Game Theoretical Semantics for Paraconsistent Logics

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1 Introduction

Game theoretical semantics suggests a very intuitive approach to formal semantics. *The semantic verification game* for classical logic is played by two players, *verifier* and *falsifier* who we call Heloise and Abelard respectively. The goal of Heloise in the game is to verify the truth of a given formula in a given model whereas for Abelard it is to falsify it. The rules are specified syntactically based on the form of the formula. During the game, the given formula is broken into subformulas step by step by the players. The game terminates when it reaches the propositional literals and when there is no move to make. If the game ends up with a propositional literal which is true in the model in question, then Heloise wins the game. Otherwise, Abelard wins. Conjunction is ssociated with Abelard, disjunction with Heloise. That is, when the main connective is a conjunction, it is Abelard's turn to choose and make a move, and similarly, disjunction yields a choice for Heloise. The negation operator switches the roles of the players: Heloise becomes the falsifier, Abelard becomes the verifier. The major result of this approach states that Heloise has a winning strategy *if and only if* the given formula is true in the given model. The semantic verification game and its rules are shaped by classical logic and consequently by its restrictions. In this work, we first observe how the verification games change in non-classical, especially propositional paraconsistent logics, and give Hintikka-style game theoretical semantics for them. We will obtain games in which winning strategies for players are not necessary and sufficient conditions for truth values of the formulas.

Game theoretical semantics (GTS, for short) was largely popularized by Hintikka and Helsinki School researchers even though earlier pointers to similar ideas can be found in Parikh [12]. An overview of the field and its relation to various epistemic and scientific topics can be found in [15]. Moreover, [9,14,15] provide extensive surveys of GTS. A game theoretical concept of truth and its relation to winning strategies were investigated by [3]. Pietarinen considered various non-classical issues including partiality and non-competetive games within the framework of GTS with some connections to the Kleene logic without focusing on particular (paraconsistent) logics [13,16,23]. Hintikka and Sandu discussed non-classicality in GTS also without specifically offering any insight on paraconsistency [9,14]. Tulenheimo studied languages with two negation signs, which can bear some resembles to paraconsistent ideas on weak and strong negations [26]. Additionally, there

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were some technical work discussing the intersection of GTS and intuitionism including some work on type-theoretical foundations [21]. An epistemic, first-order extension of GTS, called "Independence-Friendly" logic, was suggested by Hintikka and Sandu relating GTS to Henkin quantifiers [8,11]. Some discussions on intuitionism from the viewpoint of GTS are worth noting. Tennant argued that some aspects of GTS do not work intuitionistically [25]. Similarly, Hintikka noted that the law of excluded middle may not hold in some instances since the lack of a winning strategy for a player does not entail the existence of a winning strategy for the other player [7]. However, Hintikka himself, perhaps with the exception of independence-friendly logic, is not very clear on GTS and intuitionism, especially when it comes to negation [25]. GTS relates directly to various issues in programming languages, yet, this will not be our focus here.

In this work, we consider propositional paraconsistent logics. We define paraconsistent logic as any formal system that does *not* satisfy the explosion principle: $\varphi, \neg\varphi \vdash \psi$ for any φ, ψ. There exists a wide variety of paraconsistent logics, and there are numerous ways to construct them [5,18,19]. Apart from its proof-theoretical definition, paraconsistency can also be described semantically suggesting that in paraconsistent logic some formulas and their negations can both be true.

Apart from studying the underlying logic, GTS can also be approached from a game theoretical perspective. It is then worthwhile to consider verification games where i) Abelard and Heloise both may win, ii) Abelard and Heloise both may lose, iii) Heloise may win, Abelard may not lose, iv) Abelard may win, Heloise may not lose, v) There is a tie, vi) There is an additional player, vii) Players do not take turns. Such different possibilities can occur, for instance, when both p and $\neg p$ are true, so that both players can have winning strategies. We can also imagine verification games with additional truth values and additional players beyond verifiers and falsifiers, and also construct games where players may play simultaneously.

This paper investigates the logical conditions which entail such game theoretical conditions, and aims at filling the gap in the literature between GTS and paraconsistency. In what follows, we consider a variety of well-known paraconsistent logics, offer a game semantics for them and observe how different logics generate different verification games. This is also important philosophically especially when winning strategies are seen as constructive proofs for truth in an intuitionistic sense or when they are seen as verifications [3]. Therefore, by focusing on inconsistent formulas and associated winning strategies, we offer (constructive) proofs for inconsistencies (cf. appendix) and expand the computational discussions on the connection between proofs, strategies and truth.

2 Game Semantics for Logic of Paradox

Logic of paradox (LP, for short) introduces an additional truth value P, called *paradoxical*, which intuitively stands for both true and false [17].

