Paraconsistency and Games

a quick course

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April 5th & 6th, 2016

CLE, UNICAMP - Campinas, Brazil

Slogan: Paraconsistency for Game Theory!

Paraconsistency helps us understand

game theory better,

and

game theory helps us understand

paraconsistency better.

Lecture One

Paraconsistent Game Sematics

Outlook of Lecture One

- Logic of Paradox
- First-Degree Entailment
- Belnap's 4-valued Logic
- Connexive Logic

Classical Game Semantics

During the semantic verification game, the given formula is broken into subformulas by **two** players (Abelard and Heloise) **step by step**, and the game **terminates** when it reaches the propositional atoms.

If we end up with a propositional atom which is true, then Heloise the verifier wins the game. **Otherwise**, Abelard the falsifier wins. We associate **conjunction with Abelard**, **disjunction with Heloise**.

A win for the verifier is when the game terminates with a true statement. The verifier is said to have a winning strategy if she can force the game to her win, regardless of how her opponent plays.

Classical Game Semantics

Just because the game may end with a true/false atom does not necessarily suggest the truth/falsity of the given formula in general.

In classical logic, however, the major result of game theoretical semantics states that the verifier has a winning strategy **if and only if** the given formula is true in the model.

Classical Games

Classical semantic games are

- Two-player,
- Determined,
- Sequential,
- Zero-sum,
- Complete: winning strategies necessarily and sufficiently guarantee the truth value.

Question How do these attributes of semantical games depend on the underlying logical structure? How can we give game semantics for *deviant* logics?

Logic of Paradox

Logic of Paradox and GTS

Consider Priest's Logic of Paradox (LP) (Priest, 1979).

LP introduces an additional truth value P, called *paradoxical*, that stands for both true and false.

	$\neg$
T	F
P	P
F	T

$\wedge$	T	P	F
T	T	P	F
P	P	P	F
F	F	F	F

$\vee$	T	P	F
T	T	T	T
P	T	P	P
F	T	P	F

Game Models

We define the verification game as a tuple $\Gamma = (\pi, \rho, \delta, \sigma)$ where

- π is the set of players,

- ρ is the set of well-defined game rules,

- δ is the set of designated truth values: the truth values preserved under validities: they determine the theorems of the logic.

- σ is the set of positions: subformula and player pairs.

It is possible to extend it to concurrent games as well.

Game Rules for LP

The introduction of the additional truth value *P* requires an additional player in the game, let us call him *Astrolabe* (after Abelard and Heloise's son).

Since we have three truth values in LP, we need three players forcing the game to their win. If the game ends up in their truth set, then that player wins.

Then, how to associate moves with the connectives?

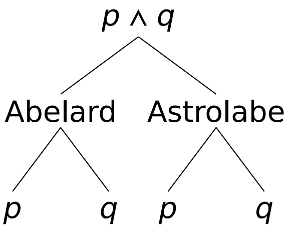
Game Rules for LP

Denote this system with GTS^{LP}.

p	whoever has p in their extension, wins
$\neg F$	Abelard and Heloise switch roles
$F \wedge G$	Abelard and Astrolabe choose between F and G simultaneously
$F \vee G$	Heloise and Astrolabe choose between F and G simultaneously

An Example

Consider the conjunction. Take the formula $p \wedge q$ where p, q are P, F respectively. Then, $p \wedge q$ is F.



Abelard makes a move and chooses q which is false. This gives him a win. Interesting enough, Astrolabe chooses p giving him a win.

In this case both seem to have a winning strategy. Moreover, the win for Abelard does not entail a loss for Astrolabe.

Correctness

Theorem

In GTS^{LP} verification game for φ,

- Heloise has a winning strategy if φ is true,
- Abelard has a winning strategy if φ is false,
- Astrolabe has a winning strategy if φ is paradoxical.

The contra-positive of the above theorem is also useful.

Correctness

Theorem

In a $\mathrm{GTS}^{\mathrm{LP}}$ game for a formula φ in a LP model M,

- If Heloise has a winning strategy, but Astrolabe does not, then φ is true (and only true) in M,
- If Abelard has a winning strategy, but Astrolabe does not, then φ is false (and only false) in M,
- If Astrolabe has a winning strategy, then φ is paradoxical in in M.

First-Degree Entailment

First-Degree Entailment

Semantic valuations are *functions* from formulas to truth values.

If we replace the valuation function with a valuation *relation*, we obtain *First-degree entailment* (FDE) which is due to Dunn (Dunn, 1976).

We use $\varphi \mathbf{r} 1$ to denote the truth value of φ (which is 1 in this case).

Since, $\mathbf{r}$ is a relation, we allow $\varphi \mathbf{r} \emptyset$, and both $\varphi \mathbf{r} 0$ and $\varphi \mathbf{r} 1$.

Thus, FDE is a paraconsistent (inconsistency-tolerant) and paracomplete (incompleteness-tolerant) logic.

