Chapter 9
Public Announcements and Inconsistencies: For a Paraconsistent Topological Model

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Abstract In this paper, we discuss public announcement logic in topological context. Then, as an interesting application, we consider public announcement logic in a paraconsistent topological model.

Keywords Public announcement logic • Topological semantics • Homotopy • Paraconsistent logic

9.1 Introduction

9.1.1 Motivation

Public announcement logic is a formal framework that strives to express various dynamic aspects of knowledge change. Considered a kind of dynamic epistemic logic, public announcement logic works as follows. An external agent makes a truthful and public announcement, then the agents update their epistemic states by eliminating the possible worlds that do not agree with the announcement. For example, you may think that today is either Tuesday or Wednesday, then on TV you hear that it is actually Tuesday today. Then, you eliminate the possibility that today is Wednesday and come to know that today is Tuesday. Thus, after an announcement, you come to know the announcement.

Traditionally, public announcement logic (PAL, henceforth) adopts Kripke semantics (Plaza 1989; Gerbrandy 1999). Kripke frames and semantics enjoy a simplistic approach to modal logics in general, and makes it quite feasible to express various epistemic issues. However, Kripke semantics is not the only way to express truth in public announcement logic. In a relatively recent work, a topological semantics for public announcement logic was given (Başkent 2012). In that paper, the completeness and decidability results of PAL with respect to the topological semantics in several multi-agent frameworks were proven. Furthermore,

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it was shown that topological semantics changes some aspects of PAL compared to Kripke semantics. For example, announcements may stabilize in more than ω steps in topological models, which cannot be the case in Kripke models. Moreover, topological models exhibit some unexpected properties when it comes to formal analysis of rationality and backward induction. In topological game models, where we consider a topology based on a game tree, under the assumption of rationality, the backward induction procedure can take more than ω steps (ibid).

In this work, we extend such results by focusing on the relation between topologies, public announcements and inconsistency-friendly logics, particularly paraconsistent logic. By paraconsistent logic, we mean the logical systems in which the explosion principle (which says that from a contradiction, everything follows) fails. Therefore, in paraconsistent systems, there are some formulas that do *not* follow from a contradiction. Paraconsistent logics help us build inconsistent but non-trivial theories. As we shall make it clear in due course, from an epistemological perspective, paraconsistency and dynamic epistemology show an appealing interaction. If the given universe admits ontological contradictions (namely, if *some* things *are* and *are not* at the same time), how can knowledge and the dynamic *change* of knowledge be expressed logically? How do they interact? What kind of dynamic semantics do we need, if we want a universal framework that can work with some adjustments both in classical and non-classical (paraconsistent, and also intuitionistic) structures?

One of the main motivation of this work comes from *impossible worlds* – worlds which satisfy contradictions. Adopting a model that admits some impossible worlds immediately raises some questions about the possibility of expressing dynamic epistemologies in such a model. That is what we achieve in this paper.

The organization of the current work is as follows. First, we briefly remind the reader the basic topological concepts and structures which we will need throughout the paper. Then, from a rather technical perspective, we will show that topological models indeed present a rich and wide variety of possibilities of mathematical modeling of dynamic epistemologies. Next, we will present paraconsistent public announcement logic with some examples.

9.1.2 Basics

Let us now start with some basic definitions to make this work more self contained. Here, we define the classical PAL with topological semantics following (Başkent 2012).

Given a non-empty set S, a topology σ is defined as a collection of subsets of S satisfying the following conditions.

- The empty set and S are in σ,
- The collection σ is closed under finite intersections and arbitrary unions.

We call the tuple (S, σ) a *topological space*. The members of the topology is called *opens*. Complement of an open set (with respect to the classical set theoretical complement) is called a *closed set*. A function defined on a topological space is *continuous* if the inverse image of an open is an open; *open* if the image of an open is an open. A function is called *homeomorphism* if it is a continuous function between topological spaces with a continuous inverse. Homeomorphic spaces possess the same topological properties.

The above definition of topological space is given based on open sets. A dual definition can be given with closed sets as the primitives. In this case, for a given set S, we define the topology σ as a collection of subsets of S with the following condition.

- The empty set and S are in σ,
- The collection σ is closed under arbitrary intersections and finite unions.

We will refer to the topological spaces defined this way as *closed set topologies*. In this case, members of the topology will be closed sets. Notice that this is a dual definition for topological spaces.

