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Some topological properties of paraconsistent models

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Abstract In this work, we investigate the relationship between paraconsistent semantics and some well-known topological spaces such as connected and continuous spaces. We also discuss homotopies as truth preserving operations in paraconsistent topological models.

Keywords Paraconsistent logic · Topological semantics · Modal logic

1 Introduction and motivation

The well-studied notion of deductive explosion describes the situation where any formula can be deduced from an inconsistent set of formulas (or from its deductive closure). In other words, in deductively explosive logics, we have $\{\varphi, \neg\varphi\} \vdash \psi$ for all formulas φ, ψ where $\vdash$ is a logical consequence relation. In this respect, both "classical" and intuitionistic logics are known to be deductively explosive. Paraconsistent logic, on the other hand, is the umbrella term for the logical systems where the logical consequence relation is *not* explosive.

A variety of philosophical and logical objections can be raised against paraconsistency, and almost all of these objections can be dismissed in a rigorous fashion. In this work, we will not be concerned about the philosophical implications of paraconsistency, and refer the reader to the following references for a detailed overview of the

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subject from philosophical, logical and historical perspectives (da Costa et al. 2007; Priest 1998, 2002, 2007).

As the definition implies, paraconsistency has largely been approached from a proof theoretical point of view. Most of the work in this field concerns proof theoretical concepts, such as derivability of certain formulas and tableaux methods in various paraconsistent logics. In this work, however, we will consider semantical aspects of paraconsistency. A central theme in the semantics of paraconsistency is *true contradictions*. Therefore, perhaps it would be wiser to call our approach *dialetheic* rather than paraconsistent. Dialetheism is the view which asserts that there are true contradictions. It does not mean that *all* contradictions are true, but it maintains that there are *some* true contradictions. In this paper, in order not to make the terminology more complicated for the reader, we will use the term paraconsistent and dialetheism interchangeably.

Here, our object of study is topological models. We define them classically, and show how a paraconsistent logic can be defined in this framework following the usual approach. More specifically, we investigate the relationship between paraconsistency and some well-known topological spaces. We achieve this by considering paraconsistent topological semantics.

We need to make it clear at the beginning of our paper that the notions paraconsistency and dialetheism do not refer to the meta-logical (such as set theoretical or arithmetical) properties of the formal systems which we are discussing.[1] For that reason, some of our structural definitions and proof methods are *classical*.[2] Namely, paraconsistency does not claim that all primitives should be defined inconsistently nor inconsistencies should be allowed at meta-theoretical level. Paraconsistency, we underline, is a system where we control the inconsistencies and allow sensible deductions under their presence. For instance, in our system, we define most of the mathematical primitives (topology, subset, membership etc.) in the classical sense. One of our goals in this paper is to demonstrate how paraconsistency occurs at the object level in such classically defined models. Paraconsistency then will occur when we introduce a non-classical operator within the classical framework. This is indeed the usual strategy for paraconsistent logics. For instance, in some well known paraconsistent logics, truth values or negations were defined paraconsistently where the rest of the system retained its classical properties (da Costa 1974; Priest 1979). A similar strategy should also be familiar from intuitionistic logics where only some connectives and definitions are introduced non-classically (Mints 2000).

The use of topological semantics for paraconsistent logic is not new. To our knowledge, the earliest work discussing the connection between inconsistency and topology goes back to Goodman (1981).[3] In his paper, Goodman discussed "pseudo-complements" in a lattice theoretical setting and called the topological system he

[1] Thanks to the anonymous referee for bringing this point to my attention.

[2] For instance, classical oriented readers of paraconsistency may find it quite puzzling when paraconsistent logicians employ *proof by contradiction* as a proof method. Paraconsistent logic or dialetheism, note again, does not claim that *all* contradictions are acceptable.

[3] Thanks to Chris Mortensen for pointing this work out. Even if the paper appeared in 1981, the work had been carried out around 1978. In his paper, Goodman indicated that the results were based on an early work that appeared in 1978 only as an abstract.

obtains "anti-intuitionistic logic". In a recent work, Priest discussed the dual of the intuitionistic negation operator and considered that operator in a topological framework (Priest 2009). Similarly, Mortensen discussed topological separation principles from a paraconsistent and paracomplete point of view and investigated various inconsistent theories in some topological spaces (Mortensen 2000). The relation between paraconsistency and some modal logics was discussed by Béziau as well (Béziau 2005). However, none of the aforementioned works investigated the relation between topological spaces and paraconsistent semantics in detail, and it is our task here to provide that.

We also underline that similar investigations have been carried out extensively for various classical modal logics (Aiello et al. 2007, 2003; van Benthem and Bezhanishvili 2007). What we achieve here is to provide the reader with observations on paraconsistent topological spaces in such a way that similarities and differences between classical and non-classical logics can be seen more clearly from a semantic angle. One direction of this research program (namely the classical one) has been pursued thoroughly, and now it is time to do justice to the whole research program.

The organization of the paper is as follows. First, we present the topological basics of our subject in a nutshell. Then, we point out the connections between topological semantics and paraconsistency and make some further observations between different types of topologies and paraconsistency, and present our results. Finally, after a brief remark on the modal version of paraconsistent topological semantics, we conclude with possible research directions for future work, underlining the fact that the field is largely unexplored.

2 Basics

2.1 Definitions

The history of topological semantics for (classical modal) logics can be traced back to the early 1920s, making it the first semantics for a variety of modal logics (Goldblatt 2006). The major revival of topological semantics for modal logics is due to McKinsey and Tarski when they initiated a study of topological semantics and algebras for topological spaces (McKinsey and Tarski 1944, 1946). Following the same research program, the extension of topological semantics to intuitionistic logic can also be achieved in a natural way (Mints 2000).

In this section, we briefly mention the basics of topological semantics to set the basis of our discussion. We give two equivalent definitions of topological spaces here for our purposes. We call the first one "the open set" definition, and the second "the closed set" definition.

Definition 2.1 (*Open set definition*) The structure $\langle S, \sigma \rangle$ is called a topological space if it satisfies the following conditions.