LP is a conservative extension of the classical logic, thus preserves the classical truth. The logics LP and Kleene's three valued system K3 have the same truth tables. However, they differ on the truth values that they preserve in valid inferences, and how they read P. It is read as *over-valuation* in LP and as *under-valuation* in K3. The truth values that are preserved in validities are called *designated truth values* [20]. In LP, it is the set $\{T, P\}$; in K3, it is the set $\{T\}$. Designated truth values can be thought of as extensions of the classical notion of truth. Even if the truth tables of two logics are the same, different sets of designated truth values produce different sets of validities, thus different logics. For instance, $p \vee \neg p$ is a theorem in LP, but not in K3.

	$\neg$
T	F
F	T
P	P

$\wedge$	T	P	F
T	T	P	F
P	P	P	F
F	F	F	F

$\vee$	T	P	F
T	T	T	T
P	T	P	P
F	T	P	F

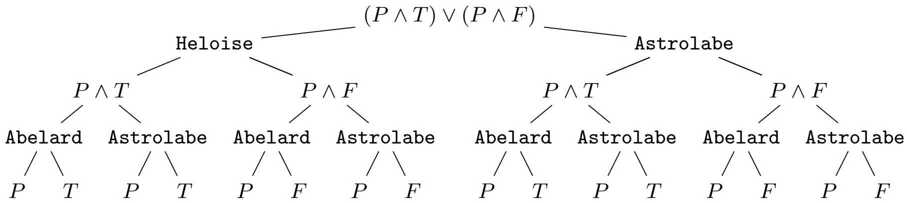
Fig. 1. *The truth table for LP and K3.*

We stipulate that the introduction of the third truth value requires an additional player that we call *Astrolabe* after Abelard and Heloise's son. Astrolabe is the *paradoxifier* in the game forcing the game to an end with P.

In GTS for LP, the first problem is to determine the turns of the players at each connective. For instance, if the formula $T \wedge P$ is considered, the problem becomes evident. In this quick game, if we assume that it is Abelard's turn then he will not have a move that can bring him a win. From the truth table, it can be seen that the formula evaluates to P, so Astrolabe can be expected to have a winning strategy. In order to make it possible, then, Astrolabe must be allowed to make a move at a conjunction, too. Similarly, if $F \vee P$ is considered, which evaluates to P, Eloise cannot make a move that can bring him a win, and Astrolabe needs to be given a turn to make a move to win the game. Therefore, we associate disjunction with Heloise and Astrolabe, and conjunction with Abelard and Astrolabe. This modification introduces parallel play where the players may make moves in a parallel, concurrent fashion. In the case of a negation, Heloise and Abelard will switch their role, and Astrolabe will keep his role as P is a fixed-point for negation in LP. Astrolabe's role always remains as the paradoxifier.

Let us now formally define GTS for LP following the terminology in [14]. First, we take the language $\mathcal{L}$ of propositional logic with its standard signature. A model M is a tuple (S, v) where S is a non-empty domain on which the game is played, and the valuation function v assigns the terms in $\mathcal{L}$ to truth values in the logic. For simplicity, we assume $\mathcal{L}$ does not have $\rightarrow$ nor $\leftrightarrow$. We define the verification game as a tuple $\Gamma = (\pi, \rho, \sigma, \delta)$ where π is the set of players, ρ is the set of well-defined game rules, σ is the set of positions, and δ is the set of designated truth values. The set of positions is determined by the subformulas of the given formula and remains unchanged in the logics we discuss as they use the same propositional syntax. We embed the turn function at the positions into the rules of the game for simplicity. A *semantic verification game* is defined as $\Gamma(M, \varphi)$ for a game Γ, model M and a formula $\varphi \in \mathcal{L}$. A *strategy* for a player is a set of rules that tells him which move to make at each position where it is his turn. A *winning strategy* is the one that guarantees a win for the player regardless of the moves of the opponent(s). A winning strategy for a player does not necessarily entail the lack of a winning strategy for the opponent(s). Let us now reconsider the following example before determining the π and ρ for LP.

Example 1. Consider the formula $(P \wedge T) \vee (P \wedge F)$ which evaluates to P. In this game, Astrolabe has a winning strategy: at each end-node ($P \wedge T$ and $P \wedge F$), he selects P. Here, we also observe that Abelard being stuck at some states (such as $P \wedge T$) does not necessarily entail a win for neither of the other players.



We call the verification game for LP as GTS^{LP}. GTS^{LP} is a non-zero sum verification game where more than one player may have a winning strategy, and making the opponent lose does not necessarily entail that it is a win for the player himself. Also, as we shall see, in GTS^{LP} admitting winning strategies does not necessarily entail the truth value of the formula in question.