First-Degree Entailment

For formulas φ, ψ, we define $\mathbf{r}$ as follows.

$$
\begin{array}{lll}
\neg\varphi\mathbf{r}1 & \textit{iff} & \varphi\mathbf{r}0 \\
\neg\varphi\mathbf{r}0 & \textit{iff} & \varphi\mathbf{r}1 \\
(\varphi \wedge \psi)\mathbf{r}1 & \textit{iff} & \varphi\mathbf{r}1 \textit{ and } \psi\mathbf{r}1 \\
(\varphi \wedge \psi)\mathbf{r}0 & \textit{iff} & \varphi\mathbf{r}0 \textit{ or } \psi\mathbf{r}0 \\
(\varphi \vee \psi)\mathbf{r}1 & \textit{iff} & \varphi\mathbf{r}1 \textit{ or } \psi\mathbf{r}1 \\
(\varphi \vee \psi)\mathbf{r}0 & \textit{iff} & \varphi\mathbf{r}0 \textit{ and } \psi\mathbf{r}0
\end{array}
$$

Game Semantics for FDE

The truth values $\{0\}, \{1\}$ and $\{0, 1\}$ work exactly as the truth values F, T, P respectively in LP. In fact, LP can be obtained from FDE by introducing a restriction that no formula gets the truth value $\emptyset$.

Recall that for GTS^{LP}, we allowed parallel plays for selected players depending on the syntax of the formula: we associated conjunction with Abelard and Astrolabe, disjunction with Heloise and Astrolabe.

Game Semantics for FDE

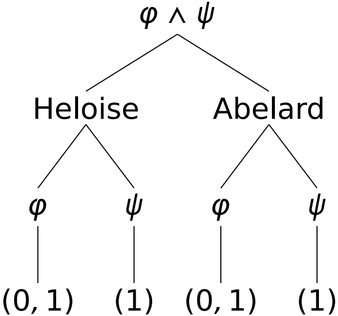
For FDE, the idea is to allow each player play at each node.

Therefore, it is possible that both players (or none) may have a winning strategy.

An Example

Consider two formulas with the following relational semantics: $\varphi\mathbf{r}0$, $\varphi\mathbf{r}1$ and $\psi\mathbf{r}1$. In this case, we have $(\varphi \wedge \psi)\mathbf{r}1$ and $(\varphi \wedge \psi)\mathbf{r}0$.

We expect both Abelard and Heloise have winning strategies, and allow each player make a move at each node.



Game Rules for FDE

p	whoever has p in their extension, wins
$\neg F$	players switch roles
$F \wedge G$	Abelard and Heloise choose between F and G simultaneously
$F \vee G$	Abelard and Heloise choose between F and G simultaneously

Correctness

Theorem

In a $\mathrm{GTS}^{\mathrm{FDE}}$ verification game for a formula φ, we have the following:

- Heloise has a winning strategy if $\varphi \mathbf{r} 1$
- Abelard has a winning strategy if $\varphi \mathbf{r} 0$
- No player has a winning strategy if $\varphi \mathbf{r} \varnothing$

Belnap's 4-Valued Logic

Belnap's 4-Valued Logic

Belnap's 4-Valued system, call it B4, introduces two non-classical truth values. Traditionally, P stands for both truth values and N stands for neither of the truth values.

	$\neg$
T	F
P	P
N	N
F	T

$\wedge$	T	P	N	F
T	T	P	N	F
P	P	P	F	F
N	N	F	N	F
F	F	F	F	F

$\vee$	T	P	N	F
T	T	T	T	T
P	T	P	T	P
N	T	T	N	N
F	T	P	N	F

Notice that P and N are the fixed-points under negation.

Game Rules for B4

From a game-semantics perspective, the problems with B4 include

- Two fixed-points for negation
- Non-monotonicity: two truth values may produce a third truth value under binary connectives

In particular, we have $P \wedge N = F$ and $P \vee N = T$.

Game Rules for B4

Let us have 4 players for 4 truth values:

The truth value T is forced by Heloise, F by Abelard, P by Astrolabe and N by Bernard[1].

Two negation-fixed-points suggest that Astrolabe and Bernard both will be the concurrent players.

[1] After Abelard's rival Bernard of Clairvaux.

Game Rules for B4

p	whoever has p in their extension, wins
$\neg F$	Heloise assumes Abelard's role, Abelard assumes Heloise's role, Astrolabe and Bernard keep their previous roles, and the game continues with F,
$F \wedge G$	if Bernard has a winning strategy for F and Astrolabe has a winning strategy for G, then Abelard wins,
$F \wedge G$	otherwise Abelard, Astrolabe and Bernard choose simultaneously between F and G,
$F \vee G$	if Bernard has a winning strategy for F and Astrolabe has a winning strategy for G, then Heloise wins,
$F \vee G$	otherwise Heloise, Astrolabe and Bernard choose simultaneously between F and G.