1. $S \in \sigma$ and $\emptyset \in \sigma$.
2. σ is closed under arbitrary unions and under finite intersections.

Definition 2.2 (*Closed set definition*) The structure $\langle T, \tau \rangle$ is called a topological space if it satisfies the following conditions.

1. $T \in \tau$ and $\emptyset \in \tau$.
2. τ is closed under finite unions and under arbitrary intersections.

Collections σ and τ are called topologies. The elements of σ are called *open* sets whereas the elements of τ are called *closed* sets. A set is open if its complement in the same topology is closed and vice versa. Functions can easily be defined on topological spaces. A function is called *continuous* if the inverse image of an open (respectively, closed) set is open (respectively, closed), and a function is called *open* if the image of an open (respectively, closed) set is open (respectively, closed). Moreover, two topological spaces are called *homeomorphic* if there is a continuous bijection with a continuous inverse from one space to the other. Two continuous functions are called *homotopic* if there is a continuous deformation between the two—namely, if one function can be continuously transformed to the other. Homotopy is an equivalence relation and gives rise to the subject of homotopy groups which is a foundational subject in algebraic topology.

It should be underlined that our definitions are the standard textbook definitions of classical topological primitives. We believe this is important for two reasons. First, it shows on a more philosophical and *ideological* level that paraconsistency is quite *natural* and *expected* even under the very basic classical notions, and that there is a way to handle true contradictions with the given classical primitives. Second, more pragmatically, we build our current (and future) work on paraconsistent topologies on the already existing body of mathematical work in the field, so that we can extend our research program to various other subfields of topology (mereotopology), algebra (Co-Heyting algebras) and game theory (epistemic game theoretical paradoxes) in the future.

2.2 Syntax and semantics

Let us now define our syntax before discussing the semantics. We denote a countable set of propositional variables with P. The symbol $\bot$ stands for contradiction. We use a language of propositional modal logic with the modality $\Diamond$, and define the dual modality $\Box$ in the usual sense. Informally, we construct the language of the basic unimodal logic recursively in the standard fashion by using the symbols $\bot$, $\sim$, $\wedge$ and $\Diamond$. When we discuss negation in a classical or intuitionistic frameworks, we will use the $\neg$ symbol in order to distinguish it.

Since our focus is paraconsistent logic, we need to be precise about the $\bot$ symbol. As we will indicate later on, $\bot$ is a constant which is true nowhere. Note that in paraconsistent logics, we do not define contradiction as $p \wedge \sim p$ since some contradictory statements may be satisfiable.

Let us now fix some notation and terminology. The *extension* of a formula φ in the model M is defined as the points in M at which φ is satisfied, and denoted as $[\varphi]^M$. We will omit the superscript if the model we are working in is obvious. Moreover, by a *theory*, we mean a deductively closed set of formulas. We define a *subtheory* in the usual sense by using the classical subset relation.[4]

In topological semantics, the modal operator $\Box$ for necessitation corresponds to the topological *interior* operator Int where $\mathsf{Int}(O)$ is the largest open set contained in set O. Furthermore, one can dually associate the topological closure operator Clo with the possibility modal operator $\Diamond$ where the closure $\mathsf{Clo}(O)$ of a given set O is the smallest closed set that contains O. In this framework, the extensions of Boolean cases are obvious. The extension of a modal formula $\Box\varphi$ will then be associated with an open set in the topology. Thus, we have $[\Box\varphi] = \mathsf{Int}([\varphi])$. Similarly, we will put $[\Diamond\varphi] = \mathsf{Clo}([\varphi])$ associating Clo operator with the $\Diamond$ modality. Therefore, in the classical setting, modalities necessarily produce topological entities such as open or closed sets. However, the extension of Booleans may or may not be topological entities.

At this stage we can take one step further, and suggest that extension of *any* propositional variable will be an open set (Mints 2000; Mortensen 2000). In that setting, conjunction and disjunction work as before for finite intersections and unions. Nevertheless, the negation can be problematic as the complement of an open set is generally not an open set, thus may not be the extension of a formula in the language. For this reason, we need to define the negation operator topologically as the *open complement* (interior of the complement) of a given set. In this semantics, $\neg p$ is true if and only if p is true at the interior of the complement of the extension of p. By this method, we obtain an intuitionistic logic that supports the incomplete theories where such systems can also be called *paracomplete*. It is easy to show that in paracomplete logics, $p \vee \neg p \rightarrow \top$ is not necessarily true. Topological semantics for paracomplete logics (particularly for intuitionistic logic) are well known and well studied, and we will not pursue them here.

A similar idea can be used to obtain paraconsistent logics where we stipulate that the extension of *any* propositional variable will be a closed set. In order to avoid a similar problem with the negation operator, we define the negation as the closed complement. Namely, in this semantics, $\sim p$ is true if and only if p is true at the closure of the complement of the extension of p. We call such systems paraconsistent topological modal logics. They are our main focus in this work.

Now, let us consider the boundary $\partial(\cdot)$ of a set O where $\partial(O)$ is defined as $\partial(O) := \mathsf{Clo}(O) - \mathsf{Int}(O)$. Then for a formula φ, consider the boundary of its extension $\partial([\varphi])$ in a topology. We will now briefly explain how paracomplete and paraconsistent semantics differ underlining the importance boundary points in the semantics.

Let us start with paracomplete logics. Let $x \in \partial([\varphi])$. Since $[\varphi]$ is open in paracomplete logics, $x \notin [\varphi]$ by definition. Similarly, $x \notin [\neg\varphi]$ as the open complement is also open by definition. Thus, neither φ nor $\neg\varphi$ is true at the boundary points. We conclude that in paracomplete topological semantics, any theory with formulas which have both nonempty extensions and nonempty open complements is incomplete. Consequently, we can make a similar observation about the boundary points using paraconsistent topological semantics. Now, take $x \in \partial([\varphi])$ where $[\varphi]$ is closed in paraconsistent topological models. By the above definition, since we have $x \in \partial([\varphi])$, we obtain

[4] Remember that our meta-theory is classical, thus the subset relation we resort to is also classical.