Definition 1. *The tuple $\Gamma_{\text{LP}} = (\pi, \rho, \sigma, \delta)$ is an LP verification game for LP where $\pi = \{$Astrolabe, Heloise, Abelard$\}$, σ is as in classical logic, δ is $\{T, P\}$ and ρ is given as follows inductively for a game $\Gamma_{\text{LP}}(M, \varphi)$.*
— *If φ is atomic, the game terminates, and Heloise wins if φ is true, Abelard wins if φ is false and Astrolabe wins if φ is paradoxical,*
— *if $\varphi = \neg\psi$, Abelard and Heloise switch roles, Astrolabe keeps his role, and the game continues as $\Gamma_{\text{LP}}(M, \psi)$,*
— *if $\varphi = \chi \wedge \psi$, Abelard and Astrolabe choose between χ and ψ simultaneously,*
— *if $\varphi = \chi \vee \psi$, Heloise and Astrolabe choose between χ and ψ simultaneously.*

Correctness theorem for GTS^{LP} follows.

Theorem 1. *In a GTS^{LP} verification game $\Gamma_{\text{LP}}(M, \varphi)$*
— *Heloise has a winning strategy if φ is true in M,*
— *Abelard has a winning strategy if φ is false in M,*
— *Astrolabe has a winning strategy if φ is paradoxical in M.*

LP distinguishes different trues and falses: trues that are only true (T), falses that are only false (F), and trues that are also false (P) and falses that are also ture (P). In GTS, this carries over to games allowing Astrolabe making moves alongside Heloise and Abelard. In, GTS^{LP} there are winning strategies that causes a loss for the opponent, and there are winning strategies do not. Additionally, there are winning strategies that cannot guarantee the logical truth of formulas. A game for $P \wedge F$ illustrate this point, where both Abelard and

Astrolabe can have a winning strategy. But, this does not directly say anything about the truth value of $P \wedge F$. Therefore, in GTS^{LP}, the immediate connection between the existence of winning strategies and truth values becomes slightly more complicated as the following theorem identifies.

Theorem 2. *In a GTS^{LP} verification game $\Gamma_{\text{LP}}(M, \varphi)$,*
— If Heloise has a winning strategy, then φ is true in M;
— If Abelard has a winning strategy, then φ is false in M;
— If Astrolabe has a winning strategy, but not the other players, then φ is paradoxical in in M.

Theorem 2 also indicates that Astrolabe's strategy is the strictly dominated in a sense that if some other player also has a winning strategy, then Astrolabe's strategy will not bring him a win. Based on this observation, it is possible to change some game rules in order to give a biconditional correctness theorem for GTS^{LP} by prioritizing some players over the others. This will allow some players to dominate the others reflecting the truth table for LP. In this new and extended reading of GTS^{LP}, such a *move priority* is given to the *parents* (Abelard and Heloise), they are let to play first, then Astrolabe makes his move. This extension prevents parallel moves and incorporates winning strategies into the game rules. These additional rules are given as follows.

1. For propositional letters and negation, the rules are as before.
2. Disjunction belongs to Heloise and Astrolabe; conjunction belongs to Abelard and Astrolabe.
3. If Heloise (resp. Abelard) has a winning strategy in the sub-game they choose, the game proceeds with her (resp. his) move.
4. Otherwise, Astrolabe makes a move.

Example 2. Let us consider the formula in Example 1. Given $(P \wedge T) \vee (P \wedge F)$, Heloise first attempts to choose either of them only to realize that she does not have a winning strategy in either of the sub-games with $P \wedge T$ or $P \wedge F$. So, she cannot make a move, and it becomes Astrolabe's turn. Astrolabe chooses $P \wedge T$. Now, Abelard attempts to choose either P or T only to realize that neither brings him a win. So, he cannot make a move. Astrolabe makes a move, chooses P, and wins - this is Astrolabe's winning strategy. If Astrolabe chose $P \wedge F$, then first Abelard would make a move and choose F for a win. Yet, Abelard still does not have a winning strategy in this game.

As we mentioned earlier, such a twist on GTS^{LP} is *ad-hoc*. It incorporates possessing winning-strategies, which is a meta-logical condition, into game rules, which are supposed to be syntactic. This modification naturally provides a biconditional Theorem 1 at the expense of violating the pure syntacticality of the game rules, resulting in completely *ad-hoc* game rules.

3 Game Semantics for First-Degree Entailment

Semantic evaluations are generally thought of as *functions* from logical formulas to truth values. This ensures that each and every formula is assigned a *unique* truth value. However, it is possible to replace the valuation function with a valuation *relation* which can produce multiple truth values for logical formulas. The system obtained in this manner is called *First-degree entailment* (FDE, for short), and is due to Dunn [1,6].

For the given propositional language $\mathcal{L}$, the valuation relation $\mathbf{r}$ is defined on $\mathcal{L} \times \{0,1\}$. By $\varphi\mathbf{r}\emptyset$, we will denote the situation where φ is not related to any truth value. By $\varphi\mathbf{r}\{0,1\}$, we denote the situation when φ is related to both truth values. FDE is a paraconsistent (inconsistency-tolerant) and paracomplete (incompleteness-tolerant) logic. For formulas $\varphi, \psi \in \mathcal{L}$, the valuation $\mathbf{r}$ is defined inductively as follows.