Correctness

Theorem

For the evaluation games for a formula φ in Belnap’s 4-valued logic, we have the following:

- Heloise the verifier has a winning strategy if φ evaluates to T,
- Abelard the falsifier has a winning strategy if φ evaluates to F,
- Astrolabe the paradoxifier has a winning strategy if φ evaluates to P,
- Bernard the nullifier has a winning strategy if φ evaluates to N.

Connexive Logic

McCall's Connexive Logic

Connexive logic is a "comparatively little-known and to some extent neglected branch of non-classical logic" (Wansing, 2015). Even if it is under-studied, its philosophical roots can be traced back to Aristotle and Boethius.

Connexive logic is defined as a system which satisfies the following two schemes of conditionals:

- **Aristotle's Theses**: $\neg(\neg\varphi \to \varphi)$
- **Boethius' Theses**: $(\varphi \to \neg\psi) \to \neg(\varphi \to \psi)$

In this work, we discuss one of the earliest examples of connexive logics CC, which is due to McCall (McCall, 1966).

McCall's Connexive Logic

CC is axiomatized by adding the scheme $(\varphi \to \varphi) \to \neg(\varphi \to \neg\varphi)$ to the propositional logic. The rules of inference for CC is modus ponens and adjunction, which is given as $\vdash \varphi, \vdash \psi \therefore \vdash \varphi \wedge \psi$.

The semantics for CC is given with 4 truth values: *T*, *t*, *f* and *F* which can be viewed as “logical necessity", “contingent truth", “contingent falsehood", and “logical impossibility" respectively (Routley & Montgomery, 1968).

In CC, the designated truth values are *T* and *t*.

McCall's Connexive Logic

	$\neg$
T	F
t	f
f	t
F	T

$\wedge$	T	t	f	F
T	T	t	f	F
t	t	T	F	f
f	f	F	f	F
F	F	f	F	f

$\vee$	T	t	f	F
T	t	T	t	T
t	T	t	T	t
f	t	T	F	f
F	T	t	f	F

First, we introduce 4 players for 4 truth values: T is forced by Heloise, F by Abelard, t by Aristotle and f by Boethius.

Game Rules for CC

As the *trues* and *falses* are closed under the binary operations respectively, we suggest the following coalitions.

Truth-maker Coalition:

Heloise (T) and Aristotle (t)

False-maker Coalition:

Abelard (F) and Boethius (f)

Game Rules for CC

p	whoever has p in their extension, wins
$\neg F$	switch the roles: Heloise assumes Abelard's role, Aristotle assumes Boethius' role, Boethius assumes Aristotle's role, Abelard assumes Heloise's role, and the game continues with F
$F \wedge G$	false-makers coalition chooses between F and G
$F \vee G$	truth-makers coalition chooses between F and G

Correctness

Theorem

For the evaluation games for a formula φ in McCall's Connexive logic, we have the following:

- truth-makers have a winning strategy if and only if φ has the truth value t or T in M,
- false-makers have a winning strategy if and only if φ has the truth value f or F in M.

What Have We Observed?

- Failure of the biconditional correctness
- Multiplayer semantic games in a nontrivial way
- Non-sequential / paralel / concurrent plays
- Variable sum games
- Coalitions

If winning strategies are proofs, game semantics for paraconsistent logics present a constructive way to give proofs for inconsistencies.

Difficult Logics

- Da Costa systems, Logics of Formal Inconsistency
- Preservationism
- First-order paraconsistent logics
- Infinitary, fixed-point non-classical logics

Reference

CB, *Game Theoretical Semantics for Paraconsistent Logics*, in “Preceedings of the Fifth International Conference on Logic, Rationality and Interaction" (LORI-V), Edited by W. van der Hoek and W. Holliday and W. Wang, pp. 14-26, Springer, 2016.

Lecture Two

A Self-Referential Paradox in Games

Outlook of Lecture Two

- The Brandenburger - Keisler Paradox
- Paraconsistent Models
- A Countermodel

The Brandenburger-Keisler Paradox

The Paradox

The Brandenburger-Keisler paradox (BK paradox) is a two-person self-referential paradox in epistemic game theory (Brandenburger & Keisler, 2006).

The following configuration of beliefs is impossible:

The Paradox

Ann believes that Bob assumes that Ann believes that Bob’s assumption is wrong.

The paradox appears if you ask whether “Ann believes that Bob’s assumption is wrong".

Notice that this is essentially a 2-person Russell’s Paradox.

The Model

Brandenburger and Keisler use belief sets to represent the players' beliefs.

The model (U^a, U^b, R^a, R^b) that they consider is called a *belief structure* where $R^a \subseteq U^a \times U^b$ and $R^b \subseteq U^b \times U^a$.

The expression $R^a(x, y)$ represents that in state x, Ann believes that the state y is possible for Bob, and similarly for $R^b(y, x)$. We will put $R^a(x) = \{y : R^a(x, y)\}$, and similarly for $R^b(y)$.

At a state x, we say Ann believes $P \subseteq U^b$ if $R^a(x) \subseteq P$.

