A Geometrical - Epistemic Approach to Lakatosian Heuristics

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1 Introduction

Lakatos's *Proofs and Refutations* (PR, henceforth) exhibits a careful analysis of a very significant mathematical development, namely the evolution of Euler's Theorem $V - E + F = 2$ for three dimensional polyhedra where V, E and F are the number of vertices, edges, and faces of the polyhedron respectively. In his analysis, Lakatos introduced several notions in order to be able to give a rational account of the historical and the methodological development of the theorem. In this regard, for Lakatos, the *development* of a mathematical theorem together with its proof was a very significant aspect of the growth of mathematical knowledge.

In this work, our aim is to utilize a rather weak but expressive geometrical epistemic logic to formalize the Lakatosian heuristics. First, we will recall Lakatos's methodology of mathematics and briefly introduce the epistemic logic which we will use along these lines. After providing the necessary background, we will establish the connection between the two.

1.1 Lakatosian Heuristics

The Lakatosian method of proofs and refutations consists of the following steps [3]: 1.Primitive conjecture; 2. Proof (a rough thought experiment or argument); 3. Global counterexamples; 4. Proof is re-examined and the guilty lemma is spotted. 5. Proofs of the other theorems are examined to see if the newly found lemma occurs in them; 6. Hitherto accepted consequences of the original and now refuted conjecture are checked; 7. Counterexamples are turned into new examples, and new fields of inquiry open up.

Lakatos employed three main strategies to implement his method of *proofs and refutations*: *monster-barring*, *exception-barring* and *lemma incorporation*. The method of "monster-barring" deals with the objects which are *not in mind* when the conjecture is put forward. They are, in this sense, monsters and should be excluded from our domain of discourse. "Exception-barring" accepts that the theorem in its stated form is not valid due to the emergence of some genuine counterexamples targeting the correctness of the theorem itself. The last method is called "lemma incorporation" which characterize the way the counterexamples are turned into new examples. This methodology shows that 'in Lakatos's heuristics, the theorem is not ready when we start to prove it. It is stated in a possibly false generality, and it can be formulated several times in the process [of its development]" [4].

1.2 *Proofs and Refutations*

In this work, we focus on *Proofs and Refutations*. The main conjecture which was discussed in PR is the following.

$$V - E + F = 2$$

for all polyhedra, where V, E and F denote the number of the vertices, the edges and the faces of the given polyhedron respectively. The integer value which is obtained from the equation $V - E + F$ is called the *Euler characteristic*. The proof of this theorem was due to Cauchy. Let us briefly recall it step by step.

Step 1 Imagine that the polyhedron is hollow and made of a rubber sheet. Cut out one of the faces and, stretch the remaining faces to a flat surface (or board) without tearing it apart. In this process, V and E does not change. However, as we removed a face, the Euler characteristics of the polyhedron decreases by one. Thus, we now need to show that $V - E + F = 1$.

Step 2 The remaining map is triangulated. Drawing diagonals for the curvilinear polygons will not alter $V - E + F$ since E and F increase simultaneously by the same amount while V does not change.

Step 3 Remove the triangles. It can be done in two ways: either one edge and one face are removed simultaneously; or one face, one vertex and two edges are removed simultaneously. During this process, $V - E + F$ remains unchanged. Consequently, at the end of this process, we will end up with an ordinary triangle for which $V - E + F = 1$ holds trivially.

Observe that there are three lemmas that have been used implicitly throughout the proof.

Lemma 1 Any polyhedron, after a face is removed, can be stretched flat on a flat surface.

Lemma 2 In triangulating the map, one will always get a new face for every new edge.

Lemma 3 There are only two alternatives for removing a triangle out of the triangulating map: the disappearance of one edge or else of two edges and a vertex - when one decreases the number of triangles by one. Furthermore, one will end up with a single triangle at the end of this process.

Now, the very first counterexamples emerge upon the three lemmas which were used in the proof. *Local counterexamples* deny the specific lemmas or the constructions which were utilized in the proof without targeting the main conjecture itself. However, *global counterexamples* deny the main conjecture.