$x \in [\varphi]$ as $[\varphi]$ is closed. Yet, $\partial([\varphi])$ is also included in $[\sim\varphi]$ which we have defined as a closed set. Thus, by the same reasoning, we conclude $x \in [\sim\varphi]$. Thus, $x \in [\varphi \wedge \sim\varphi]$ yielding that $x \models \varphi \wedge \sim\varphi$. Therefore, in paraconsistent topological models, any theory with formulas which have nonempty extensions and nonempty closed complements is inconsistent. This summarizes our exposition of the subject; so far, we have recalled how paracomplete and paraconsistent semantics can be defined and understood in a topological setting with classically defined primitives.

Let us be more precise now. Given a topological space $\langle T, \tau\rangle$, we define a paraconsistent topological modal model as $M = \langle T, \tau, V\rangle$ where T and τ are as before, and $V : P \mapsto \wp(T)$ is a valuation function mapping propositional variables to subsets of T. An important distinction of paraconsistent topological modal models is that we stipulate that $V(P) \subseteq \mathsf{K}(\tau)$ where $\mathsf{K}(\tau)$ is the set of closed sets in topology τ. In other words, the valuation of propositional variables are closed sets. Here, also note that the topological space can be "larger" than what can be covered by the model and its semantics. In other words, if we consider the topological space as the real world, the model (with its language and semantics) represents what we can discuss about the real world with the given language and semantics.

Finally, let us make a notational convention. Throughout the paper, we will use the notation in Definition 2.1 when we are referring to topological spaces defined with respect to opens, and we will use the notation in Definition 2.2 for the topological spaces defined with respect to closed sets such as paraconsistent topological models. In short, $\langle T, \tau\rangle$ will be reserved for paraconsistent topological models to distinguish them from the classical or paracomplete ones.

3 Topological properties and paraconsistency

In this section, we investigate the relation between various basic topological properties and paraconsistency. Our work can be seen as a continuation and an expansion of Mortensen's earlier work where he briefly discussed some topological separation axioms and their relation to the inconsistent theories (Mortensen 2000). Here, we extend his approach to some other topological properties and spaces, and discuss the behavior of such spaces under some special functions.

Let us now clarify the syntax and semantics of our system which we call *paraconsistent topological logic* (PTL). Given a countable set of propositional variables P, we define the language of PTL recursively as follows for $p \in P$.

$$\varphi ::= \bot \mid p \mid \sim\varphi \mid \varphi \wedge \varphi \mid \Diamond\varphi$$

Note that the language of PTL has the contradiction symbol $\bot$ which is not necessarily logically equivalent to $p \wedge \sim p$ as we have observed earlier.

Given a paraconsistent topological model $M = \langle T, \tau, V\rangle$, we define the semantics in terms of the extensions of the formulas. For a set X, we denote its closed complement (closure of its complement) by X^c.

- $[\bot] = \emptyset$
- $[p]$ is a closed set in T

- $[\sim\varphi] = [\varphi]^c$ $(= \mathsf{Clo}(T - [\varphi]))$
- $[\varphi \wedge \psi] = [\varphi] \cap [\psi]$
- $[\Diamond\varphi] = \mathsf{Clo}([\varphi])$

We denote the satisfaction of a formula φ at a point w in a model M by $M, w \models \varphi$. The proof theoretical rules we employ are modus ponens (from φ and $\varphi \rightarrow \psi$ derive ψ) and necessitation (from $\vdash \varphi$ derive $\vdash \Box\varphi$). We use the symbol $\equiv$ for logical equivalence. Furthermore, De Morgan laws do not necessarily hold in PTL as the closure and intersection operators do not commute in this semantics.[5]

Note that in this paper, our concern is not to axiomatize PTL or present its relevant completeness or decidability results. What we are trying to achieve is to observe the behavior of certain formulas in certain topological spaces within the paraconsistent framework.

A word of caution is necessary here. Since we stipulated that the extension of Booleans are closed sets, and $\mathsf{Clo}(\mathsf{Clo}(X)) = \mathsf{Clo}(X)$ by basic topology, the diamond modality is redundant. In other words, because of the way we defined negation and the topological connection between the $\Diamond$ and Clo operators, it is easy to see that $p \leftrightarrow \Diamond p$ is valid in PTL. Nevertheless, for the reasons of completeness of our structure, we still keep the standard modal operator $\Diamond$ in our language similar to the reason as to why the public announcement operator is kept in the language of public announcement logic even if it does not increase the expressibility of the language but provides succinctness (van Ditmarsch et al. 2007).

Finally, observe that the semantics of PTL makes it clear how we understand the symbol $\bot$ paraconsistently. In PTL, $\bot$ is true nowhere. Another way of seeing this is the fact that in PTL, $p \wedge \sim p$ is not logically equivalent to $\bot$. It is very easy to provide a PTL model for this case. Take an arbitrary w from the nonempty extension $[p \wedge \sim p]$ for some $p \in P$. Then, $w \in [p]$ and $w \in [\sim p]$. Thus, $w \in [p]$ and $w \in [p]^c$, namely $p \in \partial([p])$. In short, as long as the boundary of $[p]$ is nonempty, we cannot deduce $\bot$ from $p \wedge \sim p$. Thus, PTL is a paraconsistent logic, as expected.

When topological spaces and inconsistencies are concerned, discrete topologies where every subset is closed (or dually open) provide an interesting case. We conclude this section with Mortensen's observation.

Proposition 3.1 (Mortensen 2000) *If a topological space $\langle S, \sigma\rangle$ is discrete, then every theory on $\langle S, \sigma\rangle$ is consistent and complete.*

Proof See the aforementioned reference. □

Similarly, at the other extreme of the spectrum, in a trivial topology (where only closed sets are the empty set and the space itself), every theory is consistent and complete as well.