The Semantics

A modal logical semantics for the interactive belief structures can be given.

We use two modalities $\Box$ and $\heartsuit$ for the belief and assumption operators respectively with the following semantics.

$$x \models \Box^{ab}\varphi \quad \textit{iff} \quad \forall y \in U^b.R^a(x,y) \textbf{\textit{ implies }} y \models \varphi$$
$$x \models \heartsuit^{ab}\varphi \quad \textit{iff} \quad \forall y \in U^b.R^a(x,y) \textbf{\textit{ iff }} y \models \varphi$$

Note the bi-implication in the definition of the assumption modality!

Completeness

A belief structure (U^a, U^b, R^a, R^b) is called *assumption complete* with respect to a set of predicates Π on U^a and U^b if for every predicate $P \in \Pi$ on U^b, there is a state $x \in U^a$ such that x assumes P, and for every predicate $Q \in \Pi$ on U^a, there is a state $y \in U^b$ such that y assumes Q.

We will use special propositions $\mathbf{U}^a$ and $\mathbf{U}^b$ with the following meaning: $w \models \mathbf{U}^a$ if $w \in U^a$, and similarly for $\mathbf{U}^b$. Namely, $\mathbf{U}^a$ is true at each state for player Ann, and $\mathbf{U}^b$ for player Bob.

Incompleteness

Brandenburger and Keisler showed that no belief model is complete for its (classical) first-order language.

Therefore, “not every description of belief can be represented" with belief structures (Brandenburger & Keisler, 2006).

Incompleteness

The incompleteness of the belief structures is due to the *holes* in the model. A model, then, has a hole at φ if either $\mathbf{U}^b \wedge \varphi$ is satisfiable but $\heartsuit^{ab}\varphi$ is not, or $\mathbf{U}^a \wedge \varphi$ is satisfiable but $\heartsuit^{ba}\varphi$ is not.

Namely, φ is true for b, but cannot be assumed by a (or vice versa).

Some Remarks

- BK paradox is a game theoretical example of a self-referential paradox
- It is a simple step towards the possibility of paraconsistent games - a broader research program in progress
- It raises the possibility of discussing discursive/dialogical logics within game theory proper
- Provides an interesting take on Hintikka's interrogative theory - how to inquire about a paradoxical sentence?

Paraconsistent Topological Approach

What is a Topology?

Definition

The structure $\langle S, \sigma \rangle$ is called a topological space if it satisfies the following conditions.

1. $S \in \sigma$ and $\emptyset \in \sigma$
2. σ is closed under finite unions and arbitrary intersections

Collection σ is called a topology, and its elements are called *closed* sets.

Paraconsistent Topological Semantics

Use of topological semantics for paraconsistent logic is not new. To our knowledge, the earliest work discussing the connection between inconsistency and topology goes back to Goodman (Goodman, 1981).

In classical modal logic, only modal formulas produce topological objects.

However, if we stipulate that:

extension of *any* propositional variable to be a closed set (Mortensen, 2000; Mortensen, 2010), we get a paraconsistent system.

Problem of Negation

Negation can be difficult as the complement of a closed set is not generally a closed set, thus may not be the extension of a formula in the language.

For this reason, we will need to use a new negation symbol $\sim$ that returns the closed complement (closure of the complement) of a given set.

Topological Belief Models

The language for the logic of topological belief models is given as follows.

$$\varphi := p \mid {\sim}\varphi \mid \varphi \wedge \varphi \mid \Box_a \mid \Box_b \mid \boxplus_a \mid \boxplus_b$$

where p is a propositional variable, $\sim$ is the paraconsistent topological negation symbol which we have defined earlier, and $\Box_i$ and $\boxplus_i$ are the belief and assumption operators for player i, respectively.

Topological Belief Models

For the agents a and b, we have a corresponding non-empty type space A and B, and define closed set topologies τ_A and τ_B on A and B respectively. Furthermore, in order to establish connection between τ_A and τ_B to represent belief interaction among the players, we introduce additional constructions $t_A \subseteq A \times B$, and $t_B \subseteq B \times A$. We then call the structure $F = (A, B, \tau_A, \tau_B, t_A, t_B)$ a paraconsistent topological belief model.

A state $x \in A$ *believes* $\varphi \subseteq B$ if $\{y : t_A(x, y)\} \subseteq \varphi$. Furthermore, a state $x \in A$ *assumes* φ if $\{y : t_A(x, y)\} = \varphi$. Notice that in this definition, we identify logical formulas with their extensions.

