Epistemic Game Theoretical Reasoning in History Based Models

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In this work, we use Parikh and Ramanujam's history based temporal-epistemic models to reason about various epistemic game theoretical issues. First, we introduce a modal operator to express subjective preferences to history based models, and present an analysis of the Prisoners' Dilemma in this framework. Finally, we extend Aumann's celebrated agree-to-disagree result to history based models.

... you act, and you know why you act, but you don't know why you know that you know what you do.

The Name of the Rose, Umberto Eco

1 Introduction

1.1 Motivation

History based structures, proposed by Parikh and Ramanujam [16], suggest a formal framework which lies between process models, interpreted systems and propositional dynamic logics. They have been used to model epistemic messages and communication between agents, deontic obligations and the relation between obligations and knowledge [16, 14, 15]. Moreover, history based models are technically similar to interpreted systems [7, 14]. Epistemic and temporal reasoning in history based models depend on a sequence of events, called *history*.

In this work, we consider history based structures from a game theoretical point of view with some applications. In order to achieve this, we first make history-based models more *game-theory friendly* by introducing a preference modality. Then, we apply our extended formalism to a fundamental game, which is the Prisoners' Dilemma, and show how history based models can be helpful to compute the equilibrium. The choice of prisoners' dilemma is not arbitrary. Because in this game, the epistemology of the agents play a central role and the way their knowledge is formalized bear some similarities to some other formalisms of epistemic games. Building on this observation, we use history based game models to present an iteration of Aumann's well-known "agree-to-disagree" theorem.

The overall goal of this research agenda is to introduce more expressive formalism for the analysis of various foundational game theoretical issues. These issues include security games, epistemic games and how they depend on the *history* of the game and how we can *read off* strategies from such a model. We achieve this by discussing these topics in a model where histories are taken as the basic elements of the model and by introducing a modal preference relation.

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1.2 Basic Logical Structure

Different from Kripke models, history based models are constructed by using a given set of events and agents. Events can be seen as actions or moves which vary over time and affect the knowledge of the agents. In such a model, agents' epistemic capacities differ from local and global perspectives. When a history is considered as a sequence of events, it is important to tell apart which events were carried out by which agents, and which agents can see which events, and how all this affects the knowledge and preferences of the agents.

Similar attempts have been made to apply history based models to deontic and epistemic issues [15, 14]. However, in that body of work, game theoretical reasoning was never clear or of prime importance which left many interesting phenomena outside its boundaries. In this preliminary work, we take the first step to formalize epistemic games with their histories and start from history based models. For this aim of ours, we first introduce preferences. Let us proceed step by step in our formalism.

History based structures are constructed by using a fix set of events $\mathbf{E}$ and agents $\mathbf{A}$. A finite set of events is denoted as $\mathbf{E}^*$, and for each agent i, $\mathbf{E}_i \subseteq \mathbf{E}$ is the set of events which are "seen" or "accessible" by the agent i. A finite sequence of events from $\mathbf{E}$ is denoted by lowercase h, whereas a possibly infinite sequence of events is denoted by uppercase H. We call them both *histories*.

We denote the concatenation of finite history h with (possibly infinite) history H by hH. For a set of events $\mathbf{E}$, $\mathscr{H}^*_{\mathbf{E}}$ denotes the set of all finite histories with events from $\mathbf{E}$ and $\mathscr{H}_{\mathbf{E}}$ denotes the set of all histories, finite and infinite, with events from $\mathbf{E}$. By $\mathscr{H}$, we denote any set of histories. Given two histories H, H', $H \leq H'$ denotes that H is a prefix of H'. We denote the length of finite h with $\mathsf{len}(h)$. For a history H, H_t denotes that $H_t \leq H$ with $\mathsf{len}(H_t) = t$.

We define *global history* as a sequence of events, finite or infinite, where a *local history* is the history of a particular agent. For any set of histories $\mathscr{H}$, the set $\mathsf{FinPre}(\mathscr{H})$ denotes the set of finite prefixes of the histories in $\mathscr{H}$. A set of histories $\mathscr{H}$ is called a *protocol* if it is closed, under set inclusion, for all prefixes. In other words, in order for a history to make sense, its prefixes should be included in the model, and there should be no jumps.

Now we can discuss temporal and epistemic operators in this framework. Given an agent i and a global history H, the agent i can only access some of H. For two histories H, H', if the agent can access to the same parts of H and H', then H and H' are indistinguishable for i. Then, a function $\lambda_i : \mathsf{FinPre}(H) \to \mathbf{E}^*_i$ is called a *locality function* for agent i and a global history H. Based on locality functions, the epistemic indistinguishability $\sim_i$ for agent i is defined between two histories H, H' as follows: If $H \sim_i H'$, then $\lambda_i(H) = \lambda_i(H')$.