In the rest of the paper, we will investigate how the topological semantics for PTL interacts with different topological models. The proofs of our results will demonstrate how paraconsistency works. Briefly, we allow true contradictions in our logic, not in its metatheory. For this reason, for instance, in our proofs we often use proof by

[5] See Ferguson (2012) for a more detailed treatment of non-classical logics that do not satisfy DeMorgan's laws.

contradiction by obtaining contradictions towards the meta-logical or set theoretical properties of PTL. PTL is equipped to handle true contradictions at the object level by its semantics, but not at the meta level. This is essentially a design choice. It is possible to design a system with paraconsistent meta-theory in a paraconsistent set theory and arithmetic. However, in our work our focus is working with the contradictions at the object level. Some metaphysical reasons can be given for this choice, yet we will refrain ourselves from dwelling on the metaphysics of paraconsistency as our main focus here is the logical and topological properties of PTL.

3.1 Connectedness

In this section, we discuss connectedness and its relation to paraconsistency. Let us start with the basic definitions. A topological space is called *connected* if it is not the union of two disjoint non-empty closed sets. The same definition works if we replace "closed sets" with "open sets". Let us state the formal definition to be precise.

Definition 3.2 A set X is called *connected* if $A \cup B \neq \emptyset$ whenever A, B are nonempty closed subsets and $X = A \cup B$. Moreover, it is called totally disconnected if all of its subsets with more than one element are disconnected.

We note that in any connected topological space, the only subsets with empty boundaries are the space itself and the empty set (Bourbaki 1996).[6] We now define *connected formulas* as follows.

Definition 3.3 A formula φ is called connected in a model M, if for any two formulas α_1 and α_2 with nonempty closed (or dually, open) extensions in M, if $\varphi \equiv \alpha_1 \vee \alpha_2$, then we have $[\alpha_1 \wedge \alpha_2] \neq \emptyset$. We will call a theory connected if it is generated by a set of connected formulas.

In other words, connected formulas are formulas which cannot be split into two formulas with disconnected extensions within the language. For example, $\top$ is a connected formula, as we cannot find two formulas φ and $\sim\varphi$ with nonempty extensions and empty intersection. In such a case, we have $[\varphi \wedge \sim\varphi] = \partial([\varphi]) = \partial([\sim\varphi])$ which is nonempty by construction. Similarly, $\bot$ is never connected.

There is an interesting connection between connected formulas and Parikh's language splitting, and this is one of the reasons why we suggested the above definition (Parikh 1999). Parikh's work discusses situations where a theory can be written as a logical consequence of formulas from disjoint languages, making his approach syntactic. In this respect, connected formulas can be seen as the semantical anti-counterpart of language splitting. In our case, connected formulas are formulas which cannot be *split semantically*. Now, we suggest a semantical (and topological) cousin of Parikh's language splitting. First, we define *subtopologies*. Given a topology τ, a we call τ_1 a subtopology of τ if $\tau_1 \subseteq \tau$ where $\subseteq$ is the classical subset relation. We call τ_1 and τ_2 *disjoint subtopologies* of τ if both τ_1 and τ_2 are subtopologies of τ, and $\tau_1 \cap \tau_2 = \emptyset$. Now, we define semantic splitting as follows.

[6] Also, note that connectedness as a property is *not* definable in the (classical) modal language (Cate et al. 2009).

Definition 3.4 Given a topological model $M = \langle S, \sigma, V \rangle$, let σ_1 and σ_2 be disjoint subtopologies of σ. We say a formula is split into σ_1 and σ_2, if there are two formulas α_1 and α_2 with $[\alpha_1] \subseteq \sigma_1$ and $[\alpha_2] \subseteq \sigma_2$, and $\varphi \equiv \alpha_1 \vee \alpha_2$.

This definition can easily be extended to the case of $n < \omega$ subtopologies. Notice that since the arbitrary union of closed sets may not be closed, we do not define splittings into more than ω subtopologies.

Proposition 3.5 *Connected formulas cannot be split.*

Proof The proof is quite immediate. Let φ be a connected formula. In order to get a contradiction, assume it can be split into τ_1 and τ_2 in the given model $M = \langle T, \tau, V \rangle$. Therefore, there are some α_1 and α_2 with $[\alpha_1] \subseteq \tau_1$ and $[\alpha_2] \subseteq \tau_2$, and $\varphi \equiv \alpha_1 \vee \alpha_2$. However, since τ_1 and τ_2 is disjoint, $[\alpha_1] \cap [\alpha_2] = \emptyset$. Thus, φ is disconnected, contradicting the earlier assumption. □

However, we can split any formula with nonempty extension in discrete topologies.

Proposition 3.6 *Let $M = \langle T, \tau, V \rangle$ be a topological (paraconsistent) model. If τ is discrete and $|T| > 1$, then every formula with nonempty extension in M can be split.*

Proof First, recall that discrete topologies with at least two elements are totally disconnected. Let $M = \langle T, \tau, V \rangle$ be a given topological (paraconsistent) model where τ is discrete and $|T| > 1$. Take an arbitrary formula φ with a nonempty extension. Consider $[\varphi]$. Proof is by induction on φ.

The proposition holds vacuously for $\varphi \equiv \bot$ since $[\bot] = \emptyset$. Similarly, the argument is quite straight-forward for $\varphi \equiv p$ for propositional variable p. Let $\tau_1 = \{[p]\}$, and $\tau_2 = \tau - \tau_1$ to obtain $p \equiv p \vee \bot$.

For conjunction, assume $\varphi \equiv \psi \wedge \psi'$ where ψ and ψ' can both be split. Assume ψ is split into τ_1 and τ_2 with formulas α_1 and α_2 respectively. Similarly, assume that ψ' is split into τ_1' and τ_2' with the formulas α_1' and α_2'. Thus, we have $\psi \equiv \alpha_1 \vee \alpha_2$ and $\psi' \equiv \alpha_1' \vee \alpha_2'$ with the associated disjointness conditions. Then we observe that φ is split into $\overline{\tau_1}$ and $\overline{\tau_2}$ where $\overline{\tau_1}$ is the intersection topology of τ_1 and the disjoint union of $\tau_1' \cup \tau_2'$. Similarly, $\overline{\tau_2}$ is the intersection topology of τ_2 and the disjoint union of $\tau_1' \cup \tau_2'$. Since, τ_1 and τ_2 are assumed to be disjoint topologies, $\overline{\tau_1}$ and $\overline{\tau_2}$ are disjoint topologies as well. Therefore, we split φ into $\overline{\tau_1}$ and $\overline{\tau_2}$ with $\alpha_1 \wedge (\alpha_1' \vee \alpha_2')$ and $\alpha_2 \wedge (\alpha_1' \vee \alpha_2')$.

