Some Observations on Nabla Modality

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1 Introduction and Motivation

The traditional necessity and possibility operators of modal logic provide us with a direct insight about how modalities work in various frameworks. Supported with the simple-to-use Kripke semantics and its intuitive proof theory, such modalities have led to a variety of mathematical and philosophical developments in the field. Nevertheless, from a mathematical point of view, it is possible to put these two modalities together in a certain way to create a single modal operator, and obtain an equi-expressible language as the standard propositional modal logic.

In this paper, we discuss such a reformulation of the syntax of propositional unimodal modal logic with some further advancement and applications. Our focus will be the nabla modality which was introduced by Larry Moss for coalgebraic purposes [11]. Our goal here is to present several results on nabla modality spanning various fields in epistemic logic including epistemic entrenchment, language splitting and public announcements, and along the way we will present various complexity results.

Similar observations on the connection between new syntactical elements and their semantical richness has been undertaken in other epistemic logics which relate the current work to a broader framework. The dichotomy between two epistemic languages renders itself rather an interesting research area when basic epistemic logic (EL) and public announcement logic (PAL) are considered. PAL has one additional operator called the *public announcement operator*. Nevertheless, it turns out that PAL and EL are equi-expressible: every PAL formula can be reduced to an EL formula. Yet, PAL provides us with a more direct insight about dynamic epistemology of multi-agent systems in an economical and succinct fashion [16, 9].

Our contribution in this paper is as follows. We take the nabla modality as our basic modality and investigate it in several epistemic frameworks. Starting from the complexity issues raised by the nabla modality, we discuss various epistemic issues including epistemic entrenchment, where some coalgebraic properties come in handy. Then, we consider a very interesting syntactic interpolation result of Parikh, and analyze it within this new language. Next, we show the completeness of a dynamic epistemic logic that uses the nabla operator as its basic "epistemic" operator. Finally, we consider some issues regarding the persistence of sets of formulas with the nabla modality.

Related Literature Larry Moss introduced the nabla modality for coalgebraic purposes [11]. A large body of work focused on coalgebraic and category theoretical aspects of the nabla modality, and within that framework Gentzen style proofs, distribution properties, connections to algebras were investigated in detail [4, 13, 17]. Nabla modality has also been discussed within modal logic. Moss used cover sets to give weak completeness and decidability results of standard systems of modal logics [12]. French, Pinchinat and van Ditmarsch used nabla modality in their event logic [5]. Our work attempts to fill the gap between the coalgebraic treatments of this new syntactic element and applied/classical modal logic.

2 Basics

Different from the standard modal operators, the nabla modality applies to set of formulas which can be empty, infinite or inconsistent. In this work, for simplicity reasons, we restrict ourselves to finite sets of formulas.

The definition of the nabla modality is given as follows.

$$\nabla \Phi := (\bigwedge \blacklozenge \Phi) \land (\Box \bigvee \Phi)$$

where $\blacklozenge \Phi$ for a set of formulas Φ is an abbreviation for the set $\{\Diamond \varphi : \varphi \in \Phi\}$. We call ∇ *nabla* or *cover* modality interchangeably depending on the context. The set Φ is called the *cover set*. We denote the standard modal and propositional languages as $\mathcal{L}_\Box$ and $\mathcal{L}$, respectively. For a set P of countable propositional variables, the syntax of the modal language we consider is given as follows.

$$p \mid \top \mid \neg\varphi \mid \varphi \land \varphi \mid \nabla \Phi$$

where $\top$ is the truth constant, and $p \in P$. We take $\lor$, $\to$ and $\leftrightarrow$ as shorthands in the usual way. We call this language $\mathcal{L}_\nabla$.

The semantics of the nabla modality in Kripke structures is straightforward. If $M = \langle W, R, V \rangle$ is a model where W is a non-empty set, R is a binary relation on $W \times W$, and $V : P \to \wp(W)$ is a valuation function; then we define the semantics of the nabla modality as follows.

$$\begin{aligned} M, w \models \nabla \Phi \quad \text{iff} \quad & \forall \varphi \in \Phi, \exists v \text{ with } wRv \text{ such that } M, v \models \varphi, \text{ and} \\ & \forall v \text{ with } wRv, \exists \varphi \in \Phi \text{ such that } M, v \models \varphi. \end{aligned}$$

The semantics of the nabla modality states that every formula in the cover set is possible and furthermore, at each accessible state we have a satisfiable formula from the cover set. In that sense, nabla *covers* the set of accessible states with the formulas from cover set. Moreover, notice that cover set may be inconsistent.

It is also noteworthy to see that the classical modal connectives in $\mathcal{L}_\Box$ are definable in terms of the nabla modality as follows.

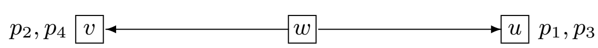
$$\Diamond \varphi \equiv \nabla\{\varphi, \top\} \qquad \Box \varphi \equiv \nabla \emptyset \lor \nabla\{\varphi\}$$

The semantics of the nabla modality can also be expressed in game theoretical terms. In a verification game between ∀belard and ∃loise, the game rule for the ∇ modality is given as follows. At the game state $(w, \nabla \Phi)$ with state w, ∀belard can make two different moves. He can either choose a formula φ from Φ after which ∃loise makes a choice for the accessible state v at which φ is true. Alternatively, ∀belard can choose an accessible state v forcing ∃loise to choose a formula φ from Φ. Let us denote the board game between ∀belard and ∃loise starting from the state w with the formula φ as $E(w, \varphi)$. The game-states in the game $E(w, \varphi)$ are of the form (w, φ) where w is a state in the model and $\varphi \in \mathcal{L}_\nabla$. The aim of the verification game is to evaluate a given formula, by reducing it step by step to its simpler subformulas, eventually reaching propositional variables for which the decision procedure for truth is trivial. The following immediate theorem summarizes this brief discussion here.