The locality function as given above is rather general. For that reason, we impose some conditions on it [14]. First, we assume that agents' clock is consistent with the global clock, that is all agents share the same clock. Second, $\lambda_i(H)$ is embeddable in H, that is the events in $\lambda_i(H)$ appear in H in the same order. In other words, "agents are not wrong on about the events that they witness" [ibid].

For obvious reasons, $\sim_i$ is an equivalence relation. Thus, the epistemic logic of history based structures is the standard multi-agent epistemic logic $\mathbf{S5}_n$.

Given a set $\mathbf{P}$ of propositional letters, the syntax of history based models can be given as follows in the Backus - Naur form where $p \in \mathbf{P}$ and $i \in \mathbf{A}$.

$$\varphi := \mathrm{p} \mid \neg\varphi \mid \varphi \wedge \varphi \mid K_i\varphi \mid \bigcirc\varphi \mid \varphi U \varphi$$

The epistemic modality for agent i is K_i and the operator $\bigcirc$ is the next-time modality. We call U the *until operator*.

A history based model M is given as a tuple $M = \{\mathscr{H}, \mathbf{E}_1, \ldots, \mathbf{E}_n, \lambda_1, \ldots, \lambda_n, \mathscr{V}\}$ where $\mathscr{V}$ is a valuation function which is defined in the standard fashion as follows: $V : \mathsf{FinPre}(\mathscr{H}) \mapsto \wp(\mathbf{P})$.

History based models semantically evaluates formulas at history-time pairs. At history H and time t, the satisfaction of a formula φ is denoted as $H,t \models \varphi$, and defined inductively as follows.

$$
\begin{array}{lll}
H,t \models_M p & \text{iff} & H_t \in V(p), \\
H,t \models_M \neg\varphi & \text{iff} & H,t \not\models_M \varphi, \\
H,t \models_M \varphi \wedge \psi & \text{iff} & H,t \models_M \varphi \text{ and } H,t \models_M \psi, \\
H,t \models_M \bigcirc\varphi & \text{iff} & H,t+1 \models_M \varphi, \\
H,t \models_M K_i\varphi & \text{iff} & \forall H' \in \mathscr{H} \text{ and } H_t \sim_i H'_t \text{ implies } H',t \models_M \varphi, \\
H,t \models_M \varphi U \psi & \text{iff} & \exists k \geq t \text{ such that } H,k \models_M \psi \text{ and } \forall l, t \leq l < k \text{ implies } H,l \models_M \varphi.
\end{array}
$$

The dual of the epistemic modality will be denoted with L_i and defined in the usual way. The expression $M \models \varphi$ denotes the truth of φ in a history based model M, independent from the current history and time-stamp.

The axioms for history based models are given as follows.

- All tautologies of propositional logic,
- $K_i(\varphi \to \psi) \to (K_i\varphi \to K_i\psi)$,
- $K_i\varphi \to \varphi \wedge K_iK_i\varphi$,
- $\neg K_i\varphi \to K_i\neg K_i\varphi$,
- $\bigcirc(\varphi \to \psi) \to (\bigcirc\varphi \to \bigcirc\psi)$,
- $\bigcirc\neg\varphi \leftrightarrow \neg\bigcirc\varphi$,
- $\varphi U \psi \leftrightarrow \psi \vee (\varphi \wedge \bigcirc(\varphi U \psi))$.

The rules of inference are modus ponens, and normalization for all three modalities:

- $\models \varphi, \varphi \to \psi \therefore \models \psi$,
- $\models \varphi \therefore \models K_i\varphi$,
- $\models \varphi \therefore \models \bigcirc\varphi$,
- $\models \varphi \to (\neg\psi \wedge \bigcirc\varphi) \therefore \models \varphi \to \neg(\varphi' U \psi)$.

Additional axioms can be introduced to history based models to formalize variety of properties including perfect recall and no learning [14]. It is also important to note that the above axiomatization does not include any axioms that govern a possible interaction between the epistemic and temporal modalities. The reason for this is the fact that the former quantifies over histories (up to a fixed t) whereas the latter ranges over the time stamp only. However, as we argued earlier, further temporal and epistemic conditions can be forced by introducing various interaction axioms.

History based models combine epistemic and temporal modalities in a complex way and they are closely related to *runs* [7]. Furthermore, histories and runs can be translated to each other effectively [14]. However, it still remains an unexplored direction to use history based models for game theoretical purposes. We will illustrate it in due time.

