A Paraconsistent Logic for Contrary-to-Duty Imperatives

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Abstract

Contrart-to-duty imperatives are those which tell us what we ought to do if we violate some of our obligations. In this work, we give an inconsistency-friendly framework for contrary-to-duty imperatives and introduce three semantics for it: static, dynamic and topological. The static semantics uses the standard modal paraconsistent approach whereas the dynamic semantics views violations as dynamic updates of the model. The topological semantics, on the other hand, underlines the depth and the richness of paraconsistent models. After discussing different paraconsistent models for contrary-to-duty imperatives, we briefly discuss some applications.

Keywords Da Costa paraconsistent logic, contrary-to-duty obligations, deontic logic, topological semantics.

1 Introduction

Various moral philosophers suggested that people who live in affluent countries are obliged to give out a certain portion of their income to those in need. Peter Singer in his monograph *The Life You Can Save* argued in favor of this argument and suggested a sliding-scale formula for those people who he thinks is obliged to give out some part of their income [40]. However, as Unger also observed, for a variety of reasons, people usually refrain from doing so in real life [43]. Unger argued that almost all of us know that millions of children are starving in many African countries at any given moment and that we feel obliged to help often times, yet we do not donate enough to NGOs working on this issue. Then, a natural question is the following. If we assume that able people are morally obliged to help the unfortunate, what happens if they simply do not help? Would they have another obligation even if they ignored the first one?

As another example, consider the tragic death of Kitty Genovese. The case of Kitty Genovese which was discussed in [36] shows the importance of our moral obligations and collective knowledge. This example also serves as a case to illustrate the close relation between (in)consistencies, knowledge and obligations. The example recalls a real life case when a woman named Kitty Genovese

in Queens, New York was stabbed to death while walking to her car. The murder was witnessed by many people from their apartments whom we can safely assume morally good. Genovese could have been saved if someone called the emergency services in time, which can be seen as a moral imperative for most of the witnesses. Yet, they did not call the police. The reason was because they thought that someone else must have already called the police.

Assuming that calling the police when witnessed a horrible murder is a moral obligation, this story illustrates another aspect of people's moral obligations. It can be put forward that if the witnesses omitted their immediate obligation of making the calls, they still have another obligation to ensure that the victim's health is in stable condition. In an idealized world, it can be thought that the witnesses have an obligation to call the police. However, for whatever reason, if they can not make the call, they still need to make sure that the victim is okay, or that they have the physical description of the murderer to aid the police.

The current work is concerned with such cases. How can we formalize the situations when it is obligatory to do certain actions, and yet when those actions are not carried out, there follows some other obligations, called contrary-to-duty imperatives? Moreover, how can we achieve this without excluding the deontic and ontological possibility of inconsistencies?

Moral philosopher Roderick Chisholm defines contrary-to-duty (CtD) imperatives as the "Exhortations [which] often take the form: 'You ought to do a, but if you do not do a, then you must by all means, do b' " [11]. Therefore, contrary-to-duty imperatives are those that tell us "what we ought to do if we neglect certain of our duties" [ibid]. Apart from its obvious connections to epistemology [8], CtD obligations relate directly to inconsistencies.

CtD obligations can be formalized in a variety of ways. Counterfactual reasoning can be seen as a plausible way to formalize them [29]. However, in this work, we suggest a broader framework that underlines the inconsistent nature of CtD obligations and deontologies.

A logic is *paraconsistent* if contradictions do not trivialize it. In paraconsistent logics, there can exist propositions that are both true and false. Paraconsistent analysis for the CtD imperatives relates the issue to a broader context of non-classical logics, and suggests a framework that can deal with inconsistent obligations.

Literature Review The problems of deontic logic have troubled many logicians and philosophers and deontic *paradoxes* have been analyzed by many [1, 21]. Use of paraconsistency in deontic logic is not new. As Priest remarked "standard deontic logic suffers badly from explosion" as in classical logic "if you have inconsistent obligations you are obliged to do everything. This is surely absurd. People incur inconsistent obligations; this may give rise to legal or moral dilemmas, but hardly to legal or moral anarchy" [38]. Various paraconsistent deontic logics were discussed in [31]. As a specific issue in deontic logic, the problems of contrary-to-duty imperatives and their formalizations have attracted some attention. Meyer achieved this formalization from

a dynamic modal perspective [33], da Costa and Carnielli suggested a modal paraconsistent point of view [15]. Gabbay proposed a reactive model based approach [19, 18] whereas Prakken and Sergot developed an extension of deontic logic to express them [37]. Mott used a stronger implication to express contrary-to-duty imperatives [35]. An in-depth overview of CtD imperatives from a deontic logical perspective can be found in [9], and [20] surveys various deontic dilemmas and some suggested solutions for them. Horty, on the other hand, discussed the relation between modal dilemmas and non-monotonic systems [23]. In [24], Horty considers background contexts with an ordering on formulas and discusses conditional obligations and compares them with van Frassen's. Kratzer joins Horty to indroduce an ordering on norms which can be used to analyze deontic conflicts [27]. Jones and Pörn introduces two deontic relations, similar to an ordering, to distinguish -what they call- ideal and subideal deontologies, and apply it to Chisholm's paradox [25]. A conditional-like dyadic deontic modality was suggested in [30]. Such conditional operators can be viewed as a way to handle the problems of material implications. Also, a multi-agent approach to conflicting group obligations was studied in [26]. An interesting combination of preference and deontic logics was presented with an application to CtD obligations in [44].

The paraconsistent logic we are concerned with in this paper has been developed by da Costa and his colleagues, and is one of the earliest and well-studied systems of paraconsistency [13, 14]. Logics of formal inconsistency (LFIs) form a broader framework for da Costa systems [10, 16], and both LFIs and da Costa systems have been discussed from a deontic logical angle [14]. Coniglio discussed a variety of deontic LFIs as well [12].