In order to see that the claim holds for negated formulas, assume $\varphi = \sim\psi$ for some ψ. Therefore, by induction hypothesis ψ can be split, namely for subtopologies τ_1, τ_2, there are two formulas α_1, α_2 with $[\alpha_1] \subseteq \tau_1$ and $[\alpha_2] \subseteq \tau_2$, and $\psi \equiv \alpha_1 \vee \alpha_2$. Consider $\sim\psi$ which is $\sim(\alpha_1 \vee \alpha_2)$. Then, φ can be split into τ_1' and τ_2' where $\tau_1' = \tau_2$ and $\tau_2' = \tau_1$.

This completes the proof. □

Note that Mortensen's result (Theorem 3.1) stated that every theory in discrete spaces is consistent. He then goes ahead and makes further observations about T_0 and T_1 separation principles and consistency. We refer the interested reader to his work (Mortensen 2000). Now, we improve his results.

Theorem 3.7 *A PTL model with no connected formulas cannot have true contradictions.*

Proof We will show that in a model $M = \langle T, \tau, V \rangle$ with no connected formula, we cannot find a formula α for which $[\alpha \wedge \sim\alpha] \neq \emptyset$.

To get a contradiction, assume that in M that has no connected formula, there is a formula α for which $[\alpha \wedge \sim\alpha] \neq \emptyset$. Call $\alpha \wedge \sim\alpha$ as φ.

By the assumption, φ is not connected. So, for every two formulas α_1 and α_2 for which $\varphi \equiv \alpha_1 \vee \alpha_2$, we have $[\alpha_1] \cap [\alpha_2] = \emptyset$ contradicting the earlier assumption. □

We can extend this result quite straight-forwardly.

Theorem 3.8 *A PTL model with totally disconnected topology cannot be inconsistent.*

Proof Let $M = \langle T, \tau, V \rangle$ be a PTL model where τ is totally disconnected. Towards a contradiction, assume that for φ, we have a true contradiction $\varphi \wedge \sim\varphi$. Consider $[\varphi] \cap [\sim\varphi]$. Since, the formula $\varphi \wedge \sim\varphi$ is a true contradiction, $[\varphi] \cap [\sim\varphi] \neq \emptyset$. This is a contradiction towards the total disconnectedness of τ. Thus, M cannot have true contradictions or inconsistencies. □

So far, we have investigated some immediate results on connected formulas and paraconsistent models. Now, let us make some further observations.

Theorem 3.9 *Every connected formula is satisfiable in some connected (classical) topological space.*

Proof Let φ be a connected formula and $M = \langle W, \nu, V \rangle$ a (classical) topological space where for some $w \in W$, $w \models \varphi$. Then, define a connected subspace $M|\varphi = \langle W_\varphi, \nu_\varphi, V_\varphi \rangle$ as follows. Let $W_\varphi = W \cap [\varphi]^M$ so that $W_\varphi = [\varphi]^{M|\varphi}$. Notice that $W_\varphi \neq \emptyset$ as $w \in W_\varphi$. The topology ν_φ then is defined as follows $\nu_\varphi = \{O \cap W_\varphi : O \in \nu\}$. It is easy to verify that ν_φ is indeed a topology (in fact the induced topology), so we skip it. Valuation V is restricted in the usual sense. Now, we need to show that ν_φ is connected.

Now, take any two formulas α_1 and α_2 with nonempty open extensions in $M|\varphi$. Observe that if $\varphi \equiv \alpha_1 \vee \alpha_2$, then $[\alpha_1 \wedge \alpha_2] \neq \emptyset$. Since $W_\varphi = [\varphi]$, and the extensions $[\alpha_1]$ and $[\alpha_2]$ are nonempty by the condition, this shows that the space W_φ is connected with respect to the topology ν_φ. □

Note that the way we obtained a topological submodel is a rather standard method in modal logics. A similar theorem within the context of dynamic epistemic logic showing the completeness of that logic in topological spaces also used a similar construction (Başkent 2011, 2012).

Corollary 3.10 *Every connected theory is satisfiable in some connected (classical) topological space.*

So far, we have made observations in classical topological spaces. Nevertheless, connected theories may be inconsistent or incomplete in some situations.

Theorem 3.11 *Every connected theory in a paraconsistent topological logic is inconsistent. Moreover, every connected theory in a paracomplete topological logic is incomplete.*

Proof Let T be a connected theory generated by a set of connected formulas $\{\varphi_i\}_i$, so $\varphi_i \in T$ for each i in a closed set topology. By the earlier corollary, T is satisfiable in some connected space, say $\langle W, \sigma \rangle$.

Consider an arbitrary φ_i from the basis of T. Since it is a connected formula, assume that we can write it as $\varphi_i \equiv \alpha \vee \beta$ for $[\alpha \wedge \beta] \neq \emptyset$. Let $x \in \partial[\alpha \wedge \beta] \subseteq [\varphi_i]$ as we are in a closed set topology and therefore $[\varphi]$ is closed. Thus, T includes φ_i which in turn includes the theories at x. By our earlier remarks in Sect. 2.2, this makes T inconsistent in σ.

As a special case, in PTL, observe that if $\top \in T$ where $[\top] = W$, then T is inconsistent as well. Take $\top \equiv p \vee \sim p$ for some propositional variable p with nonempty extension. Then, $[p \wedge \sim p] \neq \emptyset$.

The second part of the corollary about the incomplete theories and paracomplete models can be proved similarly. □

The converse direction is a bit more interesting. Do connected spaces satisfy only connected formulas?