Semantics

For $x \in A$, $y \in B$, the semantics of the modalities are given as follows with a modal valuation attached to F.

$$
\begin{array}{lll}
x \models \Box_a \varphi & \textit{iff} & \exists Y \in \tau_B \textit{ with } t_A(x, Y) \rightarrow \forall y \in Y . y \models \varphi \\
x \models \boxplus_a \varphi & \textit{iff} & \exists Y \in \tau_B \textit{ with } t_A(x, Y) \leftrightarrow \forall y \in Y . y \models \varphi \\
y \models \Box_b \varphi & \textit{iff} & \exists X \in \tau_A \textit{ with } t_B(y, X) \rightarrow \forall x \in X . x \models \varphi \\
y \models \boxplus_b \varphi & \textit{iff} & \exists X \in \tau_A \textit{ with } t_B(y, X) \leftrightarrow \forall x \in X . x \models \varphi
\end{array}
$$

The Result

Theorem

The BK sentence is satisfiable in some paraconsistent topological belief models.

Namely, we can construct a state which satisfies the BK sentence - push the *holes* that create the inconsistencies to the boundaries.

Conclusion

This was a *self-referential* paradox in games.

What about *non*-self-referential paradoxes in games?

Reference

CB, *Some Non-Classical Approaches to Branderburger-Keisler Paradox*, Logic Journal of the IGPL, vol. 23, no. 4, pp. 533-552, 2015.

Lecture Three

A *Non*-Self-Referential Paradox in Games

University of Bath

Outlook of Lecture Two

- Yablo’s Paradox
- What is the big deal?
- A Countermodel

Yablo's Paradox

Yablo's Paradox

Yablo's Paradox, according to its author, is a non-self referential paradox (Yablo, 1985; Yablo, 1993).

Yablo considers the following sequence of sentences.

$$S_1 : \forall k > 1, S_k \text{ is untrue},$$
$$S_2 : \forall k > 2, S_k \text{ is untrue},$$
$$S_3 : \forall k > 3, S_k \text{ is untrue},$$
$$\vdots$$

Why is it a Paradox?

By using *reductio*, Yablo argues that the above set of sentences is contradictory. Here, the infinitary nature of the paradox is essential as the each finite set of S_n is satisfiable.

The scheme of this paradox is not new. To the best of our knowledge, the first analysis of this paradox was suggested in 1953 (Yuting, 1953).

Impact of Yablo's Paradox

Ketland showed that the paradox is ω-inconsistent (Ketland, 2005).

Barrio showed that Yablo's Paradox in first-order arithmetic has a model and not inconsistent, but it is ω-inconsistent (Barrio, 2010).

It is easy to see how. Since every finite set of S_n sentences is satisfiable, then, by compactness there exists a model for the Yablo sentences. By ω-inconsistency, it can be argued that the model we are looking for is a non-standard model of arithmetic.

Impact of Yablo’s Paradox

As Hardy puts it “Is Yablo’s paradox Liar-like? In some ways yes, and in other ways no" (Hardy, 1995).

Priest offers another analysis regarding the infinitary language that it requires, and suggests a reading of the paradox that does indeed involve circularity (Priest, 1997).

Sorensen disagrees and point out the hierarchical view of Tarskian truth theory arguing that Yablo’s paradox in effect “exploit[s] an alternative pattern of semantic dependency" (Sorensen, 1998).

Applications of Yablo's Paradox

Goldstein presents a set theoretical yabloesque paradox for class membership (Goldstein, 1994).

Leitgeb suggests a yabloesque paradox for non-well-founded definitions underlining the set theoretical limitations of the logical toolbox (Leitgeb, 2005).

Picollo discusses the paradox in second-order logic generalizing the ω-inconsistency results (Picollo, 2013).

Non-well-founded Yablo chains form a topological space encouraging Bernardi's topological approach to the paradox (Bernardi, 2009).

Cook and Beall consider Curry-like versions of the paradox (Cook, 2009; Beall, 1999).

A Yabloesque Paradox in Epistemic Games

A Yabloesque Paradox in Epistemic Games

Consider the following sequence of assumptions where numerals represent game theoretical players.

$$A_1 : 1 \text{ believes that } \forall k > 1, k\text{'s assumption } A_l \text{ about } \forall l > k \text{ is untrue,}$$
$$A_2 : 2 \text{ believes that } \forall k > 2, k\text{'s assumption } A_l \text{ about } \forall l > k \text{ is untrue,}$$
$$A_3 : 3 \text{ believes that } \forall k > 3, k\text{'s assumption } A_l \text{ about } \forall l > k \text{ is untrue,}$$
$$\vdots$$

An Interpretation

Imagine a queue of players, where players are conveniently named after numerals, holding beliefs about each player behind them, but not about themselves. In this case, each player i believes that each player $k > i$ behind them has an assumption about each other player $l > k$ behind them and i believes that each k's assumption is false.

This statement is perfectly perceivable for games, and involves a specific configuration of players' beliefs and assumptions, which can be expressible in the language. However, as we shall show, similar to Yablo's paradox and the BK paradox, this configuration of beliefs is impossible.

Modal Logically

Let us start with an informal argument.

Now, for a contradiction, assume A_n is true for some n. Therefore, player n believes that $\forall k > n$, k's assumption is untrue. In particular, player $n+1$'s assumption is untrue. In other words, $n+1$'s assumption

$$A_{n+1} : n+1 \text{ believes that } \forall k > n+1, k\text{'s assumption } A_l \text{ about } l > k \text{ is untrue}$$

is untrue.