Theorem 1. *In the evaluation game $E(w, \varphi)$ for $\varphi \in \mathcal{L}_\nabla$ at a state w, ∃loise has a winning strategy if and only if $w \models \varphi$.*

Let us now make some further observations on the behavior of the nabla modality. First, we note that the nabla modality does not provide a complete picture of the modalities in a given model as illustrated by the following example.

Example 1. Consider the following picture. In this example, observe that the states v and u are accessible from w, and the states u and v satisfy the propositional letters p_1, p_3 and p_2, p_4 respectively. Let us take $\Phi_1 = \{p_1, p_2\}$, $\Phi_2 =$



$\{p_3, p_4\}$, $\Phi_3 = \{p_1, p_4\}$ and finally $\Phi_4 = \{p_2, p_3\}$. Thus, $w \models \bigwedge_{1 \leq i \leq 4} \nabla \Phi_i$. Notice that in this example, ∇ cannot distinguish Φ_i from Φ_j for $i \neq j$.

Regarding this example, first, notice that the truth of nabla modality is invariant under bisimulation, as expected. A straight-forward induction argument shows this. Furthermore, the nabla operator is closed under union, that is, if $w \models \nabla \Phi$ and $w \models \nabla \Psi$, then $w \models \nabla(\Phi \cup \Psi)$. However, it is not closed under intersection. To see this, consider the Example 1, and take Φ_1 and Φ_3. Even if $w \models \nabla \Phi_1$ and $w \models \nabla \Phi_3$, we observe $w \not\models \nabla\{p_1\}$ where $\{p_1\} = \Phi_1 \cap \Phi_3$. Nevertheless, by imposing an intuitive additional constraint, and a slight abuse of the language, we can formulate when nabla is closed under the superset relation. That is, if $w \models \nabla \Phi$ and $w \models \Diamond \varphi$, then $w \models \nabla(\Phi \cup \{\varphi\})$.

The last remark shows that once the current state is *covered* by the nabla operator with some set of formulas Φ, then we can expand this cover set by adding some other formulas which are possible at the current state. Then, the immediate question is the following: *Can we go backwards?* In other words, starting from a cover set Φ, can we make it smaller and smaller by eliminating some formulas each time, and obtain a minimal cover set?

3 MinimalNabla

Let us first introduce some short-hand notations. By $(\varphi)^M$ we denote the extension of φ in the model M, i.e. the set of points where φ is true in M. For a state w in M, $[w]^M$ denotes the set of accessible states from w in M. By $||w||^M$, we denote the set of formulas which are true at the state w. When it is obvious, we will drop the superscript. As we underlined earlier, for our current purposes, we consider only finite cover sets here.

Proposition 1. *For a given set of formula Φ, if there are two distinct formulas $\varphi, \psi \in \Phi$ such that $(\varphi) \cap [w] = (\psi) \cap [w]$; then we have $M, w \models \nabla(\Phi - \{\varphi\})$ if and only if $M, w \models \nabla(\Phi - \{\psi\})$.*

The above proposition is given for two formulas, and it is not difficult to generalize it to a set of formulas.

Proposition 2. *For a given set of formula Φ, if there are two disjoint sets of formulas $\Pi, \Psi \subseteq \Phi$ such that $\{\bigcup_{\pi \in \Pi}(\pi)\} \cap [w] = \{\bigcup_{\psi \in \Psi}(\psi)\} \cap [w]$; then we have $M, w \models \nabla(\Phi - \Pi)$ if and only if $M, w \models \nabla(\Phi - \Psi)$.*

Moreover, we can remove the formulas from a cover set if they can be "covered" by other formulas.

Proposition 3. *Let $w \models \nabla\Phi$. Then, $w \models \nabla(\Phi - \Psi)$ for some $\Psi \subseteq \Phi$ if the following holds $\bigcup_{\psi \in \Psi}((\psi) \cap [w]) \subseteq \bigcup_{\varphi \in \Phi - \Psi}((\varphi) \cap [w])$.*

The ideas and observations in the above propositions hint out how we can remove some formulas from the cover set to obtain a smaller cover set. However, notice that minimal sets as described above are not necessarily unique.

Definition 1. *Given $w \models \nabla\Phi$, we call Φ' a Φ-*minimal cover set *if $\Phi' \subseteq \Phi$ with $w \models \nabla\Phi'$ and there is no $\Phi'' \subset \Phi'$ with $w \models \nabla\Phi''$.*

Given $w \models \nabla\Phi$ for a fixed w, we now consider the complexity of finding a minimal cover set $\Phi' \subseteq \Phi$ with $w \models \nabla\Phi'$. Let us call the problem of finding a minimal cover set MinimalNabla.

Theorem 2. MinimalNabla *is NP-complete.*

MinimalNabla problem may have some immediate applications when, for instance, minimal knowledge/belief base of an agent needs to be computed or verified. Moreover, when a resource-bound agent needs to economize without losing much information, it is very important to know the complexity of this procedure. MinimalNabla solves this issue.