At this stage, Lakatos gives an intuitive discussion of geometric topology. For instance, the picture-frame (i.e. torus) cannot be inflated into a sphere or cannot be stretched onto an Euclidean plane. The reason for that is the fact that the *genus* of the sphere is zero (i.e. it has no holes), even if we remove a single point from the sphere. Removing a single point from a sphere makes it possible to stretch it onto the Euclidean plane. Moreover, the sphere with one single point removed is homeomorphic to the Euclidean plane - there exists a bicontinuous isomorphism between the two. Thus, it is topologically the “same” to stretch a polyhedron onto the Euclidean plane or onto a sphere with one point removed. Then, it was concluded that the picture-frame could be inflated only onto a torus, not onto a sphere. It should now be recalled that the torus has genus one, namely it has one hole passing through it. Indeed, the general formula of the Euler characteristics in (oriented) manifolds is the following.

$$V - E + F = 2 - 2.g(S)$$

where S is the surface we consider on which the polyhedron will be inflated, and $g(S)$ is the genus of the surface S.

The discussion we have presented so far exhibits the central idea of Lakatos's heuristics. We start with a proposition (Euler's Conjecture), and then restrict the domain of the objects to those for which the conjecture holds. Then, by monster-adjustment and exception-barring, we reconstruct our epistemic model. The newly reconstructed model, in this case, depends on the previous one. The existence of the geometric objects whose Euler characteristics are not 2 suggests that the use of *possible world semantics* to represent Lakatosian heuristics is worth pursuing. Moreover, the dynamic aspect of updates and the theory changes encouraged us to use an epistemic modal logic.

1.3 Subset Space Logic

In order to give a formal account of the Lakatosian heuristics, we will use *subset space logic* which was put forward to formalize reasoning about sets and points. The motivation behind this logic was the following observation about the topological notion of *closeness*: “ In order to *get close*, one needs to make some *effort*” [10]. By spending some effort, we eliminate some of the existing possibilities, and obtain a smaller set of possibilities.

The language of subset space logic $\mathcal{L}_{SSL}$ has a countable set P of proposition letters, a truth constant $\top$, the usual Boolean operators $\neg$ and $\wedge$, and two modal operators K (knowledge modality) and $\Box$ (shrinking modality). The formulae in the language $\mathcal{L}_{SSL}$ are obtained from atomic propositions by closing them under $\neg$, $\wedge$, K and $\Box$. A *subset frame* is a pair $\mathcal{S} = \langle S, \sigma \rangle$ where S is a set of points and σ is a set of subsets of S. However, note that σ is not necessarily a topology. The elements of σ are called *observations*. The triple $\mathcal{S} = \langle S, \sigma, v \rangle$ is called a subset space model where $\langle S, \sigma \rangle$ is a subset frame, $v : P \to \wp(S)$ is a valuation function. We can now define the semantics of subset spaces.

Definition 11 *For $s \in S$ and $s \in U \in \sigma$ in the subset space model $\mathcal{S} = \langle S, \sigma, v \rangle$, we define the satisfaction relation $\models_{\mathcal{S}}$ on $(S \times \sigma) \times \mathcal{L}_{SSL}$ by induction on the length of the formulae. We will drop the subscript $_{\mathcal{S}}$ when the subset space model we are in is obvious.*

$$
\begin{array}{ll}
s, U \models p & \textit{iff } s \in v(p) \\
s, U \models \varphi \wedge \psi & \textit{iff } s, U \models \varphi \textit{ and } s, U \models \psi \\
s, U \models \neg\varphi & \textit{iff } s, U \not\models \varphi \\
s, U \models \mathsf{K}\varphi & \textit{iff } t, U \models \varphi \textit{ for all } t \in U \\
s, U \models \Box\varphi & \textit{iff } s, V \models \varphi \textit{ for all } V \in \sigma \textit{ such that } s \in V \subseteq U
\end{array}
$$

The duals of $\Box$ and K are $\Diamond$ and L respectively and defined in the usual way. If we *know* φ at (s, U), then we can move from the point s to any other point t in U. On the other hand, if we arrive at the truth by spending any effort, then we shrink the neighborhood U to another subset $V \subseteq U$ keeping the point s intact.