This paper positions itself between the paraconsistent and the classical modal logical approaches to obligations, and presents a dynamic take on the subject. The idea developed in this paper is to view deontic violations as dynamic updates. In other words, when an obligation is violated, the violation will be considered as a dynamic update of the model.

This perspective relates a broader agenda of paraconsistency to deontic logic. That is a natural semantics for paraconsistent logic is topological semantics [34, 5]. Due to its own collection of mathematical toolkit, such as homeomorphisms and transformations, topological semantics is a reasonable choice to approach dynamic logics [4]. Apart from considering CtD obligations within the well-known Kripke structures, the paper discusses them first in a dynamic framework, then in a topological framework. This will introduce the semantic richness of paraconsistent logics to the study of CtD obligations.

In what follows, we start by reviewing the basic system we will use in the paper, then introduce modal, dynamic and topological methods to express CtD imperatives paraconsistently. Throughout the text, we will use the terms "imperatives" and "obligations" interchangeably if no confusion arises.

2 Logic of Contrary-to-Duty Imperatives

2.1 The Basic System

Da Costa and Carnielli suggested a paraconsistent deontic logic C_1^D which is based on da Costa's well studied paraconsistent logic C_1[15]. In this section, we will first review C_1^D, and extend it with a modal operator to describe CtD imperatives.

The propositional system C_1^D is constructed with the standard propositional syntax given a denumerably infinite set of propositional variables **P** as follows.

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \to \varphi$$

where $p \in \mathbf{P}$. We take disjunction $\vee$ and implication $\to$ as abbreviations in the usual sense. We denote this syntax with $\mathcal{L}$.

The genuine approach of C_1 (and therefore C_1^D) to paraconsistency is that it distinguishes two kinds of propositions: the ones that are classical and satisfy the law of contradiction (which suggests that contradictions are impossible), and the ones that are not classical and do not satisfy the law of contradiction. They call the prior type of propositions as *good*, and the latter ones as *bad*. The good propositions can be distinguished by the following definition.

Definition 2.1. $\varphi^\circ := \neg(\varphi \wedge \neg\varphi)$.

Therefore, contradictions are impossible only for good propositions. Then, the classical negation $\sim$ can be reclaimed by using the negation symbol $\neg$.

Definition 2.2. $\sim\varphi := \neg\varphi \wedge \varphi^\circ$.

The operator $\sim$ has all the properties of the classical negation and stronger than the negation $\neg$ in C systems. Now, the basic system C_1 admits the following axiom schemes [14].

- $\varphi \vee \neg\varphi$
- $\neg\neg\varphi \to \varphi$
- $\varphi \to (\psi \to \varphi)$
- $\varphi \wedge \psi \to \varphi$
- $\varphi \wedge \psi \to \psi$
- $\varphi \to (\psi \to \varphi \wedge \psi)$
- $\varphi \to \varphi \vee \psi$
- $\varphi \to \psi \vee \varphi$
- $(\varphi \to \psi) \to ((\varphi \to (\psi \to \chi)) \to (\varphi \to \chi))$
- $(\varphi \to \chi) \to ((\psi \to \chi) \to (\varphi \vee \psi \to \chi))$
- $\psi^\circ \to ((\varphi \to \psi) \to ((\varphi \to \neg\psi) \to \neg\varphi))$
- $\varphi^\circ \wedge \psi^\circ \to ((\varphi \to \psi)^\circ \wedge (\varphi \wedge \psi)^\circ \wedge (\varphi \vee \psi)^\circ)$

The rule of inference in C_1 is modus ponens. Notice that in the above axiomatization, the double negation rule is one-directional, and only good formulas (i.e. φ°) cannot be inconsistent. The following list includes some of the invalidities in C_1 which may help to get a better grasp of the logic [14].

- $\varphi \wedge \neg\varphi \to \psi$
- $\neg\varphi \to (\varphi \to \psi)$
- $\varphi \wedge \neg\varphi \to \neg\psi$
- $\varphi \equiv \neg\neg\varphi$
- $\neg(\varphi \wedge \neg\varphi)$
- $\neg(\varphi \vee \psi) \equiv \neg\varphi \wedge \neg\psi$
- $\varphi \to (\neg\varphi \to \psi)$
- $\neg(\varphi \wedge \psi) \equiv \neg\varphi \vee \neg\psi$

In the above list, the first two invalidities show that the logic does not collapse under contradictions and is indeed paraconsistent. It is important to note that not only the classical laws of double negation, also De Morgan's Laws do not hold in C_1 for the negation operator $\neg$ (but they do for $\sim$). Failure of De Morgan's Laws can be observed in various paraconsistent logics with weaker negation or connectives [17]. Specifically in C-systems, we have $\neg(\varphi \wedge \psi) \to \neg\varphi \vee \neg\psi$ satisfied while the converse direction is not. Nevertheless, C_1 is a conservative extension of the classical propositional logic when it is considered with the classical negation $\sim$ [14]. Furthermore, a three-valued semantics for da Costa systems was given [16, p. 11].

Based on C_1, da Costa and Carnielli constructed the system C_1^D - a paraconsistent deontic modal logic. The syntax of C_1^D introduces a unary modal operator O that stands for "it is obligatory that", and the formulas are defined in the usual way closing them under the standard connectives of C_1 and the modal operator O. The language obtained in this fashion is denoted with $\mathcal{L}_O$.

The deontic modalities of *forbidden* and *permitted* can also be defined in C_1^D by putting $F\varphi := O\neg\varphi$, and $P\varphi := \neg O\neg\varphi$, for *forbidden* and *permitted* respectively. The additional axioms of C_1^D are given as follows.

- $O(\varphi \to \psi) \to (O\varphi \to O\psi)$
- $\varphi^\circ \to (O\varphi)^\circ$
- $O\varphi \to \sim O\sim\varphi$
- $\vdash \varphi \therefore \vdash O\varphi$

It is easy to observe that following is valid in C_1^D.