Given a topological space, we can define a logical model. Let $M = (S, \sigma, v)$ be a *topological model* where (S, σ) is a topology and v is a valuation function assigning subsets of S to propositional variables. We denote the extension of φ in a model M with $|\varphi|^M$, and define it as follows $|\varphi|^M = \{s \in S : s, M \models \varphi\}$. When it is obvious, we will drop the superscript. Then, for an announcement φ, we define the *updated model* $M'_\varphi = (S', \sigma', v')$ as follows. Set $S' = S \cap |\varphi|, \sigma' = \{O \cap S' : O \in \sigma\}$, and $v' = v \cap S'$. Thus, in PAL, an announcement is made and the states that do not satisfy the announcement are eliminated in a way that preserves the topological structure. Also, the updated models are parametrized based on the extension of the announcement, in which the agents come to know the announcement in the updated model. Logically equivalent formulas, and even formulas that have the same extensions in the given original model produce the same updated model. Also, notice that the new topology σ', which we obtained by relativizing σ, is a familiar one, and is called *the induced topology*, and is indeed a topology (Başkent 2012).

The language of topological PAL includes the epistemic modality K and the public announcement modality $[\cdot]$, and they are defined recursively in the standard fashion based on a given set of propositional variables. We denote the dual of K as L, and define it as $\mathsf{L}\varphi := \neg\mathsf{K}\neg\varphi$ for a negation symbol $\neg$. For simplicity, we only give the single agent PAL here.

In a topology, for a given set, we have the *interior* operator Int and the *closure* operator Clo which return the largest open set contained in the given set, and the smallest closed set containing the given set respectively. The extensions of modal/epistemic formulas depend on such operators. We put $|\mathsf{K}\varphi| = \mathsf{Int}(|\varphi|)$. Dually, we have $|\mathsf{L}\varphi| = \mathsf{Clo}(|\varphi|)$. Intuitively, extension of a modal formula is the interior (or the closure) of the extension of the formula. It is important to note that in the classical case, epistemic modal operators necessarily produce topological entities. However, it is not necessary that $|p|$ for a propositional variable p will be open or closed, as it simply does not follow from the definition.

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The semantics of propositional variables and Booleans are standard. Let us give the semantics of the modalities here. For simplicity, we give the semantics for single agent here, and refer the reader to Başkent (2012) for various multi-agent extensions that require some more topological operations.

$$M, s \models \mathsf{K}\varphi \quad \text{iff} \quad \exists O \in \sigma.(s \in O \wedge \forall s' \in O, M, s' \models \varphi)$$
$$M, s \models [\varphi]\psi \quad \text{iff} \quad M, s \models \varphi \text{ implies } M', s \models \psi$$

The semantics of topological models makes it clear why topological models can distinguish a variety of epistemic properties that Kripke models cannot (van Benthem and Sarenac 2004). The reason is that the topological semantics for the epistemic modality K has Σ_2 complexity as it is of the form $\exists\forall-$, while Kripkean semantics offers Π_1 complexity as it is of the form $\forall-$. Also, even it does not directly fall within the scope of this paper, topological models handle infinitary cases better.

PAL with classical topological semantics admits the following standard reduction axioms.

- $[\varphi]p \leftrightarrow (\varphi \rightarrow p)$
- $[\varphi]\neg\psi \leftrightarrow (\varphi \rightarrow \neg[\varphi]\psi)$
- $[\varphi]\psi \wedge \chi \leftrightarrow [\varphi]\psi \wedge [\varphi]\chi$
- $[\varphi]\mathsf{K}\psi \leftrightarrow (\varphi \rightarrow \mathsf{K}[\varphi]\psi)$

In PAL, the rules of derivation are normalization ($\vdash \varphi \therefore \vdash \Box\varphi$) and modus ponens. Then, we have the expected completeness and decidability results.

Theorem 9.1 (Başkent 2012). *PAL in topological models is complete and decidable.*

The topological semantics for modal logics has been proposed in early 1940s even before the well-known Kripke semantics (van Benthem and Bezhanishvili 2007; McKinsey and Tarski 1946, 1944). The literature on the subject has evolved rapidly with a wide range of applications in philosophy and computer science, including various pointers to non-classical logics (Mints 2000; Goodman 1981). Within the family of non-classical logics, in this paper, we consider paraconsistent logics. We already gave a proof-theoretical definition of paraconsistency which underlines the fact that much of the work on the subject is from a proof-theoretical perspective. Yet, the current paper focuses on the semantical aspects of paraconsistency. *Dialetheism* is the view that suggests that there are true contradictions. Hence, dialetheism can be seen as a semantical counterpart of paraconsistency. In order to prevent an inflation of terminology, we will use both terms interchangeably when no confusion arises.