Now, from a modal logical point of view, the immediate question is how bisimulations can be defined within the context of history based models where we focus on events/actions as opposed to possible worlds/states and possess complex temporal modalities such as the until modality.

Definition 1.1. For history based models M, M', a bisimulation $\bowtie$ between M and M' is a tuple $\bowtie = (\bowtie_0, \bowtie_1)$ where $\bowtie_0 \subseteq M \times M'$ and $\bowtie_1 \subseteq M^2 \times M'^2$ such that
Propositional base case:

- If $H,t \bowtie_0 H',t'$, then H,t and H',t' satisfy the same propositional variable,

Temporal forth case:

- If $H,t \bowtie_0 H',t'$ and $t < u$, then there is u' in M' such that $t' < u'$, $H,u \bowtie_0 H',u'$ and $(H,t),(H,u) \bowtie_1 (H',t'),(H',u')$,
- If $(H,t),(H,u) \bowtie_1 (H',t'),(H',u')$ and if there is v' with $t' < v' < u'$, then there exists v such that $t < v < u$ and $H,v \bowtie_0 H',v'$,

Temporal back case:

- If $H,t \bowtie_0 H',t'$ and $t' < u'$, then there is u in M such that $t < u$, $H,u \bowtie_0 H',u'$ and $(H,t),(H,u) \bowtie_1 (H',t'),(H',u')$,
- If $(H,t),(H,u) \bowtie_1 (H',t'),(H',u')$ and if there is v with $t < v < u$, then there exists v' such that $t' < v' < u'$ and $H,v \bowtie_0 H',v'$,

Epistemic forth case:

- If $H,t \bowtie_0 H',t'$ and $H_t \sim_i K_l$, then there is K',l' in M' such that $K,l \bowtie_0 K',l'$ and $H'_{t'} \sim_i K'_{l'}$,

Epistemic back case:

- If $H,t \bowtie_0 H',t'$ and $H'_{t'} \sim_i K_{l'}$, then there is K,l in M such that $K,l \bowtie_0 K',l'$ and $H_t \sim_i K_l$,

In the above definition, the *interval bisimulations* we defined in the temporal cases are needed for the until modality, as the until modality is essentially an interval process equivalence. This definition clarifies how history based models can simulate state-based models or interpreted systems, and how different histories can be identified to form bisumulations. Based on this definition, the following theorem follows immediately.

Theorem 1.2. *For history based models M,M', if $M \bowtie M'$, then they satisfy the same formula.*

Proof. For the epistemic case see [3], for the temporal case see [11]. □

2 Adding Preferences

History based models provide sufficient tools to formalize simple epistemic games. If games are considered as formal representations of interactive situations in which agents make rational decisions, such decisions then must rely on those agents' subjective preferences. Moreover, these subjective preferences may change depending on what stage of the game the players are in and how far ahead in the game they have progressed. In short, preferences depend on the game history. This is the motivation behind introducing subjective preferences into history based models.

For an agent i, and possibly infinite histories H,H', the expression $H \preceq_i H'$ denotes that "the agent i (weakly) prefers H' to H". The preference relation will be taken as a pre-order satisfying reflexivity and transitivity [2, 10].

We can amend the syntax of the logic of history based models with the modal operator $\Diamond_i \varphi$ which expresses that there is a history which is at least as good as the current one and satisfies φ for agent i. We specify the semantics of this new modality as follows.

$$H,t \models \Diamond_i \varphi \text{ iff } \exists H'.H \preceq_i H' \text{ and } H',t \models \varphi$$

The dual of the above modality is denoted by $\Box_i$ with the following semantics: $H,t \models \Box_i \varphi$ whenever $\forall H'.H \preceq_i H' \rightarrow H',t \models \varphi$.

Notice that this formalism compares histories as opposed to propositions. For a history based model M, the formula $M \models \varphi \rightarrow \Diamond_i \psi$ denotes that the agent i prefers ψ to φ. In other words, each φ has an alternative history which is at least as good as the current one and satisfies ψ.

The additional axioms and rules of inference for the **S4** preference modality can be given as follows.

- $\varphi \to \Diamond_i \varphi$,
- $\Diamond_i \Diamond_i \varphi \to \Diamond_i \varphi$,

The additional rule of inference for the preference modality is the expected one.

- $\models \varphi \therefore \models \Box_i \varphi$

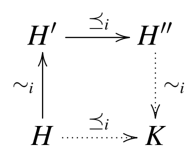
We call the logic of history based structures with preferences as HBPL after *history based preference logic*. HBPL can be supplemented with various additional axioms to express some other interactive epistemic, temporal and game theoretical properties. Here we consider a few.

