Correction: *Some non-classical approaches to the Brandenburger - Keisler Paradox*

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In my paper (Başkent, 2015), there appears to be a problem. The problem regards the algebraic and category theoretical properties of co-Heyting (Brouwerian) algebras.

Proposition 3.5 must read as follows.

Proposition 3.5 Co-Heyting algebras are co-Cartesian closed categories.

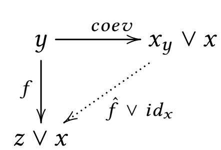
It is well-known that Heyting algebras are cartesian closed (Awodey, 2006). Therefore, their dual are co-cartesian closed: they have initial elements, co-products and co-exponentials.

In the case of co-Heyting algebras, the initial element is $\mathbf{0}$ by containment. For every x, we have $\mathbf{0} \subseteq x$.

The co-product in a co-Heyting algebra is the join $\vee$ with respect to containment. Because, by definition, for all x, y, $x \vee y$ is the unique element such that $x \leq x \vee y$ and $y \leq x \vee y$. And, for all z with $x \leq z$ and $y \leq z$, we have $x \vee y \leq z$. This shows that $\vee$ is a co-product, by definition.

The co-exponential object x_y is a bit tricky. We define x_y as $\neg x \wedge y$. It's easy to see that the dual of this is $\neg x \vee y$ and it is exponential object in Heyting algebra. Nevertheless, let us show that $\neg x \wedge y$ is indeed the co-exponential object in co-Heyting algebras.

For two objects x, y, their co-exponential is an object x_y together with a co-evaluation map $coev : y \mapsto x_y \vee x$ such that for any object z and a map $f : y \mapsto z \vee x$, there is a unique map $\hat{f} : x_y \mapsto z$ such that the following diagram commutes:



Now, we can claim that co-exponential x_y is $\neg x \wedge y$. First of all, $coev$ arrow is $y \leq (\neg x \wedge y) \vee x$. This always holds, because

$$y \leq (\neg x \wedge y) \vee x = (\neg x \vee x) \wedge (y \vee x) = 1 \wedge (y \vee x) = y \vee x.$$

We also need to show that $y \leq z \vee x$ iff $\neg x \wedge y \leq z$, reading off from the commutativity diagram above.

From left-to-right direction, let $z \vee x \leq y$. Then,

$$\neg x \wedge y \leq \neg x \wedge (z \vee x) = (\neg x \wedge z) \vee (\neg x \wedge x) = \neg x \wedge z \leq z,$$

which produces $\neg x \wedge y \leq z$.

From right-to-left direction, suppose $\neg x \wedge y \leq z$. Then, reading from right to left,

$$y \leq y \vee x = (\neg x \vee x) \wedge (y \vee x) = (\neg x \wedge y) \vee x \leq z \vee x,$$

which produces $y \leq z \vee x$. Therefore, $\neg x \wedge y$ is the co-exponential object x_y.

This completes the proof.

Similarly, the closed sets in a topology is an example of a co-CCC, *not* a CCC. In that case, argumentation in the paper is correct: the co-product (*not* the product) is the union of closed sets C_1 and C_2, and the co-exponent (*not* the exponent) is $\mathsf{Clo}(C_1^c \cap C_2)$.

Even if these corrections change the reasoning for Theorem 3.7, they don't affect the main result: Co-Heyting algebras still admit fixed-points due to Lawvere's Theorem because they have products (that is $\wedge$) and co-products (that is $\vee$).

Theorem 3.7 Co-Heyting algebras admit fixed-points. Therefore, there exists a co-Heyting algebraic model with a satisfiable BK sentence.

First, as demonstrated in (Abramsky & Zvesper, 2015, Proposition 4.3), Lawvere theorem works in any category with finite products, which includes co-Heyting algebras. Moreover, since aforementioned paper also establishes that the BK argument reduces to Lawvere's theorem, our Theorem 3.7 follows immediately.

More precisely, in co-Heyting algebras, the valuation at the boundary ∂ produces a fixed-point regardless of the truth value.

We define $\partial(p) = p \wedge \sim p$ where $\sim$ is the co-Heyting (paraconsistent) negation. The operator $\sim$ is unary, thus has to admit fixed-points. Thus, for all $x \in \partial(p)$, we have $x = \sim x$. Particularly, for all $x \in \partial(p)$, we have $\partial x = x$.

For the BK paradox, simply take p as the BK sentence.

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References

Abramsky, Samson, & Zvesper, Jonathan. 2015. From Lawvere to Brandenburger-Keisler: Interactive forms of diagonalization and self-reference. *Journal of Computer and System Sciences*, **81**(5), 799–812.

Awodey, Steve. 2006. *Category Theory*. Oxford University Press.

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