* s'$ in T_s, we have

$M, s' \models \mathbf{strategy}_\sigma \wedge \psi \wedge (\mathbf{turn}_i \rightarrow \mathbf{enabled}_\sigma)$

(by definition)

iff for all s' such that $s \Rightarrow^* s'$ in T_s, we have

$M^*, s' \models \mathbf{strategy}_\sigma \wedge \mathbf{Tr}(\psi) \wedge (\mathbf{turn}_i \rightarrow \mathbf{enabled}_\sigma)$

(by induction)

iff $M^*, s \models \mathsf{E}\Box[\mathbf{strategy}_\sigma \wedge \mathbf{Tr}(\psi) \wedge (\mathbf{turn}_i \rightarrow \mathbf{enabled}_\sigma)]$

(as $\Rightarrow^*$ denotes the reflexive and transitive closure

on the path T_s on the CTL* tree T)

The last case $(\sigma)_i : c$ is very similar. Recall that this formula is true in SL at a state s iff $c \in \sigma(s)$. Consider the enabled move c at s, and take a corresponding next time modality for c, namely X_c. Moreover, at s, the diamond-like modality X_c has to be *enabled* giving us $\mathsf{X}_c\top$ for $c \in \sigma(s)$. ■

Note that the translation we suggest is model-dependent. For example, the predicate $\mathbf{strategy}_\mu$, depends on the strategy μ, thus the model. For this reason, the suggested translation is not entirely syntactic and does not give us an immediate decidability result (using the fact that CTL* is decidable). However, model checking problem for CTL* is PSPACE-complete (Emerson, 1990). Therefore, we have an upper bound for the complexity of the model checking problem for SL. We next observe that the model checking problem for both SL and RSL are in PSPACE.

Theorem 4.3.10. *The model checking problem for SL is in PSPACE.*

Proof. In this proof, we will present two different arguments to show that the model checking problem for SL is in PSPACE.

First, we show that the translation $\mathbf{Tr}$: SL $\to$ multi-modal CTL* is polynomial-time in terms of the length of the formulas. We will denote the length of a specification or a formula by $|\cdot|$. We will show that $|\mathbf{Tr}(\psi)| \leq |\psi|^k$ for some integer k. The cases for Booleans are obvious, hence we skip them. For $\psi = \langle a \rangle \varphi$, consider $|\mathbf{Tr}(\langle a \rangle \varphi)|$ which is equivalent to $|\mathsf{X}_a \mathbf{Tr}(\varphi)| = 1 + |\mathbf{Tr}(\varphi)|$. By induction hypothesis, $|\mathbf{Tr}(\varphi)| \leq |\varphi|^l$ for some integer l. Therefore, $1 + |\mathbf{Tr}(\varphi)| \leq |\varphi|^k$ for some $k \geq 1 + l$. Therefore, for $\psi = \langle a \rangle \varphi$, we observe $|\mathbf{Tr}(\psi)| \leq |\psi|^k$ for some integer k. In a similar fashion, the case for $\psi = (\sigma)_i : c$ is obvious as $|\mathbf{Tr}((\sigma)_i : c)|$ is always constant. The case for $\psi = \sigma \leadsto_i \varphi$ is also very similar. By induction hypothesis, we immediately observe that $|\mathbf{Tr}(\sigma \leadsto_i \varphi)| \leq 11 + |\sigma \leadsto_i \varphi|^l$ (counting the parantheses as well). Therefore, for some large enough integer $k > l$, we have $|\mathbf{Tr}(\sigma \leadsto_i \varphi)| \leq |\sigma \leadsto_i \varphi|^k$.

Now, we can consider the translation $\mathtt{tr}$ for the strategy specifications. The reason why we consider $\mathtt{tr}$ after $\mathbf{Tr}$ is the fact that former depends on the latter. We will now show that, for strategy specification σ, $|\mathtt{tr}(\sigma)| \leq |\sigma|^k$ for some integer k. Consider the case where $\sigma = [\psi \to a]^i$. Then, $|\mathtt{tr}([\psi \to a]^i)| = 5 + |\mathbf{Tr}(\psi)|$. By the previous observation, we know that $|\mathbf{Tr}(\psi)| \leq |\psi|^l$ for some integer l. Therefore, $5 + |\mathbf{Tr}(\psi)| \leq |\psi|^k$ for large enough integer $k \geq l$. Thus, $\mathtt{tr}(\psi \to a]^i) \leq |(\psi \to a]^i)|^k$ for some k. The cases for the $\cdot$ and $+$ operations are obvious. This concludes the proof that the translation functions $\mathtt{tr}$ and $\mathbf{Tr}$ are polynomial-time. Therefore, the complexity of model checking for SL cannot be higher than PSPACE.