Connectedness of Preferences The connectedness property for the preference relation suggests that any two histories are comparable. Therefore, it can be formalized as $\forall H, H'. H \preceq_i H' \vee H' \preceq_i H$. The modal axiom that corresponds to it is the following axiom: $\Box_i(\Box_i \varphi \to \psi) \vee \Box_i(\Box_i \psi \to \varphi)$. This renders the frame with preference modality as a total pre-order.

Epistemic Perfect Recall The agents with perfect recall retain knowledge once they acquired it. The standard axiom for this property is given as follows: $K_i \bigcirc \varphi \to \bigcirc K_i \varphi$. It is rather easy to show that this axiom is valid in HBPL. Given an arbitrary history H and a time-stamp t, we start with assuming $H, t \models K_i \bigcirc \varphi$. Our aim is to show that $\bigcirc K_i \varphi$ holds at H, t. Now, by definition, $\forall H'.(H \sim_i H' \to H', t \models \bigcirc \varphi)$. Unfolding the temporal modality gives $\forall H'.(H \sim_i H' \to H', t+1 \models \varphi)$. Now, we can fold back, but this time starting with the epistemic modality. By definition, we first obtain $H, t+1 \models K_i \varphi$, which produces $H, t \models \bigcirc K_i \varphi$. Thus, $K_i \bigcirc \varphi \to \bigcirc K_i \varphi$ is valid in HBPL.[1]

Preferential Perfectness By *preferential perfectness*, we mean that agents do not change their preferences in time. Consider the scheme $\Box_i \bigcirc \varphi \to \bigcirc \Box_i \varphi$. It is also easy to show that this scheme is valid in HBPL, so we skip it.

Epistemic Rationality By a slight abuse of terminology we will call the axiom scheme $\Diamond_i L_i \varphi \to L_i \Diamond_i \varphi$ as the Church-Rosser axiom. The frames of HBPL which satisfies the *Church-Rosser* Property enjoys the following condition:



If $H \sim_i H'$ and $H' \preceq_i H''$, then there exists a history K such that $H \preceq_i K$ and $H'' \sim_i K$.

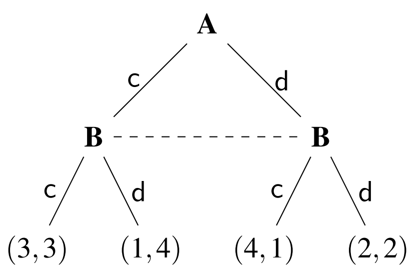
Consider the dual axiom scheme $K_i \Box_i \varphi \to \Box_i K_i \varphi$. This is valid in HBPL. Similar to above, consider $H, t \models K_i \Box_i \varphi$. Then by definition, $\forall H'.(H \sim_i H' \to H', t \models \Box_i \varphi)$. This reduces to $\forall H', H''(H \sim_i H' \wedge H' \preceq_i H'' \to H'', t \models \varphi)$. By the Church-Rosser Property, then there exists a history K such that $H \preceq_i K$ and $H'' \sim_i K$. So, by definition, $K, t \models K_i \varphi$. Thus, $H, t \models \Box_i K_i \varphi$, which shows the validity of the axiom scheme in question.

Various other combinations of the modalities, such as $\Box_i K_i \bigcirc \varphi \to \bigcirc \Box_i K_i \varphi$ or $K_i \Box_i \bigcirc \varphi \to \bigcirc K_i \Box_i \varphi$ remain valid in HBPL. Similarly, various commutativity properties of the modalities, such as $K_i K_j \varphi \leftrightarrow K_j K_i \varphi$, can be examined in order to shed light to epistemic interaction of the agents.

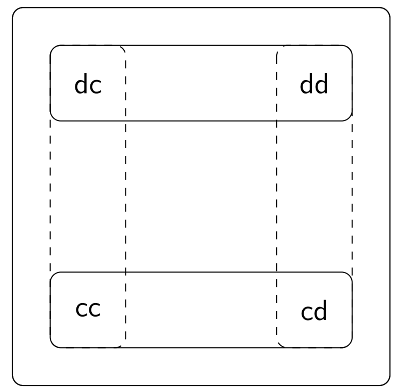
[1] However, as van der Meyden showed, the axiom $K_i \bigcirc \varphi \to \bigcirc K_i \varphi$ is not sufficient to establish the completeness of frames with respect to perfect recall [13, 12]. The additional axiom required for this task is a complicated one:
$K_i \varphi_1 \wedge \bigcirc(K_i \varphi_2 \wedge \neg K_i \varphi_3) \to \neg K_i \neg((K_i \varphi_1) U ((K_i \varphi_2) U \neg \varphi_3))$.