Modal Logically

Therefore, $n+1$ believes that for some $k' > n+1$, what k' assumes about some $l' > k'$ is true. But, this combination of players k' and l', both of which are bigger than $n+1$, thus n, is accessible from n by means of the belief-assumption modalities. We assumed A_n is true, which entails that what k' assumes about some $l' > k'$ is untrue. Contradiction. The choice of n was arbitrary, so each A_n in the sequence is untrue.

However, if each A_n is untrue, they can be assumed untrue. But, if for all n, n's assumption is untrue, then A_1 is indeed true. Yet, we just argued that each A_n is untrue.

Contradiction.

The Syntax

The Yabloesque Brandenburger - Keisler paradox ('YBK Paradox', henceforth) requires ω-many players $i \in I$. The syntax of this language is given in the Backus-Naur form as follows for a set of propositional variables **P**:

$$\varphi := p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Box^{ij}\varphi \mid \heartsuit^{ij}\varphi$$

where $p \in \mathbf{P}$ and $i \neq j$ for $i, j \in I$ with $|I| = \omega$.

The disjunction and implication are taken as abbreviations in the standard way.

The Model

The extended belief model is a tuple $M = (\{U^i\}_{i \in I}, \{R^{ij}\}_{i \neq j \in I}, V)$ where $R^{ij} \subseteq U^i \times U^j$ and V is a valuation function.

As before, the expression $R^{ij}(x, y)$ represents that in state x, the player i believes that the state y is possible for player j.

We prevent (a trivial form of) self-reference by disallowing players having beliefs about themselves.

The Semantics

The semantics for the modal operators is given as follows in a similar way.

$$x \models \Box^{ij}\varphi \quad \text{iff} \quad \forall y \in U^j.R^{ij}(x,y) \text{ implies } y \models \varphi$$
$$x \models \heartsuit^{ij}\varphi \quad \text{iff} \quad \forall y \in U^j.R^{ij}(x,y) \text{ iff } y \models \varphi$$

An Illustrative Example I

Let $w \models A_3$. Therefore, $w \models \bigwedge_{k>3} \Box^{3k} \{ \bigwedge_{l>k} \heartsuit^{kl} \neg A_l \}$.

Let us spell this out.

$$
\begin{aligned}
w &\models \bigwedge_{k>3} \Box^{3k} \{ \bigwedge_{l>k} \heartsuit^{kl} \neg A_l \} \\
w &\models \Box^{34} (\heartsuit^{45} \neg A_5 \wedge \heartsuit^{46} \neg A_6 \wedge \heartsuit^{47} \neg A_7 \wedge \ldots) \wedge \\
&\quad\ \Box^{35} (\heartsuit^{56} \neg A_6 \wedge \heartsuit^{57} \neg A_7 \wedge \heartsuit^{58} \neg A_8 \wedge \ldots) \wedge \\
&\quad\ \vdots
\end{aligned}
$$

An Illustrative Example II

In particular, for example, $w \models \Box^{34}\heartsuit^{47}\neg A_7$ and $w \models \Box^{35}\heartsuit^{57}\neg A_7$. Therefore, it can be seen that for all $3 < a < 7$, $w \models \Box^{3a}\heartsuit^{a7}\neg A_7$. Simply put, from agent 3, through each other agent between 3 and 7, it is possible to reach $\neg A_7$ via belief-assumption chain. We simply focused on player 7 and his assumption A_7, but the argument works for any player $n > 4$ in our example.

The contradiction simply occurs when A_7 is hit by two different players in two different ways. In order to see it, consider A_5 (which can also reach A_7 by a belief-assumption chain).

An Illustrative Example III

So, we have

$$w \models \Box^{34} \heartsuit^{45} \neg A_5$$

where $\neg A_5$ is given as follows:

$$\bigvee_{k>5} \Diamond^{5k} \{ \bigvee_{l>k} \neg \heartsuit^{kl} \neg A_l \}.$$

Therefore, for all v, if $R^{34}wv$ then $v \models \heartsuit^{45} \neg A_5$. Then, for all u, $R^{45}vu$ if and only if $u \models \neg A_5$. The use of the assumption modality here is crucial. It associates the set of states that falsifies A_5 with what is accessible from v with R^{45}.

An Illustrative Example IV

Spelling this out, we have the following.

$$u \models \Diamond^{56}[\neg\heartsuit^{67}\neg A_7 \vee \neg\heartsuit^{68}\neg A_8 \vee \ldots] \vee$$
$$\Diamond^{57}[\neg\heartsuit^{78}\neg A_8 \vee \neg\heartsuit^{79}\neg A_9 \vee \ldots] \vee \ldots$$

The first disjunct above suggests that there is a t such that $R^{56}ut$ and $t \models \neg\heartsuit^{67}\neg A_7 \vee \neg\heartsuit^{68}\neg A_8 \vee \ldots$.