Paraconsistent logics span a very broad field with applications in computer science, philosophy and mathematical logic (Carnielli et al. 2007; da Costa et al. 2007; Priest 2002, 2008). We need to underline it at the beginning that, in this work, paraconsistency does not refer to the meta-logical (such as set theoretical, topological or arithmetical) properties of the models. For that reason, our definitions, proof methods and meta-logic are classical, and paraconsistency occurs at an object level. Within the pluralistic world of paraconsistent logic, this is indeed one of the methods to introduce non-trivial inconsistencies into models.

Next, we first discuss various topological results for the classical PAL to show the strength of the topological semantics and the richness of the applications it provides. Then, we will take an additional step and discuss PAL in inconsistent models.

9.2 Topological Announcements

9.2.1 Homotopic Announcements

One of the advantages of working with topological models is the fact that a variety of topological tools can be used within this framework to express a broad range of epistemic and model theoretical situations. In this section, we will observe various strengths of topological semantics for public announcements.

We define *functional representation* of announcements with respect to a topological model $M = (S, \sigma, v)$ as follows. For a public announcement φ, we say φ is "functionally representable in M" if there is an open and continuous function $f_\varphi^M : (S, \sigma) \mapsto (S', \sigma')$ where $M'_\varphi = (S', \sigma', v')$ is the updated model. We will drop the superscript or subscript when they are obvious. Notice that open or continuous functions deal with only open (or dually, closed) sets. However, the extensions of each and every formula in the language (such as the extensions of ground formulas) are not necessarily an open set. Therefore, open or continuous functions do not take such formulas into account. Nevertheless, in a model where each formula necessarily has an open (or equivalently closed) set extension, functional representation still works.

We observe that public announcements are special kind of functional representations.

Theorem 9.2. *Every public announcement is functionally representable.*

Proof. Given $M = (S, \sigma, v)$, construct $M'_\varphi = (S', \sigma', v')$ with respect to the public announcement φ. Then, for every open $O \in \sigma$ in M, assign $f(O) = O'$ where $O' = O \cap S'$ in σ' in M'. Here, notice that O' can be the empty set for some $O \in \sigma$ which is perfectly OK as f is not imposed to be an one-to-one function. We claim f functionally represents φ.

Note that modal formulas necessarily produce open (or dually closed) sets as their extensions, and they are taken care of by the given function f. However, we may still have Boolean formulas which do not have open or closed extensions in the model. However, notice that they do not violate functional representation as the definition of functional representation quantifies over open sets.

Now, since both, O and O' are open, so f is an open map. Take $U' \in \sigma'$. Since, $U' = U \cap S'$ for some $U \in \sigma$, the inverse of image of U' under f is U which is an open in σ showing that f is continuous.

Thus, we conclude that f functionally represents φ. □

The converse of the above theorem is not true in general. Not every open and continuous function represents an announcement as it may not respect the valuation in the model. Now, we can use functions to represent the relation between the given (original) epistemic model and the updated model. This is indeed another way to represent the *dynamic* aspects of knowledge change in topological models.

Corollary 9.1. *In PAL, given topological models and the updated topological models may not be homeomorphic in general.*

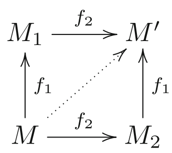
Proof. Functional representation of an announcement is not necessarily one-to-one, therefore may not be a homeomorphism. □

This is quite interesting. The above result indicates that not just knowledge may change after an announcement, but also the topological qualities of the model may alter. This is perhaps not surprising, as we would like the announcement to have an epistemic impact which may change some model theoretical properties of the model. This observation suggests the following definition.

Definition 9.1. Given two models $M = (S, \sigma, v)$ and $M' = (S'\sigma', v')$. We call M and M' *homeomorphic φ-models* if M' is the updated model of M with the public announcement φ, and there is a homeomorphism f from (S, σ) into (S', σ') that functionally represents φ.

Notice that homeomorphic model relation is not symmetric, but it is reflexive and transitive. Homeomorphic φ-models enjoy the same topological qualities after a specific public announcement (here, φ). In this context, arbitrary announcements (Balbiani et al. 2008) can be considered a generalization of homeomorphic φ-model to *homeomorphic models* that remain homeomorphic after *any* announcement.

For a given model M, consider two different announcements φ_1 and φ_2 representable by f_1 and f_2 respectively. Then, as $[\varphi_1][\varphi_2]\psi \leftrightarrow [\varphi_2][\varphi_1]\psi$, we have the following situation illustrated in the diagram.