3 Case Study: An Epistemic Analysis of the Prisoner's Dilemma

Viewed as histories with imposed subjectives preferences, HBPL is helpful in formalizing epistemic games. As an application, we consider how HBPL computes best responses in Prisoners' Dilemma (PD, for short).



(a) *Extensive form representation*



(b) *Equivalence classes of histories*

Figure 1: Prisoners' Dilemma

Let us consider PD in its extensive normal form where the utility pair (u_A, u_B) denotes the utility of the players **A** and **B** respectively. Epistemic indistinguishability of the states for player **B** is denoted by the dashed line given in Figure 1a. Based on the extensive normal form, we reproduce the epistemic model of PD below where agents' knowledge is represented by the equivalence classes in the standard way in Figure 1b [2]. In the history xy, the first event denotes Player **A**'s move while the second one denotes Player **B**'s move. Also, due to the utilities associated with the players at the possible end states of the game, we have cc $\preceq_B$ cd and dc $\preceq_B$ dd. Similarly, cc $\preceq_A$ dc and cd $\preceq_A$ dd. The HBPL model for PD can easily be read off from Figures 1a and 1b, hence skipped.

We define best response relation for agent i in a two-player game as follows where $-i$ denotes the players other than i.

$$BR_i = \sim_{-i} \cap \preceq_i$$

By a slight abuse of notation, we will use the same notation to denote the intersection modality. Put informally, in this context, best response for an agent is a move that is *indistinguishable* by the opponent yet more preferable for the agent himself.

Now let us see how we can verify the best responses of the players. Recall that for both players, the best response is defect (the move d). What follows is a direct computation of best responses for each players based on the game history and the subjective preferences of the players. Since PD is a one-shot game, we use a fixed-time stamp t.

We start with Player **A**.

cc, $t \not\models BR_A$	since	there is dc such that dc $\sim_B$ cc and cc $\preceq_A$ dc
cd, $t \not\models BR_A$	since	there is dd such that dd $\sim_B$ cd and cd $\preceq_A$ dd
dc, $t \models BR_A$	since	there is no compatible history with these properties. The only alternative cc fails to bring a higher utility
dd, $t \models BR_A$	since	there is no compatible history with these properties. The only alternative cd fails to bring a higher utility

Similarly for player **B**:

$\mathsf{cc}, t \not\models BR_B$	since	there is cd such that $\mathsf{cd} \sim_A \mathsf{cc}$ and $\mathsf{cc} \preceq_A \mathsf{cd}$
$\mathsf{dc}, t \not\models BR_B$	since	there is dd such that $\mathsf{dd} \sim_A \mathsf{dc}$ and $\mathsf{dc} \preceq_A \mathsf{dd}$
$\mathsf{cd}, t \models BR_B$	since	there is no compatible history with these properties. The only alternative cc fails to bring a higher utility
$\mathsf{dd}, t \models BR_B$	since	there is no compatible history with these properties. The only alternative dc fails to bring a higher utility

Based on the above analysis, Nash equilibrium can be observed at dd which is the state where neither of the agents can unilaterally benefit by diverging from. If **A** diverges, then the history cd is obtained which is not preferable for him. Similarly, if **B** diverges, then the history dc is obtained which is not preferable for him either. Thus, dd is the Nash equilibrium of PD.

It can be noticed that we have not discussed *strategies* in HBPL. Therefore, the Nash equilibrium in HBPL is simply a game history formed if the players follow a particular equilibrium strategy constructed with respect to their best responses. Therefore, the equilibrium is expressed in terms of a game history.

This is a model of prisoner's dilemma in HBPL.

4 Case Study: Set Based Analysis of Histories and Decisions - An Agree-to-Disagree Result

The above analysis of PD considered epistemic states of the game as sequences of moves, or histories. However, there was an additional layer of formalization on top of the histories, which considered the *structure* of histories and their relation to each other in the form of equivalence classes. We can now develop this idea further, and relate it to a well-known and foundational result in epistemic game theory.

Aumann's celebrated agree-to-disagree theorem is a mile-stone in epistemic game theory [1]. Several iteration of the agree-to-disagree result have been given in the literature [4]. In this section, we take one of such variations, which is due to Dov Samet, and apply it to history based models. Samet's model uses a non-probabilistic model together with a set algebra where the knowledge is formalized using a set operator [17]. In that case, Aumann's original statement of the theorem becomes a special case of Samet's generalized formalism.