Proposition 3.12 *Let X be a connected topological space of closed sets with a paraconsistent topological model on it. Then the only subtheory that is not inconsistent is the empty theory.*

Proof As we mentioned earlier, in any connected topological space, the only subsets with empty boundary are the space itself and the empty set. Thus, all other subsets will have a boundary, and their theories will be inconsistent by the earlier observations. By Theorem 3.11, the space itself produces an inconsistent theory. Therefore, the only theory which is not inconsistent is the empty theory. □

Based on this observation, we can prove a more general result.

Proposition 3.13 *Let X be a connected topological space of closed sets. Then for a collection of nonempty theories $T_1, \ldots, T_n$ with nonempty intersection $\bigcap_i T_i$, then we conclude $\bigcup_i T_i$ is inconsistent.*

Proof Each theory T_i will have a closed set of points X_i that satisfies it in the given topology. Since, $\bigcap_i T_i \neq \emptyset$, we observe $\bigcap_i X_i \neq \emptyset$. Therefore, $\bigcup_i X_i$ is connected and not equal to X. Thus, $\bigcup_i X_i$ has a nonempty boundary and the theories generated at the boundary points will be inconsistent. □

A basic property of the boundary operator yields the following observation.

Proposition 3.14 *Let $X = \langle T, \tau \rangle$ be an arbitrary connected topological space of closed sets. Define $\overline{X} = \{C : C = T \setminus B$ for some B in $X\}$. Then, X and $\overline{X}$ have the same inconsistent boundary theories.*

Proof The proof simply observes the fact that a set and its complement share the same boundary. A similar result can be shown for paracomplete theories, and we leave it to the reader. □

The proofs in this chapter illustrate our understanding of paraconsistency well. We allow contradictions at the object level, yet at the meta-level, as exemplified by our frequent use of the method of proof by contradiction, we do not allow contradictions. Simply put, it is the PTL which is inconsistent, not the set theory or arithmetic it is built on.

3.2 Continuity

A recent research program that considers topological modal logics with continuous functions was suggested to give a unified account of temporal and topological aspects of modal logic (Artemov et al. 1997; Kremer and Mints 2005). In their work, authors associated the modalities with continuous functions as follows $\bigcirc p = f^{-1}(p)$ where $\bigcirc$ is the temporal next time operator and f is a continuous function.

In our work, we tend to diverge from the classical modal logical approach, and focus on the connection between continuous or homeomorphic functions and modal logics with an agenda of applying it to paraconsistency.

An immediate theorem, which was stated and proved in various papers, would also work for paraconsistent logics, particularly for PTL (Kremer and Mints 2005). Consider two closed set topologies τ and τ' on a given set T, and a homeomorphism $f : \langle T, \tau\rangle \mapsto \langle T, \tau'\rangle$. Akin to a previous theorem of Kremer and Mints, we have a simple way to associate the respective valuations between two models M and M' which respectively depend on τ and τ', so that we can have a truth preservation result. So, define $V'(p) = f(V(p))$. Then, we have $M \models \varphi$ iff $M' \models \varphi$.

Theorem 3.15 *Let $M = \langle T, \tau, V\rangle$ and $M' = \langle T, \tau', V'\rangle$ be two paraconsistent topological models (where τ, τ' are closed set topologies) with a homeomorphism f from $\langle T, \tau\rangle$ to $\langle T, \tau'\rangle$ with $V'(p) = f(V(p))$. Then $M \models \varphi$ iff $M' \models \varphi$ for all φ.*

Proof The proof is by induction on complexity of formulas.

Let $M, w \models p$ for some propositional variable p. Then, $w \in V(p)$. Since we are in a paraconsistent topological model, $V(p)$ is a closed set and since f is a homeomorphism $f(V(p))$ is closed as well, and $f(w) \in f(V(p))$. Thus, $M', f(w) \models p$. The converse direction is similar and based on the fact that the inverse function is also continuous.

Negation is less immediate. Let $M, w \models \sim\varphi$. Then w is in the closure of the complement of $V(\varphi)$. So, $w \in \mathsf{Clo}((V(\varphi))^c)$. Then, $f(w) \in f(\mathsf{Clo}(V(\varphi))^c)$. Moreover, since f is bicontinuous as f is a homeomorphism, we observe that $f(w) \in \mathsf{Clo}(f((V(\varphi))^c))$. By the induction hypothesis, $f(w) \in \mathsf{Clo}((V'(\varphi))^c)$ yielding $M', f(w) \models \sim\varphi$. The converse direction is also similar.

We leave the conjunction case to the reader and proceed to the modal case. Assume $M, w \models \Diamond\varphi$. Thus, $w \in V(\Diamond\varphi)$. Thus, $w \in \mathsf{Clo}(V(\varphi))$. Then, $f(w) \in f(\mathsf{Clo}(V(\varphi)))$. Since f is a homomorphism, we have $f(w) \in \mathsf{Clo}(f(V(\varphi)))$. By the induction hypothesis, we then deduce that $f(w) \in \mathsf{Clo}(V'(\varphi))$ which in turn yields that $f(w) \in V'(\Diamond\varphi)$. Thus, we deduce $M', f(w) \models \Diamond\varphi$.

The converse direction is as expected and we leave it to the reader. □

Notice that the above theorem also works in paracomplete/intuitionistic topological models, and we leave the details to the reader.

Assuming that f is a homeomorphism may seem a bit strong. We can separate it into two, one direction of the biconditional is satisfied by continuity of f whereas the other direction is satisfied by the openness of f.

Corollary 3.16 *Let $M = \langle T, \tau, V\rangle$ and $M' = \langle T, \tau', V'\rangle$ be two paraconsistent topological models with a continuous f from $\langle T, \tau\rangle$ to $\langle T, \tau'\rangle$. Define $V'(p) = f(V(p))$. Then $M \models \varphi$ implies $M' \models \varphi$ for all φ.*

Corollary 3.17 *Let $M = \langle T, \tau, V\rangle$ and $M' = \langle T, \tau', V'\rangle$ be two paraconsistent topological models with an open f from $\langle T, \tau\rangle$ to $\langle T, \tau'\rangle$. Define $V'(p) = f(V(p))$. Then $M' \models \varphi$ implies $M \models \varphi$ for all φ.*

Proofs of both corollaries depend on the fact that Clo operator commutes with continuous functions in one direction, and it commutes with open functions in the other direction. Similar corollaries can be given for paracomplete/intuitionistic frameworks as the Int operator also commutes in one direction with open functions and in the other direction with continuous functions, and we leave this to the reader as well.