- $O\varphi \to O(\varphi \vee \psi)$
- $\sim(O\varphi \wedge \sim O\varphi)$
- $O(\varphi \wedge \psi) \equiv O\varphi \wedge O\psi$
- $O\varphi \wedge O(\varphi \to \psi) \to O\psi$

The following is invalid in C_1^D, reflecting the paraconsistency of the system.

- $O\neg(\varphi \wedge \neg\varphi)$
- $O\varphi \wedge O\neg\varphi \to O\psi$
- $O(\varphi \wedge \neg\varphi) \to O\psi$
- $\neg(F\varphi \wedge P\varphi)$

It is possible to give a modal semantics for C_1^D. The model is a standard one: $M = (W, R, V)$ where W is a non-empty set, R is a serial binary accessibility relation[1] defined on W, and V is a valuation function that assigns subsets of

[1] A relation on W^2 is serial if $\forall w \exists v. wRv$. In other words, in serial frames there are no end states.

W to propositional variables in **P**. We give the semantics as follows for $w \in W$ [15].

$w \models \neg\varphi$	when	$w \not\models \varphi$,
$w \models \varphi$	when	$w \models \neg\neg\varphi$,
$w \models \varphi \wedge \psi$	iff	$w \models \varphi$ and $w \models \psi$,
$w \models \varphi \vee \psi$	iff	$w \models \varphi$ or $w \models \psi$,
$w \models \varphi \rightarrow \psi$	iff	$w \not\models \varphi$ or $w \models \psi$,
$w \models \varphi^\circ, \psi \rightarrow \varphi,$ $\psi \rightarrow \neg\neg\varphi$	then	$w \not\models \psi$
$w \models \varphi^\circ \wedge \psi^\circ$	then	$w \models (\varphi \rightarrow \psi)^\circ \wedge (\varphi \wedge \psi)^\circ \wedge (\varphi \vee \psi)^\circ$
$w \models O\varphi$	when	$\forall w' \in W$, wRw' implies $w' \models \varphi$,
$w \models \varphi^\circ$	implies	$w \models (O\varphi)^\circ$

Based on this semantics and axiomatization, da Costa and Carnielli showed that C_1^D is sound, complete and decidable. The completeness proof is based on maximal non-trivial set of formulas, which are familiar from da Costa's C systems [14, 7].

So far is the system of da Costa and Carnielli.

2.2 Contrary-to-Duty Imperatives

In this section, we extend C_1^D with a modal operator to express contrary-to-duty imperatives. In order to achieve this, we add the the following dyadic modality $C(\varphi, \psi)$ for well-formed formulas φ, ψ. The expression $C(\varphi, \psi)$ reads "it is obligatory that φ, yet if φ is not the case, then it is obligatory that ψ". The possible world semantics for the contrary-to-duty operator is given as follows.

$$M, w \models C(\varphi, \psi) \quad \text{iff} \quad M, w \models O\varphi \wedge \neg\varphi \rightarrow O\psi$$

It is important to argue why we need a paraconsistent framework to express CtD obligations. First reason is metaphysical. Paraconsistent logics are proper formal tools to discuss (deontic) paradoxes and dilemmas if we commit to the fact that inconsistent obligations are ontologically possible. Classical logics simply ignore or collapse under the presence of paradoxes, thus we need a formal system that can actually handle paradoxes and enable us to reason with them. Second, moral dilemmas exist. As Sartre profoundly described, they are part of our *existence* [39]. Notice that CtD obligations can produce dilemmas as well. In order to see this, take ψ as $\neg\varphi$. Then, $C(\varphi, \neg\varphi)$ reduces to $O\varphi \wedge \neg\varphi \rightarrow O\neg\varphi$. If $O\varphi \wedge \neg\varphi$ and $C(\varphi, \neg\varphi)$ are the case, then, by modus ponens (which is a rule of derivation in C-systems as we mentioned earlier) and basic modal logic, we have $O(\varphi \wedge \neg\varphi)$, which is contradictory (and satisfiable in C_1^D). This justifies our use of paraconsistent logic to approach this issue in order to analyze the relation between inconsistencies and conflicting CtD obligations.

We call the extension of C_1^D with the $C(\cdot, \cdot)$ modality as C_1^{DD}. The models of C_1^{DD} are defined in the standard way as a tuple $M = (W, R, V)$ where W is a non-empty set, R is a binary relation defined on W^2, and V is a valuation function assigning the subsets of W to propositional variables.

Let us now observe some immediate validities of the $C(\cdot,\cdot)$ operator.

- $C(\varphi, \top)$
- $C(\varphi,\psi) \wedge C(\psi,\chi) \to C(\varphi,\chi)$
- $C(\varphi,\psi) \wedge C(\chi,\psi) \to C(\varphi \wedge \chi, \psi)$
- $C(\varphi, \psi \wedge \chi) \leftrightarrow C(\varphi,\psi) \wedge C(\varphi,\chi)$
- $\varphi^\circ \wedge \psi^\circ \to (C(\varphi,\psi))^\circ$
- $C(\varphi,\psi) \wedge C(\psi,\chi) \wedge \neg\varphi \wedge \neg\psi \to O(\psi \wedge \chi)$: if ψ is $\neg\varphi$ and χ is φ, this reduces to $C(\varphi,\neg\varphi) \wedge \varphi \to O(\varphi \wedge \neg\varphi)$

The following are invalid which can easily be shown.

- $C(\varphi,\neg\varphi) \to O\psi$
- $C(\top,\varphi)$
- $C(\varphi \wedge \chi, \psi) \to C(\varphi,\psi) \wedge C(\chi,\psi)$

As we argued above, $C(\cdot,\cdot)$ modality can produce inconsistencies in a non-trivial way.

Proposition 2.3. *The statement $\neg C(\varphi,\neg\varphi)$ is not valid in C_1^{DD}.*

The logic C_1^{DD} distinguishes *good* and *bad* propositions as we underlined earlier. We clarify this issue with the following proposition.