3.1 Epistemic Entrenchment

The procedure to obtain the minimal cover set does not depend on any ordering of the formulas. In some applications, however, formulas in a cover set may need to be ordered with respect to their epistemic value. As Gärdenfors and Makinson stated it "Even if all sentences in a knowledge set are accepted or considered as facts, this does not mean that all sentences are of equal value for planning or problem-solving purposes. Certain pieces of our knowledge and beliefs about the world are more important than others when planning future actions, conducting scientific investigations, or reasoning in general" [6]. Following the same approach, we now assume an order on the knowable formulas in a given cover set with aim of constructing the most entrenched cover set. We start by briefly mentioning the fundamentals of *epistemic entrenchment*.

The relation $\varphi \leq_i^M \psi$ denotes that "ψ is at least as epistemically entrenched as φ for agent i in model M". This relation can be lifted to the level of cover sets. We say "the set of formulas Φ is at least epistemically entrenched as Ψ" and write $\Psi \preccurlyeq \Phi$ if for all $\psi \in \Psi$, there is a $\varphi \in \Phi$ such that $\psi \leq \varphi$. We will apply epistemic entrenchment to the set of formulas Φ to obtain a smaller set $\Phi' \subseteq \Phi$ such that $\Psi \preccurlyeq \Phi'$ for other minimal cover set $\Psi \subseteq \Phi$. We will call Φ' minimal entrenched subset of Φ. Then, a natural question is the complexity of this procedure.

Theorem 3. *The problem of selecting the minimal and the epistemically most entrenched subset $\Phi' \subseteq \Phi$ of a given cover Φ that can cover the all accessible states from any given state is NP-complete.*

4 Further Applications

4.1 Distribution Property

Various observations about the algebraic properties of the nabla operator in the coalgebras have been made elsewhere [13]. It was shown that the nabla operator possesses a distribution property and forms a special algebra. Let us start with defining relation lifting which is an essential element of nabla algebras.

Definition 2. *Given a relation $R \subseteq S_1 \times S_2$, its power lifting relation $\mathsf{P}(R) \subseteq \wp(S_1) \times \wp(S_2)$ is defined as follows*

$$\begin{aligned}\mathsf{P}(R) := \{(X, X') : \forall x \in X, \exists x' \in X' \textit{ such that } (x, x') \in R \textit{ and} \\ \forall x' \in X', \exists x \in X \textit{ such that } (x, x') \in R\}\end{aligned}$$

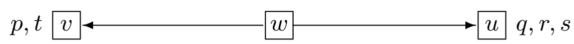
A relation R is called full on S_1 and S_2 if $(S_1, S_2) \in \mathsf{P}(R)$, and we write $R \in S_1 \bowtie S_2$.

Then, the distribution property of nabla algebras is given as follows.

$$\nabla\Phi \wedge \nabla\Psi \equiv \bigvee_{R \in \Phi \bowtie \Psi} \nabla\{\varphi \wedge \psi : (\varphi, \psi) \in R\}$$

Correctness of the distribution property can be verified by using the semantics of the cover modality and Definition 2 (See [13] for details). The following example presents a quick application of the distribution property.

Example 2. Let $\Phi = \{p, q\}$ and $\Psi = \{r, s, t\}$ where each p, q, r, s, t are propositional variables with the truth values as depicted in the given picture. Let us assume that $w \models \nabla\Phi$ and $w \models \nabla\Psi$. Then we have $w \models \nabla\Phi \wedge \nabla\Psi$. Note that here $\Phi \cap \Psi = \emptyset$. By the distribution property, we then observe the following. Let



R be the relation for $(\varphi, \psi) \in R$ such that $\varphi \wedge \psi$ is true at some accessible state from w. Then, by the given property we observe $w \models \nabla\{p \wedge t, q \wedge r, q \wedge s\}$. Now let us take $\Phi' = \{p, q, r\}$ and $\Psi' = \{t, r, s\}$ such that $w \models \nabla\Phi' \wedge \nabla\Psi'$. Notice that $\Phi' \cap \Psi' \neq \emptyset$. Now, we have $w \models \nabla\{p \wedge t, q \wedge s, r \wedge s, r \wedge q\}$.

Example 2 hints at how the distribution property can be used in knowledge representation among agents. For this purpose of ours, we will index the cover sets in Example 2 with respect to agents allowing expressions such as $w \models \nabla\Phi_i \wedge \nabla\Phi_j$ for agents i, j. Then, the distribution property makes it possible to identify the set of knowable formulas for each agent. After Φ_i and Φ_j are distributed, each formula in the joint cover set will have a conjunct per each agent. In other words, even if each agent can cover the epistemically accessible states on their own, it is possible to cover them jointly in an interactive fashion. In due time, we will make it precise how we can split the language for the same purpose for different agents given a theory.

The following theorem summarizes our discussion so far, and we state it without a proof which directly follows from the definitions.

Theorem 4. *Let Φ_i be a set of formula indexed by agent i. Let $w \models \bigwedge_{i \in I} \nabla\Phi_i$. Then, we can construct a set of formulas Φ based on Φ_is such that each formula φ in Φ is a conjunction of the form $\varphi = \bigwedge_i \varphi_i$ such that $\varphi_i \in \Phi_i$ for each i, and $w \models \nabla\Phi$.*

The following example illustrates how nabla modality can be useful in representing interaction between epistemic agents as we have argued.