Furthermore, any topological operator that commutes with continuous, open and homeomorphic functions will reflect the same idea and preserve truth. Thus, these results can easily be generalized.

We can now take one step further to discuss homotopies in paraconsistent topological modal models. To our knowledge, the role of homotopies as transformations between truth preserving continuous isomorphisms or bisimulations under some restrictions has not yet been discussed within the field of topological models of (classical or non-classical) modal logic. Therefore, we believe our treatment is the first introduction of homotopies in topological semantics of modal logics. The reason why we start from paraconsistent (paracomplete) modal logics is the simple fact that the extension of each propositional variable is a closed (open) set making our task relatively easy and straightforward.

Recall that a *homotopy* is a description of how two continuous functions can be deformed to each other. We now state the formal definition.

Definition 3.18 Let S and S' be two topological spaces with continuous functions $f, f' : S \to S'$. A homotopy between f and f' is a continuous function $H : S \times [0, 1] \to S'$ such that if $s \in S$, then $H(s, 0) = f(s)$ and $H(s, 1) = f'(s)$.

In other words, a homotopy between f and f' is a family of continuous functions $H_t : S \to S'$ such that for $t \in [0, 1]$ we have $H_0 = f$ and $H_1 = g$ and the map $t \to H_t$ is continuous from $[0, 1]$ to the space of all continuous functions from S to S'. Notice that homotopy relation is an equivalence relation. Thus, if f and f' are homotopic, we denote it with $f \approx f'$. Then the immediate question is the following. Why do we need homotopies in logic? We will now use homotopies to obtain a generalization of Theorem 3.15.

Assume that we are given two topological spaces $\langle S, \sigma \rangle$ and $\langle S, \sigma' \rangle$ and a family of continuous functions f_t for $t \in [0, 1]$. Define a model M as $M = \langle S, \sigma, V \rangle$. Then, for each f_t with $t \in [0, 1]$, define $M_t = \langle S, \sigma, V_t \rangle$ where $V_t = f_t(V)$. Then, by Theorem 3.15, we observe that $M \models \varphi$ iff $M_t \models \varphi$. Now, what is the relation among M_ts? The obvious answer is that their valuation form a homotopy equivalence class. Let us now see how it works.

Define $H : S \times [0, 1] \rightarrow'$ such that if $s \in S$, then $H(s, 0) = f_0(s)$ and $H(s, 1) = f_1(s)$. Then, H is a homotopy. Therefore, given a (paraconsistent) topological modal model M, we generate a family of models $\{M_t\}_{t \in [0,1]}$ whose valuations are generated by homotopic functions.

Definition 3.19 Given a model $M = \langle S, \sigma, V \rangle$, we call the family of models $\{M_t = \langle S, \sigma, V_t \rangle\}_{t \in [0,1]}$ generated by homotopic functions and M homotopic models. In the generation, we put $V_t = f_t(V)$.

Theorem 3.20 *Homotopic paraconsistent (paracomplete) topological models satisfy the same modal formulas.*

Proof See the above discussion. □

In the above discussion, we have focused on continuous functions and the homotopies they generate. We can also discuss homeomorphisms and their homotopies which generate homotopy equivalences between spaces. In that case, homotopic equivalent spaces can be continuously deformed to each other. This would give us, under the correct valuation, a stronger notion of bisimulation that we call *continuous topo-bisimulation*. We will first start with the definition of topo-bisimulation before introducing continuous topo-bisimulation (Aiello and van Benthem 2002).

Definition 3.21 For topological models $\langle S, \sigma, V \rangle$ and $\langle S', \sigma', V' \rangle$, two points $s \in S$ and $s' \in S'$ are said to be topologically bisimular (topo-bisimular, for short) if they satisfy the following conditions.

1. The points s and s' satisfy the same propositional variables
2. For $s \in O \in \sigma$, there is $O' \in \sigma'$ such that $s' \in O'$ and for all $t' \in O'$, there is $t \in O$ such that t and t' are topo-bisimular
3. For $s' \in O' \in \sigma'$, there is $O \in \sigma$ such that $s \in O$ and for all $t \in O$, there exists $t' \in O'$ such that t and t' are topo-bisimular

Now we can extend it to continuity.

Definition 3.22 Let $M = \langle S, \sigma, V \rangle$ and $M' = \langle S', \sigma', V \rangle$ be two paraconsistent (paracomplete) topological models. We say M, w and M', w' are continuously topo-bisimular if M, w and M', w' are topo-bisimular and there is a homeomorphism f between $\langle S, \sigma \rangle$ and $\langle S', \sigma' \rangle$ such that $V' = f(V)$.

Note that in the above definition, we need a stronger notion of homeomorphism rather than just continuity as the bisimulation is a symmetric relation.

Theorem 3.23 *Continuously bisimular states satisfy the same modal formulas.*

Proof The proof is a routine induction on the complexity of formulas in the standard sense. □

What about the converse? Can we have a property akin to Hennesy–Millner property so that for some topologies that satisfy exactly the same formulas, we can construct a homeomorphism in between? Clearly the answer to this question is positive if we are in finite spaces and the construction is essentially the same as in the classical case. We refer the interested reader to a textbook treatment of classical modal logic to see how Hennesy-Millner property is treated (Blackburn et al. 2001).

A next step towards homotopy groups and their use in modal logic can be anticipated. Note that homotopy groups essentially classifies the spaces with regard to their continuous deformability to each other, and it seems feasible to import such a concept to modal logics. Nevertheless, in order not to diverge our focus here, we will not pursue that path and leave it for future work.