Proposition 2.4. $\varphi^\circ \to \neg C(\varphi^\circ, \neg\varphi^\circ)$.

Proofs of Propositions 2.3 & 2.4. We first show that $C(\varphi,\neg\varphi)$ is satisfiable. In order to see that at a state w, assume φ holds at each accessible state from w. Assume further that $\neg\varphi$ holds at w and also at each accessible state from w. Thus, for such a w, we have $w \models C(\varphi,\neg\varphi)$.

However, if $w \models \varphi^\circ$, then we cannot have $w \models O(\varphi^\circ \wedge \neg\varphi^\circ)$. □

The $C(\cdot,\cdot)$ operator provides succinctness in a sense that it expresses complex statements about CtD obligations in a more compact way. However, it is syntactically redundant. Similar to the reduction of the public announcement operator in dynamic epistemic logics to the static epistemic operator, it can be reduced to the basic deontic modality efficiently.

Theorem 2.5. *The logic C_1^{DD} is complete and decidable with respect to the given axiomatization of C_1^D.*

Proof. The proof is a reduction of $C(\cdot,\cdot)$ terms to formulas without the $C(\cdot,\cdot)$ modality. The procedure is an efficient induction on terms, and follows immediately given C_1^D is complete and decidable. □

In C_1^{DD}, inconsistent obligations do not necessarily generate absurdities.

Proposition 2.6. *In C_1^{DD}, $C(\varphi,\neg\varphi) \to \psi$ and $C(\varphi \wedge \neg\varphi, \psi)$ are not valid. However, $\neg C(\varphi^\circ, \neg\varphi^\circ)$ is valid.*

Proof. Proof follows directly from the definitions. □

3 A Dynamic Semantics for Contrary-to-Duty Imperatives

As we hinted out earlier, a *violation* of an obligation can be viewed as a deontic update in the model. When an obligation is violated, this can be seen as an *action* or *event* which triggers an update in the model. After an update, then the CtD obligations are evaluated in the updated model.

Updating paraconsistent logical structures is not a new idea. As indicated in [6] (in the context of a certain dynamic epistemic logic, called public announcement logic (PAL)), paraconsistent logics introduce a certain advantage for update semantics:

> The classical PAL heavily depends on the law of non-contradiction. An external and truthful announcement is made. Then, the agents update their epistemic models by eliminating the states in their model which do not agree with the announcement, followed by the reducing the epistemic accessibility relation or the topology and the valuation with respect to the new, updated model. Therefore, the classical PAL does not control the inconsistencies, it completely eliminates them. Yet, in paraconsistent spaces, some contradictions need not be eliminated as they do not trivialize the theory. In short, the main problem caused by inconsistencies is that they trivialize the theory due to the choice of the underlying logic. Therefore, if there exists some contradictions that do not trivialize the theory (again, due to the choice of the underlying logical framework), there seems to be no need to eliminate them. This is our pivotal point for paraconsistent PAL.
>
> [6]

Therefore, as long as the violations do not trivialize the system but only create inconsistencies, paraconsistent logic offers a good handle on the matter. Moreover, a dynamic approach relates C_1^D to a wider research programs in classical modal logics, dynamic epistemic logics and classical dynamic deontic logics, and attempts at filling the gap in the literature merging dynamic modal logics and paraconsistent deontic logics.

Let us now define the formal matters.

Given a C_1^{DD} model $M = (W, R, V)$, we define a submodel $M|\varphi = (W', R', V')$ as follows. We first set $[\varphi]^M = \{w \in M : \forall v.wRv \rightarrow M, v \models \varphi\}$. In other words, $[\varphi]^M$ is the possibly empty set of states at which φ is obligatory. Then, $W' := [\varphi]^M$, $R' := R \cap (W' \times W')$, and similarly $V' := V \cap W'$. We call $M|\varphi$ *the updated model*. The update operator will be taken as associative to left, therefore $M|\varphi|\psi$ will read $(M|\varphi)|\psi$. Also, we have $M|\varphi|\psi = M|\psi|\varphi$. Let us now state some consequences of the updated models and CtD imperatives.

The following first observations make sure that what we have a paraconsistent logic - indeed a paraconsistent logic which can recover the classical negation.

Proposition 3.1. *It is not necessarily the case that* $[\varphi]^M \cap [\neg\varphi]^M = \emptyset$.

Proposition 3.2. $[\varphi^\circ]^M \cap [\sim\varphi^\circ]^M = \emptyset$.

Proofs of Proposition 3.1 & 3.2. Follows directly from definitions. □

Proposition 3.3. $M|\varphi, w \models O\varphi$ *for any* φ.

Proof. Given a model $M = (W, R, V)$ and a formula φ. Construct $M|\varphi = (W', R', V')$ as described. Then, for $w \in W'$, we observe $w \in [\varphi]^M$. Thus, $M|\varphi, w \models O\varphi$ follows by definition. □

It is now important to underline that updates create obligations in a sense that in an updated model, the update becomes an obligation.

Proposition 3.4. $M, w \models O\varphi \to \psi$ *if* $M|\varphi, w \models \psi$ *for any* φ, ψ *and* $w \in W'$.

Proof. If $M|\varphi, w \models \psi$, then by Proposition 3.3 we observe $M|\varphi, w \models O\varphi \wedge \psi$ which entails that $M, w \models O\varphi \wedge \psi$ which in turn is equivalent to what is desired. □

We now present a dynamic semantics for $C(\cdot, \cdot)$ as follows.

$$M, w \models C(\varphi, \psi) \text{ iff } M, w \models \neg\varphi \text{ implies } M|\varphi, w \models O\psi$$

This approach makes it clear how a *violation* of an obligation can be viewed as a dynamic and deontic *update*. The dynamic semantics provides a neutral reading of violations, entailing that sometimes "we can live with them". More importantly, dynamic semantics is an important aspect of modal logic, therefore the dynamic semantics of C_1^{DD} brings CtD imperatives closer to familiar territories of modal logic. In $M|\varphi$, the formula φ becomes obligatory after the model is updated accordingly. Also, due to the proof theory of C_1^{DD}, if we have $M, w \models \varphi$, it does *not* follow that $C(\varphi, \psi)$ holds vacuously. However, if $w \notin [\varphi]^M$ (if φ is not obligatory at the current state w), then $C(\varphi, \psi)$ fails at w.