For simplicity, we assume that M and M' are homeomorphic models. Then, what about the connection between M_1 and M_2? We can easily generalize this question to n many models. For public announcements φ_i functionally represented by f_i, and the updated models M_i obtained after announcing φ_i, one can ask about the relation between M_is? In order to give an answer to this question, we need homotopies.

Definition 9.2. Let S and S' be two topological spaces with continuous functions $f, f' : S \mapsto S'$. A homotopy between f and f' is a continuous function $H : S \times [0, 1] \mapsto S'$ such that for $s \in S$, $H(s, 0) = f(s)$ and $H(s, 1) = g(s)$.

The definition of homotopy can easily be extended to topological models. Given a topological model $M = (S, \sigma, v)$ we call the family of models $\{M_t = (S_t, \sigma_t, v_t)\}_{t \in [0,1]}$ generated by M and homotopic functions *homotopic models*. In the generation of valuation function v_ts of M_ts, we put $v_t = f_t(v)$. Homotopic models preserve truth, and they can be used to extend the definition of bisimulations in topological spaces (Başkent 2013).

Theorem 9.3. *Given M, consider a family of updated homeomorphic models $\{M_i\}_{i<\omega}$ each of which is obtained by an announcement φ_i representable by f_i. Then f_is are homotopic.*

Proof. Immediate. □

The converse of the above statement is not always true. Clearly, not each pair of updated models in a class of homotopic models can be obtained from one another by an update. Given M, consider the updated models M_1 and M_2 where the prior is obtained by an announcement of p while the latter $\neg p$. Even if there is a continuous transformation between M_1 and M_2, this transformation is not a public announcement.

Namely, there exists a smooth topological transformation from one updated model to another. Then, what is the epistemic meaning of it? Can we preserve truth under such a transformation?

We can make use of an earlier result here (Başkent 2013). Let $M = (S, \sigma, v)$ be a given model. Suppose $M_1 = (S_1, \sigma_1, v_1)$ and $M_2 = (S_2, \sigma_2, v_2)$ are updated models obtained after the announcements φ_1 and φ_2 respectively. Let the functions f_1 and f_2 represent φ_1 and φ_2 respectively. Then, there exists a homotopy $H : S \times [0, 1] \mapsto S$ such that $H(s, 0) = f(s)$ and $H(s, 1) = g(s)$ where $s \in S$. Now, observe that we also have $v_2 = f_2 f_1^{-1}(v_1)$. More importantly, we have another homotopy J such that $J(s, 0) = v_1$ and $J(s, 1) = v_2$. It is easy to notice that $J = v(H)$. Here, we discuss this example with only two updates, but the results can easily be generalized to n different updates.

In other words, the transformation between two updated models require a *renaming* or *restructuring* the real world.

Notice that homotopies discuss the topological connection between different announcements. The epistemic significance of this concept is the fact that now, at least in topological models, we can express how differentiated opinions can be transformed into each other under certain assumptions. This directly relates to *belief polarization* (Kelly 2008; Başkent et al. 2012). Thomas Kelly summarizes this phenomenon as follows.

> Suppose that two individuals – let us call them "You" and "I" – disagree about some nonstraightforward matter of fact. (...) Suppose next that the two of us are subsequently exposed to a relatively substantial body of evidence that bears on the disputed question. (...) What becomes of our initial disagreement once we are exposed to such evidence? (...)

> Exposure to evidence of a mixed character does not typically narrow the gap between those who hold opposed views at the outset. Indeed, worse still: not only is convergence typically not forthcoming, but in fact, exposure to such evidence tends to make initial disagreements even more pronounced. Kelly (2008)

This interesting, yet very common and basic phenomenon can easily be formalized in terms of public announcements. In this case, the announcement ("the substantial body of evidence") creates different updates on different agents. So far, this is perfectly normal. What is interesting is that the updated models of two agents are not transformable to each other – that is they are not homotopic. Thus, they are polarized.

In this case, homotopic models represent *degrees* of belief or knowledge where the models can be, step by step, translated to each other, and such a translation follows a topologically meaningful pattern – it preserves the topological and ideally (if it is a special kind of homotopy) model theoretical properties of the models in question. However, polarized beliefs and knowledge of two agents, in this case, cannot be transformed into each other, by the mere definition of polarization. Thus, they cannot be homotopic. This is a simple but direct application of homotopic public announcements.