3.3 Modal direction

In this section, we briefly review the modal approaches to paraconsistency in order to make our work more self-contained.

One possible modal interpretation of paraconsistency focuses on the negation operator (Béziau 2005). Under the usual alethic reading of $\Box$ and $\Diamond$ modalities, one can define an additional operator $\sim$ as $\neg\Box$, or equivalently $\sim\varphi \equiv \Diamond\neg\varphi$. Notice that this definition corresponds to our earlier definition of negation being the closed complement. For this interpretation, recall that $\Diamond$ operator needs to be taken as the Clo operator.

The Kripkean semantics of the new paraconsistent negation operator $\sim$ is as follows (Béziau 2005). Let us take a modal model $M = \langle W, R, V \rangle$ where R is a binary relation on the non-empty set of states W and V is valuation. Take an arbitrary state $w \in W$.

$\sim\varphi$ is false at w if and only if φ is true at every v with wRv

More technically, we have the following reasoning in S5.

$$\begin{aligned} w \not\models \sim\varphi \quad & \text{iff} \quad w \not\models \neg\Box\varphi \\ & \text{iff} \quad w \models \Box\varphi \\ & \text{iff} \quad \forall v.(wRv \rightarrow v \models \varphi) \\ & \text{iff} \quad w \models \varphi \end{aligned}$$

Furthermore, as it was observed, $\sim$ modality is indeed S5, and an S5 logic can be given by taking $\sim$ as the primitive negation symbol with the intended interpretation. Nevertheless, for our current purposes, the S4-character of that modality is sufficient and we will not go into the details of such a construction. We refer the interested reader to the following references for a further investigation of this subject (Béziau 2002, 2005).

It is easy to notice the similarity of modal negation we presented here and the topological negation that we used throughout his paper. Therefore, it is a nice exercise to import our topological results from topological semantics to Kripke semantics with the aforementioned negation at hand.

For this reason, we offer a transformation from topological models to Kripke models which is similar to the standard translation between classical topological models and Kripke models (Aiello and van Benthem 2002). Given a topological paraconsistent model $M = \langle T, \tau, V \rangle$, we put $sR_\tau t$ when $s \in \mathsf{Clo}(t)$ to get a Kripke model $M_\tau = \langle T, R_\tau, V \rangle$. This transformation is truth preserving.

Theorem 3.24 *Given a topological paraconsistent model M, if $M, w \models \varphi$ then $M_\sigma, w \models \varphi$ where M_σ is obtained from M by the transformation that $wR_\sigma v$ when $w \in \mathsf{Clo}(v)$.*

Proof The proof is by induction on the complexity of formulas. We will prove the negation case, and leave the other cases to the reader. Note that we use $\sim$ for both paraconsistent Kripkean negation and PTL negation; nevertheless, the context will make it clear which one we mean.

Let $M = \langle T, \tau, V \rangle$ be given. Assume $M, w \models \sim\varphi$. Since, the topological negation $\sim$ is the closure of the set theoretical complement, we observe that $M, w \models \Diamond\neg\varphi$. Therefore, for every closed set $U \in \tau$, there is a point $v \in U$ such that $M, w \models \neg\varphi$. Observe that since $v \in U$ for closed U, we observe that $w \in \mathsf{Clo}(v)$. Then, put $wR_\tau v$. So, in the model $M_\tau = \langle T, R_\tau, V \rangle$, we have $M_\tau, w \models \exists v(wR_\tau v$ and $M_\tau, v \models \neg\varphi)$. Then, by the usual semantics of modal logic, we observe $M_\tau, w \models \Diamond\neg\varphi$ which is nothing but $M_\tau, w \models \neg\Box\varphi$. Finally, by definition, we conclude $M_\tau, w \models \sim\varphi$. □

A well-known transformation from Kripke frames generates a topological space. In that case, opens are downward (or upward) closed sets (subtrees) in the Kripke model. It is also easy to prove that this transformation respects the truth of the formulas in paraconsistent Kripke models.

Theorem 3.25 *Given a paraconsistent Kripke model M, if $M, w \models \varphi$ then $M_R, w \models \varphi$ where M_R is obtained from M by the transformation that the closed sets are downward closed subsets with respect to the accessibility relation R.*

This establishes the correspondence between paraconsistent topological models and paraconsistent Kripke models in a natural way.

4 Conclusion and future work

In this work, we focused on the connection between some topological spaces and paraconsistent logic. There are many open questions that we have left for further work. Some of them can be summarized as follows.

- How can we logically define homotopy and cohomotopy groups in paraconsistent or paracomplete models?
- How would paraconsistency be affected under various topological products?
- What is the (paraconsistent) logic of regular sets?
- What is the connection between paraconsistency and mereotopology, in particular between PTL and mereotopology?
- Which topological properties are definable in PTL?

The aforementioned questions form yet another research program in which tools from algebraic topology and algebraic geometry can be employed in a non-classical sense. The interaction between truth and inconsistency in such frameworks exhibits a novel research program where such tools can be useful. Moreover, region based modal logics and mereotopology have presented a variety of results about the logic of space (Pratt-Hartman 2007). Considering the use of regular sets within the framework of region based modal logics, it is not difficult to see a connection between region based modal logics and paraconsistent logics as they both share a similar algebraic structure.

The strong algebraic connection between topological models points out to a very interesting research direction. Considering the dual relation between intuitionistic and paraconsistent logics and their algebraic structures, the relation between such non-classical logics and modal algebras was already investigated (Rauszer 1977). Extending such work using topological algebras is yet another research direction for future work.

Another possible application of such systems is epistemic logics where the knowers or agents can have inconsistent or incomplete belief bases. The intuitive connection between AGM style belief updates and paraconsistency is yet to be established within our framework. One important strength of our system is that topologies have a stronger set of tools to deal with infinite cases (such as infinite conjunctions) which is important in some situations within the theme of belief revision. In the domain of epistemic logic, there has been some recent work in the field applying topological ideas to dynamic epistemologies (Başkent 2012). A next step would be to expand those classical systems to paraconsistent ones. We leave such stimulating discussions to future work.

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