However, this is impossible. The first disjunct ($\neg\heartsuit^{67}\neg A_7$) cannot be the case at t.

An Illustrative Example V

Because it reduces to the following.

$$t \models \neg\heartsuit^{67}\neg A_7 \text{ iff } \exists y \in U^7.[\ (R^{67}(x,y) \wedge y \models A_7) \vee (\neg R^{67}(x,y) \wedge y \models \neg A_7)\ ]$$

But, this is impossible by our earlier observation: there is a state accessible via R^{67} that satisfies $\neg A_7$, and all the states accessible from u satisfies A_7 due to the definition of the $\heartsuit$ modality.

The argument can easily be extended to other disjuncts and their disjuncts. Thus, contradiction. Therefore, each A_n is false. As we observed earlier, then A_ns are also true by definition.

This is the paradox.

Further Results

Some further results:

Lemma

1. $\heartsuit^{ij}\varphi \rightarrow \Box^{ij}\varphi$
2. $\heartsuit^{ij}(\varphi \wedge \psi) \equiv \heartsuit^{ij}\varphi \wedge \heartsuit^{ij}\psi$
3. $\Box^{ij}(\heartsuit^{jk}\varphi \wedge \heartsuit^{jl}\psi) \equiv \Box^{ij}\heartsuit^{jk}\varphi \wedge \Box^{ij}\heartsuit^{jl}\psi$.
4. $\Diamond^{ij}(\heartsuit^{jk}\varphi \vee \heartsuit^{jl}\psi) \equiv \Diamond^{ij}\heartsuit^{jk}\varphi \vee \Diamond^{ij}\heartsuit^{jl}\psi$
5. $\heartsuit^{ij}(\Box^{jk}\varphi \wedge \Box^{jl}\psi) \equiv \heartsuit^{ij}\Box^{jk}\varphi \wedge \heartsuit^{ij}\Box^{jl}\psi$

Further Results

Lemma

If $w \models A_n$, then for all p, q with $n < p < q$;
$w \models \Box^{np} \heartsuit^{pq} \neg A_q$.

Theorem

If $w \models A_n$, then for all p, p', q with $n < p < q$ and $n < p' < q$; we have $R^{pq}(v) = R^{p'q}(v')$ for all $v \in U^p$ and all $v' \in U^{p'}$.

Further Results

Corollary

If $w \models A_n$, then $\Box^{np} \heartsuit^{pq} \varphi \longleftrightarrow \Box^{np'} \heartsuit^{p'q} \varphi$ for $n < p < q$ and $n < p' < q$.

Discussion and Further Remarks

Assumption Modality and the Diagonal Formula

Assumption modality $\heartsuit^{ij}$ is an essential part of the construction of the paradox. Without it, it is not possible to generate the YBK paradox.

For example, the following set of sentences about players' beliefs is not inconsistent.

$$A'_1 := \bigwedge_{k>1} \Box^{1k}\{\bigwedge_{l>k} \Box^{kl}\neg A'_l\}$$
$$A'_2 := \bigwedge_{k>2} \Box^{2k}\{\bigwedge_{l>k} \Box^{kl}\neg A'_l\}$$
$$A'_3 := \bigwedge_{k>3} \Box^{3k}\{\bigwedge_{l>k} \Box^{kl}\neg A'_l\}$$
$$\vdots$$

Non-Well-Foundedness

As Yablo also argued, what we would have is a “downward facing tree with ω branches descending from each node" (Yablo, 2004).

The set of sentences above, in other words, generate trees that are infinitely-branching which satisfy it.

Categoricity

As argued by Ketland, the set of Yablo sentences is not satisfiable on the standard model of arithmetic, thus they are "ω-inconsistent" (Ketland, 2005).

This observation suggests that the YBK paradox can be satisfied in a game with $\omega + 1$ players. As every finite set of A_ns in the YBK Sentence are satisfiable, by compactness, there must exist a model for the Yablo sentences.

Countermodels

Topological Countermodels

Now we construct paraconsistent models for the YBK paradox. For agent i, we take the corresponding non-empty type space S_i and define topologies with closed sets σ_i. For example, for player 3, the type space will be denoted by S_3 with a topology σ_3. In order to make this approach interactive, we define a function $s^{ij} \subseteq S_i \times S_j$ which associates states for player i with the states of j. For example, for player i, at states from S_i, s^{ij} returns a closed set $K \in \sigma_j$. We write $s^{ij}(w, K)$ means that at state $w \in S_i$, player i believes that states $k \in K$ are possible for j.

The Model

The model is a tuple $(\{S_i\}_{i \in I}, \{\sigma_i\}_{i \in I}, \{s^{ij}\}_{i,j \in I}, V)$ where V is a valuation defined in the standard way. The syntax for this system is similar to what we have given earlier with the paraconsistent negation symbol $\sim$. Here $p \in \mathbf{P}$ where $i \neq j \in I$:

$$\varphi := p \mid \sim\varphi \mid \varphi \wedge \varphi \mid \Box^{ij}\varphi \mid \heartsuit^{ij}\varphi$$

The dual modalities are defined as usual with the paraconsistent negation.