In what follows, we clarify some further properties of updated models. The following propositions underline the paraconsistent aspects of C_1^{DD} models.

Proposition 3.5. $M|\varphi, w \models O\neg\varphi$ *is satisfiable for some* M *and* w.

Proof. We can reconsider the model we suggested in the proof of Propositions 2.3 & 2.4. In such a model, we have $M|\varphi, w \models O\varphi \wedge O\neg\varphi$, in particular $M|\varphi, w \models O\neg\varphi$. □

Consistent formulas can still not create inconsistencies in dynamic semantics as the following argument shows.

Proposition 3.6. $\not\models \varphi^\circ \to C(\varphi^\circ, \neg\varphi^\circ)$.

Additionally, we can put the updates together in order to deal with consecutive updates more easily.

Proposition 3.7. $M|\varphi|\psi = M|(\varphi \wedge \psi)$.

Proposition 3.8. $M|\varphi^\circ|\neg\varphi^\circ = \emptyset$.

Proofs of Propositions 3.6 & 3.7 & 3.8. Follows directly from the definitions. □

Proposition 3.9. *The domain of $M|\varphi|\neg\varphi$ (or $M|\varphi \wedge \neg\varphi$) is not necessarily the empty set in C_1^{DD}.*

Proof. It is possible to construct a model where, for a given w, we impose that w and all the states accessible from w satisfies $\varphi \wedge \neg\varphi$. Then, it is easy to see that such a model remains non-empty after the updates $\varphi \wedge \neg\varphi$. So, the domain of $M|\varphi|\neg\varphi$ is not necessarily empty. □

Proposition 3.9 helps us identify the domain of contradictory obligations, given M. The model $M|\varphi|\neg\varphi$ is exactly the model in which $O(\varphi \wedge \neg\varphi)$ is valid. This is how we characterize contradictory contrary-to-duty obligations in this model.

By a slight abuse of notation, if we denote the universe of a model M by itself, then we have the following.

Proposition 3.10. *$M|\varphi \models O\psi$ iff $M|\varphi|\psi = M|\varphi$, for any model M and formulas φ, ψ.*

Proof. For a given model M and for all states in M, let us assume that $M|\varphi \models O\psi$. Then, take a random w in the domain of $M|\varphi|\psi$. By Proposition 3.7, we observe that $w \in M|\varphi \wedge \psi$, then $w \in M|\psi \wedge \varphi$ which gives $w \in M|\psi|\varphi$. So, $w \in [\varphi]$. Thus, $M|\varphi|\psi \subseteq M|\varphi$.

Similarly, take a random $w \in M|\varphi$. Then, since $M|\varphi \models O\psi$, we observe that $w \in [\psi]$. So, $w \in M|\varphi \cap M|\psi$. Thus, $w \in M|\varphi \wedge |\psi$ which yields that $w \in M|\varphi|\psi$. Thus, $M|\varphi \subseteq M|\varphi|\psi$. We then conclude that $M|\varphi|\psi = M|\varphi$.

For the other direction, if $M|\varphi|\psi = M|\varphi$, then by Proposition 3.3, we observe that $M|\varphi|\psi \models O\psi$. By the assumption, we conclude $M|\varphi \models O\psi$. □

The Proposition 3.10 seems very much like a fixed-point result. This is indeed the case. A model M is said to have a fixed-point for dynamic deontic updates at φ if $M = M|\varphi$.

Theorem 3.11. *For any model M and any formula φ, $M \models O\varphi$ if and only if M has a fixed-point for dynamic deontic updates at φ, i.e $M|\varphi = M$.*

Proof. Let $M = (W, R, V)$ be an arbitrary model. For any formula φ, assume $M \models O\varphi$. Thus, $[\varphi] = W$. Now, consider the updated model $M|\varphi = (W', R', V')$. Then, by definition, $W' = W$, which yields that $M|\varphi = M$.

The reverse direction follows immediately, thus skipped. In this case, Proposition 3.10 can be viewed as a corollary to Theorem 3.11. □

Let us now make the connection between updates and CtD imperatives more clearer.

Theorem 3.12. *$M, w \models C(\varphi, \psi)$ if $M|\varphi|\psi, w \models \neg\varphi$, for any model M and formulas φ, ψ.*

Proof. Assume $M|\varphi|\psi, w \models \neg\varphi$. By definition, this amounts to $M|\varphi, w \models \neg\varphi \wedge O\psi$ which in turn reduces to $M, w \models \neg\varphi \wedge O\psi \wedge O\varphi$. Then, $M, w \models C(\varphi, \psi)$ follows. □

The above theorem describes which updates produce CtD imperatives dynamically and paraconsistently in C_1^{DD}. This can be considered one of the main contributions of this paper.

4 Topological Semantics for CtD Obligations

As we argued earlier, paraconsistency comes with a natural topological semantics. Additionally, this semantics was extended to da Costa systems, which are the basis of our approach [7].

Given a non-empty set W, a collection τ is a topology if (*i*) the empty set and W are in τ, and (*ii*) τ is closed under arbitrary unions and finite intersections. The elements of τ are called *opens*, and the complement of an open is *closed*. For a set U, $\mathsf{Clo}(U)$ returns the smallest closed set containing U, and $\mathsf{Int}(U)$ returns the largest open set in U.