Our second argument is a sketch of direct reasoning. Similar to the arguments for the complexity of basic modal logic, we just need to check the branches of the tree model one by one using a depth-first search (Blackburn *et al.*, 2001). Now, fix a SL formula φ and a model M. Since SL models are branching tree models, we can check M, branch by branch, one at a time without any need of considering the other branches. Namely, the procedure does not need to *remember* the previous searches making it effective in the use of space. Moreover, the length of the branches in M is polynomial in $|\varphi|$ because of the construction of the SL model M, thus it shows that model checking for SL is in PSPACE. This concludes the proof that the complexity of the model checking problem for SL is in PSPACE. ■

Corollary 4.3.11. *The model checking problem for RSL is in PSPACE.*

Proof. We observed that the RSL formulas can be reduced to SL formulas. The reduction is clearly polynomial, as can be seen from the axiomatization (See Section 4.3.3). Therefore, we can translate any given RSL formula to a SL formula in a SL model, which in turn can be translated into a multi-modal CTL* formula and model. Then, by Theorem 4.3.10, we deduce that the complexity of the model checking problem for RSL is also PSPACE. ■

4.4 Related Work and Conclusion

From a logical point of view, the idea of treating strategies as the "unsung heroes of games" can be traced back to van Benthem (van Benthem, 2002; van Benthem, 2008). In line with this program, Ramanujam, Simon, and Paul (Paul *et al.*, 2009; Ramanujam & Simon, 2008a; Ramanujam & Simon, 2008b; Ramanujam & Simon, 2008c) have initiated a study within game theory and dynamic logic by taking strategies as the focus of concern and treating them as the primitives of games. They discuss strategies as a way of reasoning within and about games, and investigate how games and strategies behave under some further assumptions. Recent work by Ramanujam and Simon discusses another form strategy restriction due to resource limitations in a different setting.

The game-theoretical approach to the centipede game was initiated by Rosenthal (Rosenthal, 1981), which was followed by several solution methods (Aumann, 1995; Halpern, 2001). More recently, the role of rationality in its solution methods has been discussed (Baltag *et al.*, 2009; van Benthem & Gheerbrant, 2010) and extended to some other cases (Piccione & Rubinstein, 1997; Parikh *et al.*, 2012).The role of knowledge as opposed to beliefs has been discussed in similar contexts by Artemov (Artemov, 2009b). Moreover, based on experimental work, Ghosh, Meijering and Verbrugge aim to observe how humans reason strategically (Ghosh *et al.*, 2010a). They consider simple centipede-like games from the cognitive point of view, and use strategy logic to formalize their findings.

This chapter is built on the aforementioned studies, and introduces a dynamic twist in formalizing strategies in games. What we suggest is a formal framework where restrictions in strategies are allowed during the game. We propose a new logic, called restricted strategy logic (RSL), to express strategy restrictions. We show its completeness and discuss model checking issues. Along the way, we also discuss model checking of SL, which has not been done so far (to the best of our knowledge). We claim that RSL presents a succinct way to represent strategy revisions. One direction for future research concerns the question on how RSL can be connected to the switching strategy frameworks. Additionally, one could come up with a restriction methodology where players change their rationality assumption. Note that, within the framework of RSL, different types of rationality (utilitiarian, deontological etc) can be seen as a different set of *restrictions*. For instance, a player can initially commit herself to a $\max\min$ type of strategy and then change her commitment to a $\max\max$ type of strategy. Therefore, such changes can be represented from switching from restriction set A to restriction set A', for instance. In short, different *types* of rationality can cause different revisions in players' strategies. We leave such analysis to future work.

Moreover, note that the way strategy restrictions work, and its completeness proof resemble public announcement logic (van Ditmarsch *et al.*, 2007; Plaza, 1989). In the case of RSL, the move restriction can be seen as a prohibitive *negative* announcement, corresponding to an elimination of moves that agree with the prohibition.

In conclusion, we believe that RSL presents a concise and natural framework for dynamic strategizing in games, and it can be extended in several thought-provoking ways.

Hayatımın en mutlu anıymış, bilmiyordum.

Orhan Pamuk, *Masumiyet Müzesi*

It was the happiest moment of my life, though I didn't know it.

Orhan Pamuk, *Museum of Innocence*

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