Our application of HBPL to agree-to-disagree theorem serves two goals. First, it shows the versatility of HBPL by considering sets of histories as equivalence classes. Second, it shows that it is possible to introduce two different levels of complexity to epistemic games. The first level of complexity deals with the game play and constructs a history which includes the moves of all players and the local knowledge of players. The second level of complexity, on the other hand, provides a global view of the model by forming equivalence classes of histories introducing additional structure. In HBPL, unlike Kripkean models, we can read off the epistemics of agents from the histories directly. This is one of the major advantages of using history based models.

However, notice that HBPL evaluates truth at time stamps. The truth of a formula depends both on the history and where we are at the history. Nevertheless, the epistemic modality and the preference modality in HBPL does not quantify over the temporal parameter. For that reasons, in what follows we assume that the time stamp is fixed and the same for all agents, for simplicity.

Let us now start with defining some standard epistemic operators following [17].

Definition 4.1. For a given set of agents **A** and a formula φ, we define $E_{\mathbf{A}}\varphi$ which reads "everyone in **A** knows φ". Formally, $E_{\mathbf{A}}\varphi = \bigwedge_{i \in \mathbf{A}} K_i \varphi$.

We define the common knowledge operator $C_{\mathbf{A}}\varphi$ which reads "φ is common knowledge among $\mathbf{A}$" as follows

$$C_{\mathbf{A}}\varphi = E_{\mathbf{A}}\varphi \wedge E^2_{\mathbf{A}}\varphi \wedge \dots \wedge \dots E^m_{\mathbf{A}}\varphi \wedge \dots$$

where $E^1_{\mathbf{A}} = E_{\mathbf{A}}\varphi$ and $E^{k+1}_{\mathbf{A}}\varphi = E_{\mathbf{A}}E^k_{\mathbf{A}}\varphi$, for $k \geq 1$.

The epistemic indistinguishability relation $\sim_i$ for agent i makes it possible to redefine history based models as epistemic set models in a way that we can compare agents' knowledge relative to a given protocol [17]. In order to achieve this, we define a set valued function which takes a set and returns a partition in that set that belongs to the agent. Given a protocol $\mathscr{H}$, we define $\kappa_i : 2^{\mathscr{H}} \mapsto 2^{\mathscr{H}}$. For simplicity, we will consider sets of finite histories, and denote the sets of histories with bold letters such as $\mathbf{h}, \mathbf{h}'$ etc. In this model, for each agent, there exists a partitioning of the given protocol $\mathscr{H}$.

Now, in a given model, let π_i denote the agent i's partitioning of the protocol $\mathscr{H}$. That is, for each i, there exists equivalence classes of histories in $\mathscr{H}$. Similarly, $\pi_i(h)$ denotes the partition for agent i that contains h. In other words, for an agent at history h, the histories in $\pi_i(h)$ are *indistinguishable*.

Now, we define $\kappa_i(\mathbf{h}) = \{h : \pi_i(h) \subseteq \mathbf{h}\}$. Simply put, for a set of histories $\mathbf{h}$, the set $\kappa_i(\mathbf{h})$ includes all the histories h whose partitions are contained in $\mathbf{h}$. The operator κ_i is a set valued operator which will express agent's knowledge. In order to achieve this, we stipulate that κ_i satisfies the following three properties, for given sets of histories $\mathbf{h}, \mathbf{h}'$ [7].

1. $\kappa_i(\mathbf{h} \cap \mathbf{h}') = \kappa_i(\mathbf{h}) \cap \kappa_i(\mathbf{h}')$
2. $\kappa_i(\mathbf{h}) \subseteq \mathbf{h}$
3. $-\kappa_i(\mathbf{h}) = \kappa_i(-\kappa_i(\mathbf{h}))$

where $-$ denotes the set theoretical complement. The above three property makes κ_i an epistemic operator where the first condition corresponds to normality, the second one to veridicality and the last one to introspection in the traditional sense. Similarly, a common knowledge operator $\mathfrak{c}$ can be defined for sets of histories to express the common knowledge modality $C_{\mathbf{A}}$.

Extending the preference relation in HBPL, it is possible to compare agents' knowledge relative to each other, given a set of histories.

Definition 4.2. Define the set of histories $[j > i]^{\mathscr{H}}$ in which agent j is at least as knowledgeable as agent i with respect to a given set of protocols $\mathscr{H}$ as follows.

$$[j > i]^{\mathscr{H}} := \bigcap_{\mathbf{h} \in 2^{\mathscr{H}}} -\kappa_i(\mathbf{h}) \cup \kappa_j(\mathbf{h})$$

By a slight abuse of notation, we will denote the proposition whose extension is the set $[j > i]^{\mathscr{H}}$ by the same symbol.