Historically, topological semantics is the first semantics for modal logic, suggested in 1938 [42]. The classical topological semantics for modal logic evaluates the formulas with respect to open sets that they occupy. In this scenario, the truth of $O\varphi$ depends on *an* open set in the topology and all the points in that topology. Briefly, a model $M = (W, \tau, V)$ is a topological model, where W is non-empty set, τ is a topology and V is a valuation. In this situation, boxed formulas, denoted as $\blacksquare\varphi$, are evaluated with respect to an open set and the points in it. Thus,

$$M, w \models \blacksquare\varphi \quad \text{iff} \quad \exists U \in \tau \text{ with } w \in U. \forall v (v \in U \rightarrow M, v \models \varphi)$$

For simplicity, the set that satisfy a formula φ in a model M is called the *extension* of φ and defined by the following set $\{w : M, w \models \varphi\}$

The major result of the topological semantics states that modal logic of topological spaces is S4 [32]. In the classical case, by definition, the extension of modal formulas are topological: an open or a closed set. This however does not apply to Boolean formulas as their extensions can be anything - not necessarily an open or closed set. However, if we stipulate that the extension of each and every proposition must be a closed set, we can obtain paraconsistent logic. The (finite) intersections and unions of closed sets are still closed, thus we can have conjunctions and disjunctions as closed sets. However, in this context, the negation is not easy to define. Because if the extension of each proposition is a closed set, then their negation will have open extensions, as the classical set theoretical complement of a closed is open, by definition. In order to overcome

this problem, paraconsistent logic defines negation as the *closure of the complement*, which is a closed set. Our approach will be rather different than this as the negation in C_1^{DD} is different.

Furthermore, for obvious reasons, topological semantics, as it is, is not and ideal framework for deontic logic as well as da Costa C-systems, since $O\varphi \to \varphi$ is not necessarily the case in deontic logics. It is however possible to extend the topological models for da Costa systems to C_1^{DD} to express CtD obligations [7].

A topological C_1^{DD}-model M is a tuple $M = (S, \sigma, V, N)$ where S is a non-empty set, σ is a (Alexandroff) topology on S, $V : \mathbf{P} \to \wp(S)$ is a valuation function, and N is a (full) function which takes possible worlds $s \in S$ as inputs and returns sets of negated propositional forms (possibly empty) in such a way that $w \in \mathsf{Clo}(v)$ implies $N(w) \subseteq N(v)$. In this semantics, we assume, for the sake of paraconsistency, that the valuation function V returns closed sets.

Then, the topological semantics for C_1^{DD} is given as follows. We abbreviate $\neg^0\varphi := \varphi$, and $\neg^{n+1}\varphi := \neg(\neg^n\varphi)$ for a φ which does not include a negation sign in the front. For notational convenience, we denote $\mathsf{Clo}(\{w\})$ by $\mathsf{Clo}(v)$. Similarly, $\partial(\cdot)$ denotes the boundary operator which is defined as $\partial(U) := \mathsf{Clo}(U) - \mathsf{Int}(U)$.

$$
\begin{aligned}
M, w \models p \quad &\text{iff} \quad \forall v. w \in \mathsf{Clo}(v), v \models p \text{ for atomic } p \\
&\text{iff} \quad w \in V(p) \\
M, w \models \varphi \wedge \psi \quad &\text{iff} \quad w \models \varphi \text{ and } w \models \psi \\
M, w \models \varphi \vee \psi \quad &\text{iff} \quad w \models \varphi \text{ or } w \models \psi \\
M, w \models \varphi \to \psi \quad &\text{iff} \quad \forall v. w \in \mathsf{Clo}(v), v \models \varphi \text{ implies } v \models \psi \\
M, w \models \neg^1\varphi \quad &\text{iff} \quad \neg^1\varphi \in N(w) \text{ or } \exists v. v \in \mathsf{Clo}(w) \text{ and } v \not\models \varphi \\
M, w \models \neg^{n+2}\varphi \quad &\text{iff} \quad \neg^{n+2}\varphi \in N(w) \text{ and } w \models \neg^n\varphi, \text{ or} \\
&\qquad \exists v. v \in \mathsf{Clo}(w) \text{ and } v \not\models \neg^{n+1}\varphi \\
M, w \models O\varphi \quad &\text{iff} \quad \exists U \in \tau \text{ with } w \in \mathsf{Clo}(U) \text{ and} \\
&\qquad \forall v. (v \in \mathsf{Clo}(U) \text{ implies } M, v \models \varphi)
\end{aligned}
$$

For a detailed exposure to the above semantics, we refer the reader to [7]. What is important for our purposes is that in the above semantics for $O\varphi$, it is not necessarily the case that $w \in U$ when $w \in \mathsf{Clo}(U)$. Therefore, it is possible that for an open set U, $w \notin U$ but $w \in \partial(U)$.

It is immediate to observe that axioms of the O modality holds with respect to the above semantics, and moreover it is paraconsistent. In other words, for example, $O\varphi \wedge O\neg\varphi \to O\psi$ remains invalid as, at a point w, there may exist open sets $U, V \in \tau$ such that $w \in \mathsf{Clo}(U) \cap \mathsf{Clo}(V)$. In that case, the set of points that satisfy $O\varphi \wedge O\neg\varphi$ is not necessarily empty. Thus, an arbitrary $O\psi$ does not necessarily follow from a contradiction.

The topological semantics of the CtD modality is given as follows.

$$
\begin{aligned}
M, w \models C(\varphi, \psi) \;\; \text{iff} \;\; & \forall v. w \in \mathsf{Clo}(v).\{ \\
& \exists U \in \tau \text{ with } v \in \mathsf{Clo}(U). \forall x (x \in \mathsf{Clo}(U) \to x \models \varphi) \\
& \text{and } \exists y \in \mathsf{Clo}(v) \text{ such that } y \not\models \varphi \text{ or } \neg\varphi \in N(v) \\
& \text{imply} \\
& \exists V \in \tau \text{ with } v \in \mathsf{Clo}(V). \forall t (t \in \mathsf{Clo}(V) \to t \models \psi) \; \}
\end{aligned}
$$

The topological semantics of $C(\cdot, \cdot)$ shows how we can obtain inconsistent contrary-to-duty obligations. If V is a subset of the set theoretical complement of the set of points that satisfy φ, then we obtain inconsistent obligations, hence CtD imperatives.