SSL is axiomatized by the $\mathsf{S5}$ axioms for the K modality and by the $\mathsf{S4}$ modality for the $\Box$ modality. However, we need an additional axiom for their interaction: $\mathsf{K}\Box\varphi \rightarrow \Box\mathsf{K}\varphi$. The rules of inferences of SSL are modus ponens and necessitation for both modalities. SSL is thus sound and complete with respect to this axiomatization.

Notice that the shrinking modality is a dynamic modality. Any improvement in the knowledge can be represented by the shrinking modality $\Diamond$. Note that $\Diamond$ modality suffers from "$\exists$-sickness". The modal $\Diamond$ states that there *exists* a subset, but which subset is it? In a previous work, we described a system in which such subsets can be specified precisely. We call such systems "Controlled Subset Spaces" [1].

We call the function f a *contraction mapping*, if for every subset U in its domain, we have $fU \subseteq U$. Furthermore, let $\mathcal{F}$ be an arbitrary collection of contraction mappings which are defined on S, and further let $F \subseteq \mathcal{F}$ be some selection of such contraction mappings. Given a subset space $\mathcal{S} = \langle S, \sigma, v\rangle$, we obtain the *image space* $\mathcal{S}_F = \langle S, \sigma_F, v\rangle$ where $\sigma_F := \{fU : f \in F, U \in \sigma\}$. In this setting, the valuation v does not change.

Intuitively, the contracting maps can be thought of as *methods* or *(thought) experiments* which result in an improvement in knowledge. Among the admissible set of methods $\mathcal{F}$, we select some by specifying the set F. Thus, we will get around the $\exists$-sickness problem.

Definition 12 *$\mathcal{S} = \langle S, \sigma, v, \mathcal{F}\rangle$ is called a controlled subset space where S is a set, σ is any collection of the subsets of S, $v : P \rightarrow \wp(S)$ is a valuation function and $\mathcal{F}$ is a collection of contraction mappings defined on S.*

We will now introduce an additional modality $[F]$ representing the controlled shrinking. The intended meaning of the statement $[F]\varphi$ is that "after the application of each contracting mapping $f \in F$, φ becomes true". The dual of $[F]$ is denoted by $\langle F\rangle$ and defined in the usual way. The semantics of the new modality is defined as follows.

$s, U \models_{\mathcal{S}} [F]\varphi$ iff $s, fU \models_{\mathcal{S}_F} \varphi$ for each $f \in F$.

2 A Formalization of Lakatosian Heuristics

Based on the above observations, we will now present a formalization of the Lakatosian approach. Let us start with the method of monster barring. Recall that monster barring restricts the domain of the objects which were initially supposed to satisfy the given conjecture. However, the "contraction" of the domain is not random as it is governed by an observation, an idea, a computation or a thought-experiment, or even some combination of them. The following function, in this setting, eliminates the states which do not agree with the formula $V - E + F = 2$.

Let f be the function which returns the input object x as the output only if the given object x satisfies the Euler conjecture $V(x) - E(x) + F(x) = 2$. More precisely, f is given as follows.

$$f(x) = \begin{cases} x & : \text{ if } V(x) - E(x) + F(x) = 2 \\ \text{undefined} & : \text{otherwise} \end{cases} \tag{1}$$

Observe that f is a well-defined contraction mapping. The underlying motivation to define f as such is to mimic the characteristic function of the set of objects whose Euler characteristics are 2.

If we happen to consider some other Euler characteristics, we only need to include them as functions. In a similar fashion, let f' be the contraction mapping for the Euler characteristics 0. Similarly, the precise definition is as follows.

$$f'(x) = \begin{cases} x & : \text{ if } V(x) - E(x) + F(x) = 0 \\ \text{undefined} & : \text{otherwise} \end{cases} \tag{2}$$

The second strategy of the methods of proofs and refutations is the method of exception barring. Exception barring works exactly the same as the method of monster barring up until the point of turning monsters into the examples of some other conjecture. Exception barring, in this sense, stops after the domain is contracted.