Since our epistemic model is based on equivalence classes and partitions, it is possible to compare agents' knowledge based on their partitions. The following lemma expresses the fact that the finer the partitions, the more the epistemic knowledge.

Lemma 4.3 ([17])**.** *$h \in [j > i]^{\mathscr{H}}$ iff $\pi_j(h) \subseteq \pi_i(h)$.*

Proof. Let $h \in [j > i]^{\mathscr{H}}$. For $\mathbf{h} = \pi_i(h)$, and by the above definition, we have $h \in -\kappa_i(\pi_i(h)) \cup \kappa_j(\pi_i(h))$. By definition, $\kappa_i(\pi_i(h)) = \pi_i(h)$ and also $h \in \pi_i(h)$. Thus, $h \in \kappa_j(\pi_i(h))$. Then, by definition of κ, $\pi_j(h) \subseteq \pi_i(h)$.

For the converse direction, let κ, $\pi_j(h) \subseteq \pi_i(h)$. Suppose for some set of histories $\mathbf{h}$, we have $h \in \kappa_i(\pi_i(\mathbf{h}))$. Then, by definition, $\pi_i(h) \subseteq \mathbf{h}$. By the initial assumption, we also have $\pi_j(h) \subseteq \mathbf{h}$ which means that $h \in \kappa_j(\pi_i(\mathbf{h}))$. Thus, for each $\mathbf{h} \in 2^{\mathscr{H}}$, $h \in -\kappa_i(\mathbf{h}) \cup \kappa_j(\mathbf{h})$. Hence, $h \in [j > i]^{\mathscr{H}}$. □

Another interesting lemma suggested by Samet shows how the comparison ordering of agents' knowledge and epistemic partitions relate to each other. Let us prove it for HBPL.

Lemma 4.4 ([17]). *$h \in \kappa_i([j > i])$ iff $\pi_i(h) = \bigcup_{h' \in \pi_i(h)} \pi_j(h')$.*

Proof. The proof directly follows from the definitions.

$h \in \kappa_i[j > i]$ amounts to $\pi_i(h) \subseteq [j > i]$ by definition. By the first lemma, this statement holds if and only if for each $h' \in \pi_i(h)$ we have $\pi_j(h') \subseteq \pi_i(h)$. This is equivalent to $\pi_i(h) = \bigcup_{h' \in \pi_i(h)} \pi_j(h')$. □

Next, we define a decision function $\delta_i : \mathscr{H} \mapsto D$ for a protocol $\mathscr{H}$, agent i and any set of decisions D. The vector $\delta = (\delta_1, \ldots, \delta_n)$ is called a decision profile for n agents. In this context, we consider D as *any* set of decisions, not necessarily probabilistic or propositional. Now, for a decision $d \in D$, we define the proposition $[\delta_i = d]^{\mathscr{H}}$ with the following set as its extension.

$$[\delta_i = d]^{\mathscr{H}} = \{H \in \mathscr{H} : \delta_i(H) = d \text{ for all } H \in \mathscr{H}\}$$

Similarly, we will use $[\delta_i = d]^{\mathscr{H}}$ to denote both the set and the proposition, if no confusion arises from the context. If obvious, we will drop the superscript.

We assume each agent knows his decision [17]. In our notation, this amounts to the following statement $[\delta_i = d]^{\mathscr{H}} \subseteq \kappa_i([\delta_i = d]^{\mathscr{H}})$. In other words, agents agree with those agents who know better. Let us put it formally and more carefully as follows.

Definition 4.5. $\kappa_i([j > i]^{\mathscr{H}} \cap [\delta_j = d]^{\mathscr{H}}) \subseteq [\delta_i = d]^{\mathscr{H}}$.

Sure Thing Principle suggests that if an agent j is at least as knowledgable as another agent i, and if j's decision is d, then i's decision is also d.

If the knowledge comparison is an intuitive order, this means that there can be postulated some agents that know less than all the other agents. Now, an agent i is called *an epistemic dummy* if all the agents are at least as knowledgeable as i. Dummy agents can be introduced to decision making process if they do not upset the sure thing principle. The following notion incorporates dummy agents into the sure thing principle.

Definition 4.6. A decision profile $\mathbf{d}$ in a model with a protocol $\mathscr{H}$ with n agents is expandable if for any additional epistemic dummy i, there exists a decision profile $\mathbf{d}'$ which satisfies the sure thing principle.

It is important to stipulate that for an expandable decision profile $\mathbf{d}$ and dummy agent i, $\mathbf{d}$ and $\mathbf{d}'$ agree on the decisions of agents who are not dummies. Expandable decision profiles play an important role for the following theorem, which we adopt from [17].