We now observe that topological semantics is sound and complete with respect to the axiomatization of C_1^{DD} given in Section 2.1.

Theorem 4.1. *Topological semantics for C_1^{DD} is sound and complete.*

Proof. The soundness is immediate, hence we skip it.

For the completeness of the base system, we refer the reader to [7]. We will only show it here for the modal operator O, which was not given before.

The topological completeness follows a familiar argument using canonical models. Given a topological model $M = (S, \sigma, V, N)$, we construct the canonical model $M' = (S', \sigma', V', N')$ where S' is the set of all maximal non-trivial set. A set X is called *maximal non-trivial* if $\varphi \notin X$, then $X \cup \{\varphi\}$ is trivial. The canonical valuation V' and the function N' is defined in the usual way as described in [ibid]. The topology σ' is generated by two bases B and D independently. First, we define the basis $B = \{\widehat{\neg\varphi} : \text{ any formula } \varphi\}$ where $\widehat{\varphi} = \{s' \in S : \varphi \in s'\}$. The basis B is used to prove completeness for the negation operator and its details can be found in [7].

Now, we define the basis $D = \{\widehat{O\varphi} : \text{ any formula } \varphi\}$. Next, we need to show that D is a basis for σ'. For this, we need to show

1. For any $U, U' \in D$ and any $x \in U \cap U'$, there is $U_x \in D$ such that $x \in U_x \subseteq U \cap U'$,
2. For any $x \in S'$, there is $U \in D$ with $x \in U$.

First condition is satisfied as $O(\widehat{\varphi \wedge \psi}) = \widehat{O\varphi} \cap \widehat{O\psi}$ since we have $O(\varphi \wedge \psi) \equiv O\varphi \wedge O\psi$ in C_1^{DD} as discussed in Section 2.1. Similarly, the classical constant $\top$ can easily be defined in C_1^{DD} as $p \vee \neg p$. Now, similar to the classical case, $\top \in x$ for any maximal consistent set x. Therefore, for each $x \in S'$, there is $O\top \in D$ that includes x. Thus, D is a basis for σ'.

For the truth lemma (towards the completeness), for any formula φ, we need to observe that $s' \models \varphi$ if and only if $s' \in \widehat{\varphi}$ for $s' \in S'$. The standard cases are obvious, the negation was proved earlier in [ibid], now we prove it for the modal formula $O\varphi$.

Let $s' \models O\varphi$. Then, as D is a basis, there is a (closed) set $O\psi \in D$ such that $s' \in \widehat{O\psi}$ and, by definition, $t' \models \varphi$ for all $t' \in \mathsf{Clo}(\widehat{O\psi})$. By the induction hypothesis for $t' \in S'$, we have $t' \in \widehat{\varphi}$ for all $t' \in \mathsf{Clo}(\widehat{O\psi})$. Therefore, $\mathsf{Clo}(\widehat{O\psi}) \subseteq \widehat{\varphi}$. Taking the closure of both sides we observe $\mathsf{Clo}(\widehat{O\psi}) \subseteq \mathsf{Clo}(\widehat{\varphi})$ as the Clo operator is idempotent. As the O operator semantically quantifies over the closure, then $\mathsf{Clo}(\widehat{\varphi}) \subseteq \mathsf{Clo}(\widehat{O\varphi})$. Thus, $\mathsf{Clo}(\widehat{O\psi}) \subseteq \mathsf{Clo}(\widehat{O\varphi})$ and $s' \in \widehat{O\varphi}$.

For the converse direction let $s' \in \widehat{O\varphi}$. Since $\widehat{O\varphi} \in D$, it is open and we have $\widehat{O\varphi} \subseteq \mathsf{Clo}(\widehat{O\varphi})$. Then, by definition, there is $U \in \sigma'$ such that $s' \in \mathsf{Clo}(U)$ and for all $t' \in \mathsf{Clo}(U)$, we have $t' \models \varphi$. Then, by the induction hypothesis, $t' \in \widehat{\varphi}$. Simply, take U as $\widehat{O\varphi}$ to see $s' \models O\varphi$.

After establishing the truth lemma, the completeness follows as expected hence skipped.

Since $C(\cdot,\cdot)$ is definable in terms of the modal operator O, the completeness of C_1^{DD} follows. □

*

Thus far, we have presented various semantics for paraconsistent CtD imperatives. This boldly underlines the breadth and depth of the subject and that how it can be viewed as a pluralistic and rich framework. The modal, dynamic and topological approaches to CtD imperatives make it possible to integrate C_1^{DD} to pre-existing logical systems that are based on the aforementioned formalisms.

5 Some Further Applications

In this section, we present some applications and extensions of paraconsistent CtD obligations.

5.1 Some Well-known Paradoxes

Adopting a paraconsistent deontic logic to express CtD obligation by no means suggests that it is a universal solution for the paradoxes of deontic logic. Now, we note that C_1^{DD} may not be helpful in some well-known deontic paradoxes.

Åqvist's Paradox The well-known Åqvist's Paradox [2] can be satisfied in C_1^{DD}. In order to see this, consider the paradox paraphrased as follows.

- The bank is being robbed.
- It ought to be the case that the guard knows that the bank is being robbed.
- So, it ought to be the case that the bank is being robbed.

It is a paradox in classical logic as it is not possible to have $OK\varphi \wedge OK\neg\varphi$ classically as it reduces first to $O\varphi \wedge O\neg\varphi$, then to $O(\varphi \wedge \neg\varphi)$. In C_1^{DD}, properly extended with the epistemic modality K, and $O(\varphi \wedge \neg\varphi)$ is satisfiable.