Example 3. Theorem 4 can be used to express different perspectives on an event. Let us take the model in Example 2, and consider cover sets as *different* perspectives (points of views) over the real world per agent. They can be different scientific explanations of a physical phenomenon, different political points of view over the same issue etc. Now, the distribution property can be considered a sort of *mediator* that puts different perspectives into a joint cover set preserving the truth. In this case, given, say, $\nabla\Phi$ and $\nabla\Psi$ for two agents with different perspectives over the same issue, we can distribute their pieces of information and facts in such a way that the resulting cover set will have pieces of information from both agents per piece of information it has.

A similar methodology can be used to discuss epistemic and computational resources of agents when they consider forming coalitions in distributed systems. Similarly, for philosophically oriented reader, this discussion directly relates to *belief polarization* [8, 3]. In order to maintain our current focus in this paper, we leave it to future work.

4.2 Language Splitting

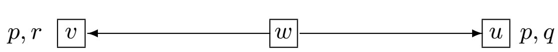
In an earlier work on belief revision and its syntax, Parikh showed that any formal propositional language with finite propositional variables can be split into a disjoint sublanguages with respect to a given theory [14]. A recent work on the subject extended Parikh's results to infinitary case allowing infinitely many propositional symbols in the language [10]. At a foundational level, this line of research relates to Craig's celebrated theorem on interpolation. Now, we first recall the notion of language splitting, and then apply some similar ideas to cover sets.

Definition 3 ([14]). *Suppose T is a theory in the language $\mathcal{L}$ and let $\{\mathcal{L}_1, \mathcal{L}_2\}$ be a partition of $\mathcal{L}$. We will say $\mathcal{L}_1, \mathcal{L}_2$* split *the theory T if there are formulas φ, ψ such that φ is in $\mathcal{L}_1$ and ψ in $\mathcal{L}_2$ and $T = Con(\varphi, \psi)$ where $Con(\cdot)$ operator takes the deductive closure of its argument. In this case, $\{\mathcal{L}_1, \mathcal{L}_2\}$ is called a T-splitting of $\mathcal{L}$.*

Theorem 5 (Parikh, [14]). *Given a theory T in the language $\mathcal{L}$, there is a unique* finest *splitting of $\mathcal{L}$, i.e. one which refines every other T-splitting.*

Let us now focus on theories formed by a (finite) cover Φ. Let w be a state and T be a theory with an extension that includes $[w]$. Then, there is a unique splitting of $\mathcal{L}_\nabla$ into $\mathcal{L}_{\nabla_i}$s such that $w \models \nabla\Phi_i$ where $\Phi_i \subseteq \mathcal{L}_{\nabla_i}$. In other words, we suggest that if a theory is satisfiable at $[w]$, then we can cover these states with different cover sets which are formed with respect to the disjoint nabla languages which are obtained by splitting $\mathcal{L}_\nabla$ with respect to T.

Example 4. Consider the following model that we discussed earlier. Let $T = Con(p, q, r)$. Then the minimum partition is $\Phi = \{p\}$ and $\Psi = \{r, q\}$ where $\Phi \cap \Psi = \emptyset$ and $w \models \nabla\Phi \wedge \nabla\Psi$. Note that for $\Phi' = \{p, q\}$ and $\Psi' = \{p, r\}$, we also have $w \models \nabla\Phi' \wedge \nabla\Psi'$. However, Φ' and Ψ' do not form a partition for obvious reasons.



Now, we generalize the observations we did in the previous example.

Theorem 6. *Let w be a state and T be a theory that is satisfied at the accessible states of w, i.e $[w] \subseteq (T)$. Then, there is a unique finest splitting of $\mathcal{L}_\nabla$ into $\mathcal{L}_{\nabla_i}$s such that $w \models \nabla\Phi_i$ where $\Phi_i \subseteq \mathcal{L}_{\nabla_i}$.*

The Theorem 6 has various possible applications in distributed systems where resources are allocated to different agents or in epistemic logic and belief revision where agents' knowledge and belief are restricted by their syntactical power (that is their language).

4.3 Dynamic Epistemic Nabla Logic

Previously, we have seen some ways to minimize the cover set without minimizing the model or set of accessible states. The dual of this procedure seems to be needed when dynamic epistemic logical issues are considered. In this section, we discuss a state-elimination based paradigm and consider epistemic announcements in a setting where we can quantify over epistemic public announcements. For this purpose of ours, we will still maintain the nabla modality as our primitive operator.

Public announcement logic (PAL) is a dynamic epistemic logic where epistemic models are updated by external and truthful announcements [15]. When a public announcement is made, the states that do not satisfy the announcement are eliminated from the model only to obtain a submodel in which the announcement is satisfied. In other words, in PAL an announcement becomes known after it is announced. The syntax of PAL includes an announcement modality, yet PAL is not more expressive than basic epistemic modal logic since the public announcement modality is reducible to the epistemic modality [16]. Therefore, PAL does not provides us with a richer language, but with a more succinct and compact language.

For our epistemic inquiry of the nabla modality, we use $\mathsf{S5}$ frames equipped with a nabla modality instead of the classical epistemic modality. We will furthermore, extend it with *arbitrary announcement operator* [1]. We will call our logic *arbitrary nabla public announcement logic*, or APAL∇ for short. The formal language $\mathcal{L}_{\boxdot}$ is a conglomerate of nabla logic and arbitrary public announcement logic, and is given as follows[1].

$$p \mid \top \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \nabla\Phi \mid [\varphi]\varphi \mid \boxdot\varphi$$

The formula $[\varphi]\psi$ reads "if φ is true, then after the announcement of φ, ψ shall be true as well". Furthermore, $\boxdot\varphi$ reads "after every possible announcement, φ is true". Announcements update the model. Therefore, after the announcement of φ, states that do not satisfy φ are eliminated, and the announcement becomes known among all the agents (it also becomes common knowledge). Based on this idea, semantics of public announcements is given as follows.