- $\neg\varphi\mathbf{r}1$ iff $\varphi\mathbf{r}0$
- $\neg\varphi\mathbf{r}0$ iff $\varphi\mathbf{r}1$
- $(\varphi \wedge \psi)\mathbf{r}1$ iff $\varphi\mathbf{r}1$ and $\psi\mathbf{r}1$
- $(\varphi \wedge \psi)\mathbf{r}0$ iff $\varphi\mathbf{r}0$ or $\psi\mathbf{r}0$
- $(\varphi \vee \psi)\mathbf{r}1$ iff $\varphi\mathbf{r}1$ or $\psi\mathbf{r}1$
- $(\varphi \vee \psi)\mathbf{r}0$ iff $\varphi\mathbf{r}0$ and $\psi\mathbf{r}0$

Notice that LP can be obtained from FDE by imposing a restriction on FDE that no formula gets the truth value $\emptyset$. We denote the GTS for FDE as GTS^{FDE}.

What does the relational semantics correspond to in verification games? If the truth value P in LP can intuitively be thought of as both true and false, and if this allows concurrent moves in GTS^{LP}, then the same approach works in GTS^{FDE} as well. In FDE, unlike LP, formulas can have no truth value which suggests that neither Heloise nor Abelard may have a winning strategy. Also, in FDE, both players can have winning strategies. We define the verification games for FDE in the standard fashion as follows.

Definition 2. *The tuple $\Gamma_{\text{FDE}} = (\pi, \rho, \sigma, \delta)$ is a FDE verification game where $\pi = \{$Heloise, Abelard$\}$, σ is as in classical logic, δ is $\{T\}$ and ρ is given as follows inductively for a game $\Gamma_{\text{FDE}}(M, \varphi)$.*
— *If φ is atomic, the game terminates, and Heloise wins if $\varphi\mathbf{r}1$, Abelard wins if $\varphi\mathbf{r}0$, neither wins if $\varphi\mathbf{r}\emptyset$,*
— *if $\varphi = \neg\psi$, players switch roles, and the game continues as $\Gamma_{\text{FDE}}(M, \psi)$,*
— *if $\varphi = \chi \wedge \psi$, Abelard and Heloise choose between χ and ψ simultaneously,*
— *if $\varphi = \chi \vee \psi$, Abelard and Heloise choose between χ and ψ simultaneously.*

The above rules determines the turn function for the GTS^{FDE} which suggests that both players make moves at all binary connectives. A simple example can be helpful.

Example 3. Consider the formula $p \wedge (q \vee r)$ where $p\mathbf{r}\{0,1\}$, $q\mathbf{r}\emptyset$ and $r\mathbf{r}0$. Then, this formula evaluates to 0. In the verification game, Abelard first chooses $q \vee r$, and then chooses r. Alternatively, he can also choose p as his winning strategy, yet this also gives Heloise a win. This is also another case where existence of winning strategies do not guarantee the truth value of the formula in question.

The correctness theorem for GTS^{FDE} is given as follows.

Theorem 3. *In a game $\Gamma_{\text{FDE}}(M, \varphi)$, we have the following:*
— *Heloise has a winning strategy if $\varphi\mathbf{r}1$,*
— *Abelard has a winning strategy if $\varphi\mathbf{r}0$,*
— *Either of the players or none of the players has a winning strategy if $\varphi\mathbf{r}\emptyset$.*

The connection between FDE and LP can further be explicated as follows.

Corollary 1. *For an LP model M and a formula φ, let M' be the model obtained from M by maintaining the same carrier set and replacing the valuation function of LP with the valuation relation of FDE as follows: $T \mapsto 1$, $F \mapsto 0$ and $P \mapsto \{0, 1\}$. If Heloise or Abelard has a winning strategy in $\Gamma_{\mathrm{LP}}(M, \varphi)$, then Heloise or Abelard has a winning strategy in $\Gamma_{\mathrm{FDE}}(M', \varphi)$ respectively. If only Astrolabe has a winning strategy in $\Gamma_{\mathrm{LP}}(M, \varphi)$, then both Heloise and Abelard have winning strategies in $\Gamma_{\mathrm{FDE}}(M', \varphi)$.*

The converse of Corollary 1 is not true. In $\mathrm{GTS}^{\mathrm{FDE}}$, for a game $T \wedge F$, both Abelard and Heloise have winning strategies. Yet, in LP for a game $T \wedge F$, Astrolabe does not have a winning strategy.

The lack of biconditional correctness theorem for $\mathrm{GTS}^{\mathrm{FDE}}$ can be seen more clearly once LP is considered as a restricted case of FDE.