Ross's Paradox Ross's Paradox is formulated by the following two sentences.

- It is obligatory that the letter is mailed.
- It is obligatory that the letter is mailed or the letter is burned.

Since $O\varphi \to O(\varphi \vee \psi)$ is a theorem in C_1^{DD}, the dilemma remains valid. The following proposition sheds some light to the issue from a dynamic deontic perspective. We skip the proof as it directly follows from the definitions.

Proposition 5.1. *For a C_1^{DD} model M, and formulas φ, ψ, we have $M|\varphi \subseteq M|\varphi \vee \psi$.*

5.2 Arbitrary Violations

Even if some obligations are violated, there may still remain some other obligations. Moreover, if *all* possible obligations are violated, it can still be claimed that there still exist some obligations to be fulfilled. For example, following *the rule of law* remains an obligation even under war or economical collapse of a nation. Then, the immediate question is whether there can exist contrary-to-duty obligations after *each and every* obligation is violated.

The dynamic - paraconsistent reading of deontic violations suggests an immediate formalism of such situations which can be familiar from dynamic epistemic logic where what can be known after every epistemic announcement is studied [3].

Let us briefly analyze this phenomenon. First, we extend the syntax of C_1^{DD} with a monadic modal operator $\Box\psi$ which express that "after every violation of $O\varphi$, ψ is still an obligation". The semantics for this operator is given as follows.

$$M, w \models \Box\psi \quad \text{iff} \quad \forall\varphi \in \mathcal{L}_O . M, w \models C(\varphi, \psi)$$

We denote this system by AC_1^{DD}. We define the dual of $\Box$ in the usual sense and denote it with $\Diamond$. Let us observe some modal logical validities of AC_1^{DD}.

Proposition 5.2. *Let $\varphi, \psi \in \mathcal{L}_O$. Then,*

- $\Box(\varphi \wedge \psi) \leftrightarrow (\Box\varphi \wedge \Box\psi)$
- $\Box\varphi \to \Box\Box\varphi$
- $\Box\varphi \to O\varphi$
- $\models \varphi$ *implies* $\models \Box\varphi$

The above results follow immediately from the definitions. A paraconsistent framework is essential in understanding the arbitrary violations as the truth of $\Box\psi$ requires that ψ remains an obligation after every possible violation of $O\varphi$ including the cases where φ can be the negation of ψ. In other words, arbitrary violations *persist* even after their own violations.

Proposition 5.3. $\Box\varphi \to C(\varphi, \varphi)$ *is satisfiable in* AC_1^{DD}.

The proof of the above proposition follows directly from the definitions. We leave the full formal treatment of AC_1^{DD} to future work.

5.3 Conditional Obligations

Van Fraassen suggested a conditional obligation operator $O(\varphi, \psi)$ which reads as "under conditions satisfying ψ, φ ought to be satisfied" [45]. In this reformulation, the unary modality $O\varphi$ is defined as $O(\varphi, \psi \to \psi)$. A connection between CtD imperatives and conditional obligations was made in [41], now we will make this connection dynamic and inconsistency-friendly.

First, CtD obligations can be represented by conditional obligations. In this case, we put $C(\varphi, \psi) := O(\psi, \neg\varphi \wedge O(\varphi, \psi \to \psi))$ or equivalently $C(\varphi, \psi) := O(\psi, \neg\varphi \wedge O\varphi)$.

Some theorems of conditional obligations hold in C_1^{DD}: $O(\varphi, \psi) \equiv O(\varphi \vee \neg\psi, \psi)$. For example, $O(\varphi, \psi) \wedge O(\chi, \psi) \to O(\varphi \wedge \chi, \psi)$ turns out to valid as well in C_1^{DD} even if φ and χ can be contradictory. Additionally, C_1^{DD} allows satisfiable statements such as $O(\varphi \wedge \neg\varphi, \psi)$ or $O(\varphi, \psi \wedge \neg\psi)$. This observation is in-line with van Fraassen's "sceptical postscript", and expands it more formally towards the direction of conflicting obligations.

The connection between CtD imperatives and conditional obligations can be taken further as follows.

Proposition 5.4. *For any model M, and a state w in it, $M|\varphi, w \models \psi$ implies $M|\varphi, w \models O(\varphi, \psi)$ for any formulas φ, ψ.*

Proof. Let $M|\varphi, w \models \psi$. Then, by Proposition 3.3 $M|\varphi, w \models O\varphi \wedge \psi$ which implies that $M|\varphi, w \models O(\varphi, \psi)$. □

The logic C_1^{DD} allows some relatively counter-intuitive statements including $O(\varphi, O\neg\varphi)$. However, some situations may necessitate its use. For example, consider a totalitarian country where it is obligatory *not* to protest the government. In that case, it can very well be claimed that under such an unacceptable restriction, *then* it becomes obligatory to protest the government, and the latter obligation surfaces only after the initial obligation that prohibits it. Thus, the formula $O(\varphi, O\neg\varphi)$ gains some context in C_1^{DD}.

6 Conclusion

What we have shown so far agrees with Lemmon's earlier observation that a conflict of obligations does not necessarily entail an ill-behaving contradiction [28]. Based on this observation, the contribution of this paper is two-fold. First, the syntactic $C(\cdot, \cdot)$ operator provides a succinct way to express an inherently paraconsistent understanding of obligations and their inconsistencies. In this way, it responds to some of the philosophical questions regarding the CtD obligations and contradictions posed in [22]. Second, the dynamic and topological readings of the aforementioned operator explicate that the CtD operator provides not only succinctness but also a dynamic take on the subject, which remains largely unexplored within paraconsistent logics. Finally, C-systems possess an internal mechanism to classify inconsistencies, colloquially called 'good'

ones and 'bad' ones. This helps us understand which propositions can never be inconsistent, and which ones can be tolerated. This distinction is quite helpful in various applications in computer science as we have briefly described.

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