Definition 4 (Semantics). *Let $M = \langle W, R, V\rangle$ be a relational modal model. The semantics of Booleans and Nabla are given already. Then, we define the semantics of dynamic modalities as follows for $w \in W$.*

$M, w \models [\varphi]\psi$ *iff* $M, w \models \varphi$ *implies* $M|\varphi, w \models \psi$
$M, w \models \boxdot\varphi$ *iff for all* $\psi \in \mathcal{L}_{\nabla}$, $M, w \models [\psi]\varphi$

[1] In the original construction, authors used $\Box$ instead of $\boxdot$ [1].

The updated model $M|\varphi$ is the model $M|\varphi = \langle W', R', V' \rangle$ where $W' = \{w : M, w \models \varphi\}$, $R' = R \cap (W' \times W')$ and $V' = V \cap W'$.

Arbitrary announcements quantify over formulas in the language $\mathcal{L}_\nabla$. Otherwise, if no restriction is made then the definition can be circular, and "it is not clear that [non-restricted definition] is well-defined" [1].

As we have pointed out already, we consider $\mathsf{S5}$ frames. Before discussing the axioms of APAL∇, let us mention the frame characterizing formulas for $\mathsf{S5}$ in $\mathcal{L}_\boxdot$. Since the translation of the usual epistemic modality $\mathsf{K}\varphi = \nabla\emptyset \vee \nabla\{\varphi\}$ is given already, we immediately observe the following translations. The normativity axiom in $\mathcal{L}_\nabla$ is given as $(\nabla\emptyset \vee \nabla\{\varphi \to \psi\}) \to ((\nabla\emptyset \vee \nabla\{\varphi\}) \to (\nabla\emptyset \vee \nabla\{\psi\}))$. The veridicality axiom T is translated as $\varphi \to \nabla\{\varphi, \top\}$. The positive introspection axiom 4 then becomes $\nabla\{\nabla\{\varphi, \top\}, \top\} \to \nabla\{\varphi, \top\}$. Finally, the negative introspection axiom 5 is translated as $\nabla\{\varphi, \top\} \to \nabla\emptyset \vee \nabla\{\nabla\{\varphi, \top\}\}$.

Proof theory of APAL∇ admits modus ponens, announcement generalization(from $\vdash \varphi$, infer $\vdash [\psi]\varphi$, arbitrary announcement generalization (for a propositional variable p, from $\vdash \varphi \to [\psi]p\zeta$ infer $\vdash \varphi \to [\psi] \boxdot \zeta$) and nabla generalization (from $\vdash \varphi$ infer $\vdash \nabla\emptyset \vee \nabla\{\varphi\}$). Soundness of these rules can be shown in the standard way.

Let us now give an axiomatization for APAL∇ based on the axiomatization of APAL which was suggested in [15, 1].

1. All instances of propositional tautologies
2. $\mathsf{S5}$ axioms for nabla modality
3. $[\varphi]p \leftrightarrow (\varphi \to p)$
4. $[\varphi]\neg\psi \leftrightarrow (\varphi \to \neg[\varphi]\psi)$
5. $[\varphi](\psi \wedge \chi) \leftrightarrow ([\varphi]\psi \wedge [\varphi]\chi)$
6. $[\varphi][\psi]\chi \leftrightarrow [(\varphi \wedge [\varphi]\psi)]\chi$
7. $\boxdot\varphi \to [\psi]\varphi$ for $\psi \in \mathcal{L}_\nabla$
8. $[\varphi]\nabla\Psi \leftrightarrow (\varphi \to \nabla[\varphi]\Psi)$ where $[\varphi]\Psi$ is an abbreviation for $\{[\varphi]\psi : \psi \in \Psi\}$
9. $\boxdot\nabla\Psi \to [\varphi]\nabla\Psi$ for $\varphi \in \mathcal{L}_\nabla$

Let us now show the soundness of nabla axioms before making further observations. We will need the following rules from the basic dynamic epistemic logic: $[\varphi]\Box\psi \leftrightarrow (\varphi \to \Box[\varphi]\psi)$ and similar for the $\Diamond$ modality: $[\varphi]\Diamond\psi \leftrightarrow (\varphi \to \Diamond[\varphi]\psi)$

For the soundness of the eighth axiom, observe the following.

$$\begin{aligned}
[\varphi]\nabla\Psi \leftrightarrow\ & [\varphi]((\bigwedge \blacklozenge\Psi) \wedge (\Box \bigvee \Psi)) \\
& [\varphi](\bigwedge \blacklozenge\Psi) \wedge [\varphi](\Box \bigvee \Psi) \\
& ([\varphi]\Diamond\psi_1 \wedge \cdots \wedge [\psi]\Diamond\psi_\omega) \wedge (\varphi \to \Box[\varphi] \bigvee \Psi) \\
& (\varphi \to (\Diamond[\varphi]\psi_1 \wedge \cdots \wedge \Diamond[\varphi]\psi_\omega) \wedge (\varphi \to \Box[\varphi](\psi_1 \vee \cdots \vee \psi_\omega)) \\
& \varphi \to ((\Diamond[\varphi]\psi_1 \wedge \cdots \wedge \Diamond[\varphi]\psi_\omega) \wedge \Box[\varphi](\psi_1 \vee \cdots \vee \psi_\omega)) \\
& \varphi \to ((\Diamond[\varphi]\psi_1 \wedge \cdots \wedge \Diamond[\varphi]\psi_\omega) \wedge \Box([\varphi]\psi_1 \vee \cdots \vee [\varphi]\psi_\omega)) \\
& \varphi \to \nabla[\varphi]\Psi
\end{aligned}$$

where $\Psi = \{\psi_1, \ldots, \psi_\omega\}$ and $[\varphi]\Psi$ is an abbreviation for $\{[\varphi]\psi : \psi \in \Psi\}$.