4 Game Semantics for A Relevant Logic

Relevant logics define negation differently by resorting to possible worlds modalizing the negation operator. The idea is due to Routley and Routley, and we will focus on their logic [22]. A *Routley model* is a structure $(W, \#, v)$ where W is a set of possible worlds, $\#$ is a map from W to itself, and v is a valuation function defined in the standard way. In this system, the semantics for disjunction and conjunction is local, whereas for negation, possible worlds are needed.

$$
\begin{array}{lll}
v(w, \neg\varphi) = 1 & \text{iff} & v(\#w, \varphi) = 0 \\
v(w, \varphi \wedge \psi) = 1 & \text{iff} & v(w, \varphi) = 1 \text{ and } v(w, \psi) = 1 \\
v(w, \varphi \vee \psi) = 1 & \text{iff} & v(w, \varphi) = 1 \text{ or } v(w, \psi) = 1
\end{array}
$$

We call Routleys' system RR, and denote its GTS as $\mathrm{GTS}^{\mathrm{RR}}$. Notice that if $\#w = w$, then we have the classical truth conditions. Further connections between RR and FDE or LP can be found in [19]. We define semantical games in RR as $\Gamma_{\mathrm{RR}}(M, \varphi, w)$ where M, φ are as before, and $w \in W$ is a possible world.

Definition 3. *The tuple $\Gamma_{\mathrm{RR}} = (\pi, \rho, \sigma, \delta)$ is a RR verification game where $\pi = \{$Heloise, Abelard$\}$, σ is in the form of (φ, w) for $\varphi \in \mathcal{L}$ and $w \in W$, δ is $\{T\}$ and ρ is given as follows inductively for a game $\Gamma_{\mathrm{RR}}(M, \varphi, w)$ where w is a possible world.*
- *If φ is atomic, the game terminates, and Heloise wins if φ is true, Abelard wins if φ is false,*
- *if $\varphi = \neg\psi$, the players switch roles, and the game continues as $\Gamma_{\mathrm{RR}}(M, \psi, \#w)$,*
- *if $\varphi = \chi \wedge \psi$, Abelard chooses between χ and ψ,*
- *if $\varphi = \chi \vee \psi$, Heloise chooses between χ and ψ.*

The correctness theorem is given as follows.

Theorem 4. *In a game $\Gamma_{\mathrm{RR}}(M, \varphi, w)$, Heloise has a winning strategy if φ is true, and Abelard has a winning strategy if φ is false.*

The converse of Theorem 4 is not correct as the # operator can create inconsistencies. In order to see this, let $w \models \neg\varphi$ and $w' \models \varphi$. If $\#(w) = w'$, then by definition φ is both true and false at w' satisfying an inconsistency.

5 Translating Games

It is possible to give a translation between three-valued logics and modal logic S5 [10]. Modal logic S5 is defined as a system (W, R, V) where W is a non-empty set, R is an equivalence relation on $W \times W$ and V is the valuation.

Now, we give a translation of LP (and K3) into S5 via GTS. The translation is built on the following observation: "In an S5-model there are three mutually exclusive and jointly exhaustive possibilities for each atomic formula p: either p is true in all possible worlds, or p is true in some possible worlds and false in others, or p is false in all possible worlds" [10].

Given the propositional language $\mathcal{L}$, we extend it with the modal symbols $\Box$ and $\Diamond$ and close it under the standard rules to obtain the modal language $\mathcal{L}_M$. GTS for modal logic is well-known. "Diamond" formulas are assigned to Heloise whereas the "Box" formulas are assigned to Abelard. Also, similar to the RR, formulas in $\mathcal{L}_M$ are associated with a possible world, and when a move is made from a modal formula, the next possible world is determined by R.

The translations $\mathsf{Tr}_{LP} : \mathcal{L} \mapsto \mathcal{L}_M$ and $\mathsf{Tr}_{K3} : \mathcal{L} \mapsto \mathcal{L}_M$ for LP and K3 respectively are given as follows where p is a propositional variable [10].

$\mathsf{Tr}_{LP}(p) = \Diamond p$	$\mathsf{Tr}_{LP}(\varphi \wedge \psi) = \mathsf{Tr}_{LP}(\varphi) \wedge \mathsf{Tr}_{LP}(\psi)$
$\mathsf{Tr}_{K3}(p) = \Box p$	$\mathsf{Tr}_{K3}(\varphi \wedge \psi) = \mathsf{Tr}_{K3}(\varphi) \wedge \mathsf{Tr}_{K3}(\psi)$
$\mathsf{Tr}_{LP}(\neg\varphi) = \neg\mathsf{Tr}_{K3}(\varphi)$	$\mathsf{Tr}_{LP}(\varphi \vee \psi) = \mathsf{Tr}_{LP}(\varphi) \vee \mathsf{Tr}_{LP}(\psi)$
$\mathsf{Tr}_{K3}(\neg\varphi) = \neg\mathsf{Tr}_{LP}(\varphi)$	$\mathsf{Tr}_{K3}(\varphi \vee \psi) = \mathsf{Tr}_{K3}(\varphi) \vee \mathsf{Tr}_{K3}(\psi)$

The translation is a co-induction, and it generates fully modalized formulas. As the authors underlined, for fully modalized formulas in S5, a formula is true somewhere in an S5 model if and only if it is true everywhere in the model. This fact is due to the frame properties of S5 [10].