The Semantics

The paraconsistent topological semantics for this language is given as follows for negation and the modal operators as the Booleans are standard. For a set X, the complement of X will be denoted by X^c.

$$|{\sim}\varphi| = \mathrm{Clo}(K^c)$$

$$|\Box^{ij}\varphi| = \{w \in S_i : \exists K \in \sigma_j \text{ with } s^{ij}(w,K) \text{ and } K \subseteq |\varphi|\}$$

$$|\heartsuit^{ij}\varphi| = \{w \in S_i : \exists K \in \sigma_j \text{ with } s^{ij}(w,K) \text{ and } K = |\varphi|\}$$

The Semantics

For easy read, we give the semantics for the modalities in the traditional sense as follows.

$$w \models \Box^{ij}\varphi \quad \text{iff} \quad \exists K \in \sigma_j \text{ with } s^{ij}(w, K) \rightarrow \forall v \in K . v \models \varphi$$
$$w \models \Diamond^{ij}\varphi \quad \text{iff} \quad \forall K \in \sigma_j \text{ with } s^{ij}(w, K), \exists v \in K \text{ such that } v \models \varphi$$
$$w \models \heartsuit^{ij}\varphi \quad \text{iff} \quad \exists K \in \sigma_j \text{ with } s^{ij}(w, K) \leftrightarrow \forall v \in K . v \models \varphi$$

The Topological Countermodel

Now we construct a counter-model using topological paraconsistent models for the YBK paradox. Let us reconsider the set of modal formulas again.

$$A_1 := \bigwedge_{k>1} \Box^{1k} \{ \bigwedge_{l>k} \heartsuit^{kl} \neg A_l \}$$
$$A_2 := \bigwedge_{k>2} \Box^{2k} \{ \bigwedge_{l>k} \heartsuit^{kl} \neg A_l \}$$
$$A_3 := \bigwedge_{k>3} \Box^{3k} \{ \bigwedge_{l>k} \heartsuit^{kl} \neg A_l \}$$
$$\vdots$$

The Topological Countermodel

We will construct a model and a state w in it that satisfy the above set of formulas step by step starting with player 1 and A_1. Now, for player 1, take $w_1 \in S_1$ and consider A_1. For each $k > 1$, construct $K_{1k} \in \sigma_k$ such that $s^{1k}(w_1, K_{1k})$ and each $v_k \in K_{1k}$ satisfies $\bigwedge_{l>k} \heartsuit^{kl} \sim A_l$. Therefore, for each $l > k$, there exists U_{kl} that $s^{kl}(v_k, U_{kl})$ such that every $u_l \in U_{kl}$ if and only if $u_l \in |\sim A_l|$. Let us unravel $\sim A_l$ as follows: $\sim A_l = \bigvee_{p>l} \Diamond^{lp} \{ \bigvee_{q>p} \sim \heartsuit^{pq} \sim A_q \}$. This is a disjunctive statement. As our goal is to construct a counter-model, we will try to satisfy only one of the disjuncts and nested-disjuncts.

The Topological Countermodel

Now at u_l, include (u_l, K_{1p})in s^{lp} for all $p > l$. Thus, we have $s^{lp}(u_l, K_{1p})$. By construction of A_1 (hence of each A_i), each $v_p \in K_{1p}$ satisfies $\bigwedge_{q>p} \heartsuit^{pq}{\sim}A_q$, hence $\heartsuit^{pq}{\sim}A_q$ for each $q > p$. Similarly, include (v_p, x_q) for each $x_q \in \partial(|A_q|)$ into s^{pq} in a way that $s^{pq}(v_p, \partial(|{\sim}A_q|))$. Thus, $v_p \models {\sim}\heartsuit^{pq}{\sim}A_q \wedge \heartsuit^{pq}{\sim}A_q$. We constructed a model in which we have $w_1 \models A_1$.

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The Topological Countermodel

This methodology can be extended inductively for each assumption A_i which in turn builds the counter-model that satisfy each and every formula in the set of game theoretical Yablo sentences.

The crucial observation is that the extension of a ♡-formula uniquely identifies with the extension of the formula in question. However, *some* of the points in that extension may also satisfy the negation of the formula in question in paraconsistent models. This makes it quite easy and straight-forward to construct the counter-model.

Conclusion

Conclusion

Yablo's Paradox is

- An interactive, ω-player paradox,
- A modal paradox,
- A paradox of well-foundedness in some ways

What to do: Develope a *Curryesque* epistemic game theoretical paradox in which negation and falsity predicates are not used.

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Closing Remarks

Conclusion

Game Theory relates non-classical logics to *homo-economicus*.

It helps us understand how we make decisions, reach equilibria, reveal preferences and put utilities in goods.

Thank you!

Slides are available at:

`www.CanBaskent.net/Logic`

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