Soundness of the other axioms can easily be shown. The completeness of APAL∇ is then immediate.

Theorem 7. *Arbitrary nabla public announcement logic is complete with respect to the given axiomatization.*

Proof. The proof is the standard completeness argument for most variants of PAL. The given axiomatization show that every formula in APAL∇ can be reduced to a formula in APAL. Since APAL is complete (as it relies on the standard modalities and we consider only finite cover sets), so is APAL∇. $\square$

Now we can ask whether we can find an *announcement invariant* cover set. In an earlier work, we showed that in some systems some formulas are invariant to announcements, and their truth does not get effected by any announcement [2]. We called such formulas persistent formulas. Now, we ask a similar question in the context of APAL∇ whether we can have a persistent cover set.

Question 1. Let $M, w \models \nabla \Phi$. Can there be a nonempty $\Psi \subset \Phi$ such that $M, w \models \boxdot \nabla \Psi$?

Before answering Question 1, let us reconsider Example 1. Let $\Phi = \Phi_1$ and announce $[\neg p_1 \wedge \neg p_2]$. Then, the states u and v will be eliminated. Thus, we will not be able to find a nonempty subset of Φ which is satisfiable under nabla.

Nevertheless, we have a weaker result.

Theorem 8. *Let $M, w \models \nabla \Phi$. Then for all φ, there is a subset $\Phi_\varphi \subseteq \Phi$ such that $M, w \models [\varphi] \nabla \Phi_\varphi$.*

Proof. For each φ simply remove all formulas which are logically equivalent to $\neg\varphi$ from Φ. Thus, define $\Phi_\varphi = \Phi - \{\psi : \psi \equiv \neg\varphi\}$. Clearly, Φ_φ exists, possibly empty. The correctness of this procedure is then immediate. $\square$

Now observe that $\bigcap_{\varphi \in \Phi} \Phi_\varphi$ can be the empty set, therefore we give a negative answer to Question 1.

5 Conclusion and Future Work

Apart from the momentum it generated in the fields of coalgebra and category theory, the nabla modality can provide insights, and perhaps even criticism, for traditional epistemic modalities within the context of standard modal logic [12]. In this paper, we investigated how nabla behaves in some well-studied epistemic situations, and discussed some relevant complexity problems.

There are some other possible research directions that we have not investigated in this paper. The idea of cover set immediately invites some topological ideas to the subject. For instance, the question whether we can come up with a similar modality that can be helpful for topological compactness is a promising one.

Last, but not least, *en passant*, we mentioned that the cover sets can be inconsistent. This relates to paraconsistent logics which consider inconsistent yet non-trivial theories. Combined with paraconsistent algebras and categories, nabla operator seems to offer further insights on paraconsistency as well.

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A Appendix of Proofs

Proof (Theorem 1). We will only prove the theorem for the modal case $\nabla\Phi$. For left-to-right direction, assume that ∃loise has a winning strategy for the game $E(w, \nabla\Phi)$ for $\Phi \subseteq \mathcal{L}_\nabla$. Thus, now at the game-state (w, Φ), ∀belard can make two different moves. In the first type of move, ∀belard chooses a formula $\varphi \in \Phi$ forcing ∃loise to pick an accessible state v from w at which φ is true. The following rule of ∃loise is still in her winning strategy. Therefore, by the induction hypothesis of the theorem, we deduce that $v \models \varphi$ for every choice of φ that ∀belard makes, thus (v, φ) is in the winning strategy of ∃loise. On the other hand, if ∀belard happens to choose an accessible state v forcing ∃loise to pick a formula $\varphi \in \Phi$ at which φ is true, by the same argumentation we see that (v, φ) is in the winning-strategy of ∃loise. Considering the fact that the initial choice by ∀belard also fits in the winning strategy of ∃loise, we deduce that $(w, \nabla\Phi)$ is in the winning strategy of ∃loise.

For the right-to-left direction, assume that $w \models \nabla\Phi$. Then, by the game rules, we see that after either choice of ∀belard, we end up with a situation where $v \models \varphi$ for accessible state v and $\varphi \in \Phi$. By the induction hypothesis, then we see that ∃loise has a winning strategy at the game-state (v, φ). Notice that whichever type of moves that ∀belard makes, we end up with a game-state at which ∃loise has a winning strategy. Then, by the definition of the notion winning strategy, we take one step to see that ∃loise has a winning strategy at the game-state $(w, \nabla\Phi)$.

This concludes the proof. □

Proof (Proposition 1). Note that we assumed that φ and ψ are supposed to be distinct elements of Φ as otherwise the result is trivial. Here, we will show one direction of the proof and leave the other direction to the reader as it is analogous. Now, assume $w \models \nabla\Phi$, and further suppose that for some $\varphi, \psi \in \Phi$, we have $(\varphi) \cap [w] = (\psi) \cap [w]$.