Given $\Gamma_{\mathrm{LP}} = (\pi, \rho, \sigma, \delta)$, we define $\Gamma_{\mathrm{S5}} = (\pi', \rho', \sigma', \delta')$ as follows: $\pi' = \{\text{Heloise}, \text{Abelard}\}$, ρ and σ' are the rules and positions of verifications games of S5, and $\delta' = \{1\}$. The correctness of the translation for LP is as follows.

Theorem 5. *Let $\Gamma_{\mathrm{LP}}(M, \varphi)$ be given. Then,*
– *if Heloise has a winning strategy in $\Gamma_{\mathrm{LP}}(M, \varphi)$, then she has a winning strategy in $\Gamma_{\mathrm{S5}}(M, \mathsf{Tr}_{LP}(\varphi))$,*
– *if Abelard has a winning strategy in $\Gamma_{\mathrm{LP}}(M, \varphi)$, then he has a winning strategy in $\Gamma_{\mathrm{S5}}(M, \mathsf{Tr}_{LP}(\varphi))$,*
– *if only Astrolabe has a winning strategy in $\Gamma_{\mathrm{LP}}(M, \varphi)$, then both Abelard and Heloise have winning strategies in $\Gamma_{\mathrm{S5}}(M, \mathsf{Tr}_{LP}(\varphi))$.*

For an LP valuation v, and a model M of S5, v and M are said to be Tr_{LP}-*equivalent* if for all $\varphi \in \mathcal{L}$ we have (i) $1 \in v^*(\varphi) \Leftrightarrow M \models_{S5} \mathsf{Tr}_{LP}(\varphi)$, and (ii) $0 \in v^*(\varphi) \Leftrightarrow M \not\models_{S5} \mathsf{Tr}_{K3}(\varphi)$, where v^* is the (truth table) function based on v

that maps *formulas* to truth values of LP. Based on various results in [10], we now prove the following, the converse of Theorem 5.

Theorem 6. *Let M be an S5 model, $\varphi \in \mathcal{L}$ with an associated verification game $\Gamma_{\mathrm{S5}}(M, \varphi)$. Then, there exists an LP model M' and a game $\Gamma_{\mathrm{LP}}(M', \varphi)$ where,*
— *if Heloise has a winning strategy for $\Gamma_{\mathrm{S5}}(M, \varphi)$ at each point in M, then Heloise has a winning strategy in $\Gamma_{\mathrm{LP}}(M', \varphi)$,*
— *if Abelard has a winning strategy for $\Gamma_{\mathrm{S5}}(M, \varphi)$ at each point in M, then Abelard has a winning strategy in $\Gamma_{\mathrm{LP}}(M', \varphi)$,*
— *if Heloise or Abelard has a winning strategy for $\Gamma_{\mathrm{S5}}(M, \varphi)$ at some points but not all in M, then Astrolabe has a winning strategy in $\Gamma_{\mathrm{LP}}(M', \varphi)$.*

For an application of Theorem 5, consider the formula $p \vee q$ where p and q have the truth values P, F respectively in LP. Then, $\mathsf{Tr}(p \vee q) = \Diamond p \vee \Diamond q$ where p, q have the truth values $\{T, F\}, \{F\}$ respectively in S5. Based on Theorem 5, we expect both players to have winning strategies. First, Heloise has a winning strategy in this game if she chooses p. Also, notice that all possible moves of Heloise brings Abelard a win without him even not making any moves, due to the truth values of p, q. Thus, both players have winning strategies in this game.

6 Conclusion

Giving a full picture of GTS for all paraconsistent logics goes beyond the limits of this article. Some well-studied logics such as da Costa's C-systems and LFIs (*Brazilian School*), 4-valued Belnap logic, the modal extensions of the logics we presented, and the *preservationist* approach (*Canadian School*) are the natural next steps of this project [4,5,2,24].

The current work can be seen as a case for logical pluralism. The classical GTS is essentially a very narrow and limited case with many additional and auxiliary game theoretical and logical presuppositions. Once those assumptions are set aside (or at least questioned) for various reasons, GTS turns out to be expressive enough for a variety of non-classical logics as we have exemplified.

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Appendix: Proofs

Proof (Proof of Theorem 1). We start with the case for Heloise. We proceed by induction on φ. Let φ be true in M.

If φ is a propositional letter p which is true in M, then Heloise wins the game by definition, hence has a winning strategy.

Let $\varphi = \neg\psi$. Then, ψ is false. By the game rules, now the game continues where Heloise is the falsifier. By the induction hypothesis (for falsifier), Heloise the falsifier has a winning strategy for ψ. Then, she has a winning strategy as the verifier for φ.

Now, let φ be a conjunction of the form $\chi \wedge \psi$. Since, φ is assumed to be true, the only way to make it true is to have χ and ψ both true. Then, by the induction hypothesis, Heloise has a winning strategy for both χ and ψ. Then, for φ, Abelard and Astrolabe make moves. Yet, whichever move they make (whichever of χ or ψ they choose), Heloise will have a winning strategy. Thus, for φ, she has a winning strategy: whatever move Abelard and Astrolabe make, she has a win.