The third strategy is perhaps the most crucial one. The method of lemma incorporation suggests us to extend our set of functions in consideration. This is precisely what we did for torus in the above illustration. We incorporated the lemma in such a way that f' will work for t. One of the ways to achieve this to introduce the conditions that stems from the modified lemma into the formulation of the function. For instance, in PR, the notion of genus was introduced to discuss non-simple polyhedra. Then the general form of the Euler conjecture, as we discussed previously, becomes $V(x) - E(x) + F(x) = 2 - 2.g(x)$ in the oriented objects such as torus where $g(x)$ is the genus (i.e. the number of holes) of the object x. Based on this reformulation, we can restate the function f given in the Equation 1 as follows.

$$h(x) = \begin{cases} x & : \text{ if } V(x) - E(x) + F(x) = 2 \wedge g(x) = 0 \\ \text{undefined} & : \text{otherwise} \end{cases} \tag{3}$$

In a similar fashion, we can incorporate the lemma which addresses the simplicity condition to the Equation 2. The following equation does the job.

$$h'(x) = \begin{cases} x & : \text{ if } V(x) - E(x) + F(x) = 0 \wedge g(x) = 1 \\ \text{undefined} & : \text{otherwise} \end{cases} \tag{4}$$

The constructions we have presented hitherto give us sufficient tools to formalize the method of proofs and refutations. Let us now consider a single case to examplify all the discussion.

2.1 An Application and Generalization

Let us now consider a torus t, and see how the Lakatosian heuristics for the Euler conjecture instantiated to torus can be formalized by using the controlled subset space logic. Based on the above illustrations and formulations, our job is now easy. We suggest that the following equation

$$t, U \models [f]\chi \vee \langle F \rangle \chi' \tag{5}$$

is sufficient to express Lakatosian heuristics in this context.

Starting off from the torus t in the observation set U, we observe and calculate that the torus does not satisfy the Euler formula $V - E + F = 2$. Thus, we need to modify the original formula f (cf. Equation 1) to get a new formula f' (cf. Equation 2) and construct the set of formula $F = \{f, f'\}$. Then, among the functions in F, we can pick the correct one that is satisfied at the actual state t.

Based on the observations and the applications we have presented hitherto, we can now give a general account of Lakatosian heuristics for arbitrary formualae. Let $T(\boldsymbol{x})$ be the theorem in question with free variables $\boldsymbol{x}$. We can formalize the Lakatosian heuristics of the development of the theorem $T(\boldsymbol{x})$ with the input vector $\boldsymbol{x}$ as follows. For simplicity let us assume that the current epistemic neighborhood situation we are in is given (s, U).

$$\varphi(\boldsymbol{x}) = \begin{cases} \boldsymbol{x} & : \text{ if } T(\boldsymbol{x}) \text{ holds} \\ \text{undefined} & : \text{otherwise} \end{cases} \tag{6}$$

The simple idea behind this formalization is to define knowledge emprically. In the contracted neighborhood situation $(s, \varphi U)$ we know the propositions which are emprically verified under the function φ. This reading of the controlled subset spaces agrees perfectly with the Lakatosian understanding of mathematics as a *quasi-emprical* science. Observe that the current focus in the controlled subset spaces is on the verification, not on the discovery. Among the set of functions that *may* work, we try to find and verify the one that works.

3 Conclusion

Applying computational and logical methods to philosophy of science is not a new idea. An early work on the very same subject suggested an algorithmic and computational model for Lakatosian philosophy of science [9]. The underlying idea for the possibility of employing such methods in Lakatosian philosophy of mathematics is Lakatos's understanding that mathematics is a quasi-empirical activity. For Lakatos, thought-experiments reflect the empiric side of the mathematical practice. However, we are very well aware of the fact that Lakatos's vague notion of heuristics cannot be fully formalized. One of the reasons for that is the mixed behavior of positive and negative heuristics. There are examples and counterexamples in PR which behave both positively and negatively [2]. Nevertheless, what we have provided here is a new and more rigorous way to understand Lakatosian heuristics: our contribution shows that Lakatosian heuristics is not random, and thus follows a pattern. Even though Lakatos's *rationally reconstructed* presentation does not perfectly reflect the actual history of Euler's conjecture [6], his approach constitutes a very significant contribution to the discussions on the methodology of mathematics, and we aimed at formalizing it.

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