Theorem 4.7. *If δ is an expandable decision profile in a model with a protocol $\mathscr{H}$ with n agents, then for any decisions $d_1, \ldots, d_n$ in D which are not identical, $C(\bigwedge_{i \leq n} [\delta_i = d_i]^{\mathscr{H}})$ is nowhere satisfiable, in other words $\mathfrak{c}(\bigcap_{i \leq n} [\delta_i = d_i]^{\mathscr{H}}) = \emptyset$.*

Proof. First, we will construct an epistemic dummy agent. Call him $n+1$. Now, define π_{n+1} as the finest partition which is coarser than any of the partitions π_i for $1 \leq i \leq n$. Then, the epistemic set operator κ_{n+1} based on the partition π_{n+1} is the common knowledge operator $C_{\mathbf{A}}$ [7].

Also $\kappa_{n+1}([j > n+1]) = \mathscr{H}$ as κ_{n+1} is common knowledge operator and $[j > n+1] = \mathscr{H}$ for each agent j for $1 \leq j \leq n$. This shows that $n+1$ is an epistemic dummy.

Now, for an expandable decision profile δ, there is δ_{n+1} such that $(\delta_1, \ldots, \delta_{n+1})$ satisfies the sure thing principle.

We will now prove the contrapositive. For this, let $h \in \mathfrak{c}(\bigcap_i [\delta_i = d_i])$. We showed that κ_{n+1} is the common knowledge operator. So, let $h \in \kappa_{n+1}(\bigcap_i [\delta_i = d_i])$.

Since κ operator satisfies the property that $\kappa(\mathbf{h} \cap \mathbf{h}') = \kappa(\mathbf{h}) \cap \kappa(\mathbf{h}')$, we have $h \in \bigcap_i \kappa_{n+1}([\delta_i = d_i])$. Therefore, for each j, $h \in \kappa_{n+1}([\delta_j = d_j])$.

By definition, π_{n+1} is coarser than π_j for any j, and $\pi_{n+1}(h) = \bigcup_{h' \in \pi_{n+1}(h)} \pi_j(h')$. By the second lemma, we observe that $h \in \kappa_{n+1}([j > i])$.

Now, we have $h \in \kappa_{n+1}([\delta_j = d_j])$ and $h \in \kappa_{n+1}([j > i])$, so that we can apply the sure thing principle to obtain $h \in [\delta_{n+1} = d_j]$ for each j. Therefore, all the decisions d_j are identical to $\delta_{n+1}(h)$. This is also why we need an epistemic dummy agent.

Thus, if the common knowledge is not an empty set, the decisions of the agents coincide.

This proves the theorem. □

So far, we have adopted Aumann's well-known theorem to history based structures via Samet's formalism [1, 17]. What is more interesting is, via our proof, the result can be extended to runs and function based knowledge structures, and expands the domain of applicability of Aumann's theorem [5, 8, 6].

Now, it is worth mentioning the potential future applications of above results. First, Theorem 4.7 provides some good handles for systems security policies. In systems' security, it can obviously be seen that attacker's and defender's decisions cannot be the same for a successful attack. Also, it is not enough that they will have different decisions, those decisions cannot be commonly known among them. The theorem specifies under which conditions, agents' decisions which are not identical cannot be common knowledge. If they are common knowledge, then some agents *cannot agree to disagree* [1].

Also, it is noteworthy that the decision set D above is given arbitrarily. Therefore, it seems possible to choose a probability measure to precise the decisions of the agents in a way close to the original set up of the theorem by Aumann [1]. Such a set-up would facilitate the introduction of probabilistic issues and mixed strategies into HBPL, which we leave to future work.

Finally, set based approaches to histories relate HBPL to topological spaces where agents' indistinguishable histories may form an open set. In such a formalization, topological transformations and paths might help us to transform histories in a continuous and knowledge-preserving fashion.

5 Conclusion

History based models provide a natural formalism for epistemic logic. In this work, we extended the standard framework by introducing modal preferences in order to reason about subjective preferences and epistemic games, and made a connection between logic and games via history based models. This opens up a broad spectrum of theoretical and applied fields for future work including process algebras, preference logics, deontological games and topologies.

History based models also seem to provide a richer understanding for agents' rationality by introducing various tools for explicate agents' decision and preferences based on the progress of the game, preferences and the time. This potential can easily be extended to a broader and utilitarian analysis of history based games, which we leave for future work.

In this preliminary work, apart from introducing a conceptual development, we argued that HBPL fits rather well within the current research on epistemic game theory, modal logic and logic of games, and provides a new and broad framework.

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