Let φ be a disjunction of the form $\chi \vee \psi$. Then, by the induction hypothesis, Heloise has a winning strategy for either χ or ψ whichever is true. Then, choosing the true disjunct is her winning strategy at φ, independent from whatever Astrolabe chooses.

The case for Abelard is almost identical to that of Heloise's, hence skipped.

For Astrolabe, we first assume that the given formula φ is paradoxical in M. If φ is a propositional letter p which is paradoxical in M, then Astrolabe has a winning strategy by definition. Similarly, if $\varphi = \neg\psi$, then, ψ is paradoxical, too. By the game rules, Astrolabe's rule remains the same. By the induction hypothesis, he has a winning strategy for ψ, and thus for for φ by simply maintaining the same role and the strategy, and proceeding with ψ.

For $\varphi = \chi \wedge \psi$. Since φ is assumed to be paradoxical, we only have two options for χ and ψ: (1) either one of them has the truth value P and the other has the truth value T, (2) both have the truth value P. Therefore, Astrolabe has winning strategy for at least one of χ and ψ, by the induction hypothesis. Then, for φ, Astrolabe chooses the conjunct that has the truth value P for which he has a winning strategy already. This forms his winning strategy for φ, independent from whatever move Abelard makes.

If $\varphi = \chi \vee \psi$, then we have two options as well: (1) one of the disjuncts has the truth value P and the other one has the truth value F, (2) both have the truth value P. By a similar argument Astrolabe has a winning strategy for either case.

Proof (Proof of Theorem 2). The proof is by induction on φ for each player, and the cases for Heloise and Abelard are very similar to the classical case. Now, assume that for φ, only Astrolabe has a winning strategy. The cases for propositional variables and negation are as above, hence skipped.

Now, let $\varphi = \chi \wedge \psi$. If only Astrolabe has a winning strategy, this means, Astrolabe has a winning strategy for either of the conjuncts (as he can choose whichever he likes), say χ without loss of generality. Then, by the induction hypothesis, χ is paradoxical. Since Abelard does not have a winning strategy, by Theorem 1, then neither of the conjuncts is false. Thus, by the truth table φ is forced to be paradoxical as χ is paradoxical. Otherwise, if Abelard had a winning strategy, and if one of the conjuncts was F, then $P \wedge F$ would return F, not P disproving the claim. This is the reason why only Astrolabe is supposed to have a winning strategy.

The case for disjunction for Astrolabe is very similar.

Proof (Proof of Theorem 3). We start with the case for Heloise. Suppose $\varphi \mathbf{r} 1$. The cases for propositional variables and negation are immediate. Let $\varphi = \chi \wedge \psi$.

If $\varphi\mathbf{r}1$, then we have both $\chi\mathbf{r}1$ and $\psi\mathbf{r}1$. By the induction hypothesis, Heloise has winning strategies for both χ and ψ. Thus, she has a winning strategy for φ. For the failure of the reverse direction, assume that Heloise has a winning strategy, that is, to choose χ (without loss of generality). Assume further that, Abelard has a winning strategy as well, that is, to choose ψ. Then, by the indiction hypothesis $\chi\mathbf{r}1$ and $\psi\mathbf{r}0$ which forces $\varphi\mathbf{r}0$. Heloise's case for disjunction is very similar.

The interesting case is for $\emptyset$. Now, assume $\varphi\mathbf{r}\emptyset$. If φ is a propositional variable, by definition, no player wins. If $\varphi = \neg\psi$, then $\psi\mathbf{r}\emptyset$, and by the induction hypothesis, no player has a winning strategy.

Let $\varphi = \chi \wedge \psi$. Then, we have two options: (1) both $\chi\mathbf{r}\emptyset$ and $\psi\mathbf{r}\emptyset$, or (2) $\chi\mathbf{r}1$ and $\psi\mathbf{r}\emptyset$ (without loss of generality). If the prior one is the case, by the induction hypothesis, no player has a winning strategy for χ or ψ. Thus, no player has a winning strategy for φ. If the latter is the case, then Heloise can have a winning strategy for φ as she can make a move at a conjunction which forms her winning strategy for φ. Dually, if $\varphi = \chi \vee \psi$, then, we have two options: (1) both $\chi\mathbf{r}\emptyset$ and $\psi\mathbf{r}\emptyset$, or (2) $\chi\mathbf{r}0$ and $\psi\mathbf{r}\emptyset$ (without loss of generality). If the prior one is the case, by the same argument as above, no player has a winning strategy for φ. If the latter is the case, as Abelard can make a move at a disjunction and choose χ, then he can have a winning strategy for φ.

Proof (Proof of Corrolary 1). The first part about Heloise and Abelard follows from Theorem 2 and Theorem 3. In other words, if Heloise has a winning strategy in an LP game, then the formula is true in LP by Theorem 2. The translation then translates T of LP to 1 of FDE. Then, by Theorem 3, Heloise has a winning strategy in the FDE game. The argument is similar for Abelard.