Suppose, $w \models \nabla(\Phi - \{\varphi\})$. Then, by definition, we have that for all $\alpha \in (\Phi - \{\varphi\})$, there is a $v \in [w]$ such that $v \models \alpha$; and on the other hand, for all $v \in [w]$, there is $\alpha \in \Phi - \{\varphi\}$ such that $v \models \alpha$. We will show that $w \models \nabla(\Phi - \{\psi\})$.

For the first conjunct, take an arbitrary $\beta \in (\Phi - \{\psi\})$. Since $(\Phi - \{\psi\}) \subseteq \Phi$, we observe $\beta \in \Phi$. Therefore, by assumption and definition, there is a $v \in [w]$ with $v \models \beta$. So, we are done with the first conjunct immediately.

In order to show the second conjunct, let us take an arbitrary state $v \in [w]$. By assumption, there is $\alpha \in (\Phi - \{\varphi\})$ such that $v \models \alpha$. If $\alpha \neq \psi$, we are done. If $\alpha = \psi$, then we observe $v \in (\psi)$ for $v \in [w]$. Thus, $v \in (\psi) \cap [w]$. By assumption, then $v \in (\varphi) \cap [w]$. Thus, $w \models \varphi$ where $\varphi \in (\Phi - \{\psi\})$. Thus, for a given $v \in [w]$, we have a formula $\varphi \in (\Phi - \{\psi\})$ with $v \models \varphi$.

Hence, $w \models \nabla(\Phi - \{\psi\})$. The converse direction is analogous, and this concludes the proof. □

Proof (Proppsition 3). Assume $w \models \nabla\Phi$. For some $\Psi \subseteq \Phi$, suppose further the following $\bigcup_{\psi \in \Psi}((\psi) \cap [w]) \subseteq \bigcup_{\varphi \in \Phi - \Psi}((\varphi) \cap [w])$.

Let $\alpha \in \Phi - \Psi$ be arbitrary. Then, $\alpha \in \Phi$, and by assumption, there is a $v \in [w]$ with $w \models \alpha$. Thus, for any $\alpha\Phi - \Psi$, there is a $v \in [w]$ such that $v \models \alpha$.

Now, take an arbitrary $v \in [w]$. Then, by assumption, there is a $\psi \in \Phi$ such that $v \models \varphi$. We have to separate cases: either $\psi \notin \Psi$ or $\psi \in \Psi$. We are immediately done if the former case holds. If the latter is the case, by assumption, we observe that at each state that the ψ holds, there exists some formula in $\Phi - \Psi$. Thus, for every $v \in [w]$, there is a formula in $\Phi - \Psi$ satisfied at w.

Hence, $w \models \nabla(\Phi - \Psi)$. □

Proof (Theorem 2). Given Φ, a nondeterministic algorithm needs to guess the subset $\Phi' \subseteq \Phi$ and check in polynomial time if it covers.

For the hardness part of the proof, we will select a known NP-complete problem. We pick the MinimumCover problem, namely, given a collection C of a set S and a natural number n, the problem of finding whether C contains a cover of S of size n or less [7]. We then translate MinimumCover to MinimalNabla by exhibiting a polynomial transformation from MinimumCover to MinimalNabla.

Take a set S and a collection C with $S \subseteq \cup C$ as an arbitrary instance of MinimumCover. We will then construct a model and a cover set Φ. Put $[w] := S$, let $|C| = I$, and for each $U_i \in C$, define a formula φ_i that holds at those states. Define $\Phi := \cup_{i \in I} \varphi_i$. Thus, we have $w \models \nabla\Phi$. Clearly, our construction is polynomial time.

Since in MinimalNabla we are not concerned with the size of the minimal cover set, we will run the MinimumCover problem for every $n \leq |\Phi|$ starting from $n = 1$. Then, the MinimumCover problem will find the minimum covering C' for S. Let us say $|C| = J$. Then, define $\Phi' = \cup_{j \in J} \varphi_j$, i.e. put those formulas φ_i whose corresponding set U_i is in C', to Φ'. Then, we will have Φ' as a minimal cover set with $w\nabla\Phi'$. Correctness of this is immediate. For each state in $[w]$ (i.e. S), there is a formula (i.e. cover) $\varphi \in \Phi'$ with $w \models \varphi$. The formula φ exists, because otherwise C' would not be a cover.

Since finding a minimum cover problem MinimumCover is NP-complete and our transformation procedure is polynomial time, it follows that MinimalNabla is NP-complete. □

Proof (Theorem 3). By using the Knapsack problem or weighted subset cover problem which are known to be NP-complete, the proof is immediate. □

Proof (Theorem 6). The proof is an extension of Parikh's original proof [14].

Let T be a theory. Assume further that $\{\mathcal{L}_1, \ldots, \mathcal{L}_n\}$ be a partition of the given language $\mathcal{L}$ and $M_1, \ldots, M_n$ be models such that each M_i is defined within $\mathcal{L}_i$.

Define $Mix(M_1, \ldots, M_n; \mathcal{L}_1, \ldots, \mathcal{L}_n)$ as the unique structure that agrees with M_i on L_i for each i. Now we need the following lemma. The structure Mix exists and is unique since the languages $\mathcal{L}_1, \ldots, \mathcal{L}_n$ partitions $\mathcal{L}$, and by definition M_i is defined within $\mathcal{L}_i$.