If only Astrolabe has a winning strategy for the LP game for φ, then by Theorem 2, φ is paradoxical. By the translation, then φ is related to both 0 and 1 in FDE. By Theorem 3, then both Heloise and Abelard has winning strategies in the FDE game.

Proof (Proof of Theorem 4). The proof is by induction on φ. Let us see the case for Heloise at w. The case for Abelard is very similar hence will be skipped.

If φ is a propositional letter p. Then, if p is true then, by definition, Heloise has a winning strategy.

Let $\varphi = \neg\psi$. Then the game continues at # for ψ with switched roles, where $v(\#w, \psi) = 0$. Thus Heloise becomes falsifier. Then, by the induction hypothesis (for Abelard), the falsifier has a winning strategy for the game at $\#w$ for ψ. Thus, Heloise has a winning strategy at w for $\neg\psi$ which forms her winning strategy for φ. The cases for conjunction and disjunction are as expected thus omitted.

Proof (Proof of Theorem 5). The theorem is given for LP and S5. Yet, a similar theorem for K3 and S5 can also be given. We will assume the correctness of such a theorem for this proof as the translation co-depends on both LP and K3.

Assume that Heloise has a winning strategy for φ in LP. Let us proceed by induction on φ. If φ is a propositional letter p, then p is true in LP. Then, it translates to S5 as $\Diamond p$, which is a turn for Heloise. Then, the game in S5 starts by Heloise with $\Diamond p$, and she makes a move to p for which she has a winning strategy.

For $\varphi = \neg\psi$, suppose Heloise has a winning strategy for $\neg\psi$ in LP. By the translation, she has a winning strategy for $\neg\mathsf{Tr}_{K3}(\psi)$ in S5. So, by the assumed similar theorem for K3 and S5, Abelard has a winning strategy in S5 for $\mathsf{Tr}_{K3}(\psi)$. Then, in S5 Heloise has a winning strategy for $\neg\mathsf{Tr}_{K3}(\psi)$ which is $\mathsf{Tr}_{LP}(\neg\psi)$. Thus, Heloise has a winning strategy for $\mathsf{Tr}_{LP}(\varphi)$ in S5.

The cases for conjunction and disjunction are immediate. Also the case for Abelard is very similar, hence skipped. The case for Astrolabe is interesting.

Assume that only Astrolabe has a winning strategy for φ in LP. As the first step of the induction, assume $\varphi = p$ for a propositional variable p. So, p is paradoxical. The translation of p into S5 is $\Diamond p$. Also, notice that for paradoxical p, we have $\neg p \equiv p$. The translation of $\neg p$ into S5 is $\Diamond\neg p$. Thus, for a paradoxical p, both players have a winning strategy in the game in S5.

Now, let $\varphi = \neg\psi$. Suppose that only Astrolabe has a winning strategy. By the game rules of GTS^{LP}, Astrolabe has a winning strategy for ψ as well as the negation of a paradoxical formula is also paradoxical. Now, we will use the co-inductive part of the argument. By the induction hypothesis for the same result for K3, Abelard and Heloise have winning strategies in the translated game in S5 for $\mathsf{Tr}_{K3}(\psi)$. Taking one step back, with their roles switched, both Abelard and Heloise have winning strategies in a game for $\neg\mathsf{Tr}_{K3}(\psi)$, too. Then, by the translation, they have winning strategies for $\mathsf{Tr}_{LP}(\neg\psi)$, which is $\mathsf{Tr}_{LP}(\varphi)$ in S5. A symmetric argument for the K3-S5 is straight forward.

The cases for the binary connectives are straight forward, hence skipped.

Proof (Proof of Theorem 6). In [10], while constructing the LP model based on a given S5 model, the authors associate the propositions that are true *everywhere* with the LP truth value T, the propositions that are true *nowhere* with F, and the propositions that are true *somewhere* with P. They also show that the given S5 model and the LP model obtained in this fashion are Tr_{LP}-equivalent [10].

Based on these observation, then, if Heloise has a winning strategy for $\Gamma_{\text{S5}}(M, \varphi)$ at all points in M, then φ has a truth value T in LP. By Theorem 1, Heloise has a winning strategy in $\Gamma_{\text{LP}}(M', \varphi)$. Similarly, if Abelard has a winning strategy for $\Gamma_{\text{S5}}(M, \varphi)$ at all points in M, then φ has a truth value F in LP. Again, by Theorem 1, Abelard has a winning strategy in $\Gamma_{\text{LP}}(M', \varphi)$. Finally, if Astrolabe has a winning strategy for $\Gamma_{\text{S5}}(M, \varphi)$ at some points in M, then φ has a truth value P in LP. By Theorem 1, Astrolabe has a winning strategy in $\Gamma_{\text{LP}}(M', \varphi)$.