Lemma 1. *$\{\mathcal{L}_1, \ldots, \mathcal{L}_n\}$ is a T-splitting iff for each model $M_1, \ldots, M_n$ with $M_i \models T$, we also have $Mix(M_1, \ldots, M_n; \mathcal{L}_1, \ldots, \mathcal{L}_n) \models T$.*

Proof. (Lemma 1) Let $T = Con(\varphi_1, \ldots, \varphi_n)$ where $\varphi_i \in \mathcal{L}_i$. Assume for each submodel $M_i \models T$, we set $M' = Mix(M_1, \ldots, M_n; \mathcal{L}_1, \ldots, \mathcal{L}_n)$. Then, for each i, M' agrees with M_i and therefore $M_i \models \varphi_i$. Thus, we conclude, $M' \models T$.

For the converse direction, define $Mod(T) = \{M : M \models T\}$. Then, Mod_i is the projection of Mod to the language $\mathcal{L}_i$. Therefore, $M \subseteq Mod_1(T) \times \cdots \times Mod_n(T)$. For the converse direction of the subset inclusion, take an arbitrary $M \in Mod_1(T) \times \cdots \times Mod_n(T)$. So for each i, there is a model M_i such that $M_i \models T$ and, M and M_i agree on $\mathcal{L}_i$. Then, $M = Mix(M_1, \ldots, M_n; \mathcal{L}_1, \ldots, \mathcal{L}_n)$, by definition. Thus, $M \models T$. Therefore, $M \in Mod(T)$. We now conclude that $Mod(T) = Mod_1(T) \times \cdots \times Mod_n(T)$.

Now, let $U_i \in \mathcal{L}_i$ such that $Mod_i(T) = Mod(U_i)$. Then, we observe that $Mod_1(T) \times \cdots \times Mod_n(T) = Mod(U_1) \times \cdots \times Mod(U_n) = Mod(T)$. Thus, $T = Con(U_1, \ldots, U_n)$. Consequently, $\{\mathcal{L}_1, \ldots, \mathcal{L}_n\}$ is a T-splitting.

This finishes the proof of Lemma 1.

Now we still need to establish the existence of finest splitting that covers the language. This part simply follows from Parikh's original proof. In order to have a complete discussion of the subject matter, we will repeat it nevertheless.

Let $P = \{\mathcal{L}_1, \ldots, \mathcal{L}_n\}$ be a maximally fine splitting. We will now show that if it is maximally fine, then it is the unique finest splitting. Suppose not. Therefore, there is another splitting $P' = \{\mathcal{L}'_1, \ldots, \mathcal{L}'_m\}$ which is not refined by P. Therefore, there is some $\mathcal{L}_i$ and $\mathcal{L}_j$ that do overlap while one does not contain the other. Without loss of generality, let us suppose that $i = j = 1$. Now, consider the following partition $\{\mathcal{L}'_1, (\mathcal{L}'_2 \cup \cdots \cup \mathcal{L}'_m)\}$. This 2-element T-splitting is not refined by P either.

Consider the splitting $S = \{\mathcal{L}_1 \cap \mathcal{L}'_1, \ldots, \mathcal{L}_n \cap \mathcal{L}'_1, \mathcal{L}_1 \cap \mathcal{L}'_2, \ldots, \mathcal{L}_n \cap \mathcal{L}'_2, \ldots, \mathcal{L}_1 \cap \mathcal{L}'_m, \ldots, \mathcal{L}_n \cap \mathcal{L}'_m\}$.

By Lemma 1, it is easy to see that S is also a T-splitting. For each intersection language $\mathcal{L}_i \cap \mathcal{L}'_j$, define a model M_{ij} that satisfies T, and consequently together the M_{ij} form a very large $Mix(\cdot;\cdot)$ structure. We now leave the tedious details to the reader.

Therefore, S is a proper and finer refinement of P even though P was assumed to be a maximally refined splitting. Thus, P is *the* finest T-splitting.

Now, to adjust the simple notation we utilized in this proof to the statement of the theorem, let $\mathcal{L}$ be $\mathcal{L}_\nabla$ and $\mathcal{L}_i$ be $\mathcal{L}_{\nabla_i}$.

We set $\Xi_i = \mathcal{L}_{\nabla_i} \cap ([w])$. In this notation, Ξ_i is the set of formulas which are true at the accessible states of w and restricted to the language $\mathcal{L}_{\nabla_i}$. However, some of the Ξ_i covers may not *cover* all accessible states. More precisely, for some j we may have $M, w \not\models \nabla \Xi_j$. If this is the case, then we will put some Ξ_is together in such a way that $M, w \models \nabla(\Xi_i \cup \Xi_j)$.

The procedure for this task is follows for each i. If $M, w \models \nabla \Xi_i$, then set $\Phi_i := \Xi_i$. If this is not the case, search for some Ξ_j such that $M, w \models \nabla(\Xi_i \cup \Xi_j)$. If no such Ξ_j exists, repeat the search procedure for $(\Xi_i \cup \Xi_j)$ to find some Ξ_k. This search procedure terminates as each Ξ_is are the subsets of the given cover set. Once such $\Xi_j, \Xi_k, \ldots, \Xi_l$ are found with $M, w \models \nabla(\Xi_i \cup \Xi_j \ldots \Xi_l)$; put $\Phi_i := \Xi_i \cup \cdots \cup \Xi_l$.

Now, it is easy to see that $w \models \bigwedge_i \nabla \Phi_i$ by the construction of Φ_i; and by definition, Φ_i's are mutually disjoint. Thus, it is a cover splitting.

This concludes the proof of Theorem 6. $\square$