In short, there is a close connection between various topological transformations and model updates after public announcements, and topological PAL models enjoy various techniques imported from pure topology. Moreover, they may correspond to various interesting epistemic concepts that are relevant for dynamic epistemic logic.

9.3 Paraconsistent Public Announcements

In classical logic, contradictions are never satisfied. However, in modal philosophical logic there is an interesting conceptual and philosophical notion, called *impossible worlds*. By *impossible worlds*, let us denote those states which satisfy some contradictions, define them as $\{x : x \models \varphi \wedge -\varphi \text{forsome} \varphi\}$ for a negation symbol $-$. Then, the natural question is how to epistemically update an epistemic model with impossible worlds.

For example, if we consider God as an impossible state in our mental model, how can we then update our mental epistemic model after we hear about a person healing the blind or splitting the Moon? Mental models may possess some contradictions, yet, they still function in a (relatively) rational and sound fashion. People believe in gods, they believe in miracles, yet they still function mostly rationally – both dynamically and epistemically. How can we portray such epistemic situations when an external announcement updates the models with impossible worlds?

Law, as a major platform for inconsistencies, exhibit similar puzzling situations.

> Suppose that there is a certain country which has a constitutional parliamentary system of government. And suppose that its constitution contains the following clauses. In a parliamentary election:

(1) no person of the female sex shall have the right to vote;
(2) all property holders shall have the right to vote.

(Priest 2006, p. 184)

Let us denote the above rules as public announcements φ_1, φ_2 respectively. Therefore, when the Law (1) was introduced, we can consider it as $[\varphi_1]$, and similarly Law (2) as $[\varphi_2]$. The introduction of new laws to the legal system can be thought of as public announcements. For simplicity, consider them as a simultaneous announcement of the form $[\varphi_1 \wedge \varphi_2]$. Therefore, when $[\varphi_1 \wedge \varphi_2]$ is announced, the states that satisfy the contradictory statement will be kept – which is the set of female property holders, in this example. This announcement does not (and should not) trivialize the system. In this case, contradictions exist, yet we are supposed to reason soundly in this model, we cannot let the model get trivialized or explode.

Another motivational example comes from a neighboring field of belief revision. Priest discusses AGM style belief revision from a paraconsistent perspective, and revises the AGM postulates (Priest 2001). Belief is defined weaker than knowledge. Therefore, the immediate next step is to consider knowledge in a paraconsistent universe, and observe how it changes.

Our goal now is to give a formal model which can descriptively and normatively express such situations.

9.3.1 Models

Topological semantics provides a versatile tool to express truth in a wide range of classical and non-classical logics. As we already showed, it is also a wise choice to express various dynamic and modal issues.

While discussing the classical topological semantics, we underlined that only the modal formulas produce topological sets (opens or closeds). Boolean formulas do not necessarily produce such sets. Let us now assume that we have a closed set topology where each member of the topology is a closed set, and stipulate further that the extensions of propositional variables are also closed sets. If propositional variables are closed sets, then their arbitrary intersections and finite unions will remain closed. Therefore, conjunctions and disjunctions of such propositional variables will still be closed sets. However, this stipulation makes an important difference for negation as the compliment of a closed set is not necessarily a closed set. For that reason, we cannot use the standard definition of negation as the set theoretical complement on the extension of the formula. So, we need to redefine it in closed set topologies. In our system, we define negation as the "closure of the complement" (Başkent 2013; Goodman 1981; Mortensen 2000). Let us denote this paraconsistent negation by $-$.

As an illustration, consider the formula $p \wedge -p$. Let us say that the extension of p is $O \in \sigma$ where σ is a closed set topology, and O is a closed set. Then the extension of $p \wedge -p$ is $O \cap \mathsf{Clo}(\overline{O})$ which is $\partial(O)$, where $\partial(\cdot)$ is the boundary operator which is defined as $\partial O := \mathsf{Clo}(O) - \mathsf{Int}(O)$, and $\overline{O}$ denotes the (classical) set theoretical compliment of O. Therefore, the contradictions are satisfied at the boundary points. Thus, we now have a paraconsistent logic. The reason why explosion fails is because for some formula φ, the extension of $\varphi \wedge -\varphi$ is not necessarily an empty set, but it is $\partial(O)$ for some closed set O. Thus, it is not necessarily a subset of every set, so not every formula follows from a contradiction in this system, failing the explosion property.

However, we need to elaborate a bit more on the epistemological meaning of the use of paraconsistent spaces in the context of public announcement logic. The classical PAL heavily depends on the law of non-contradiction. An external and truthful announcement is made. Then, the agents update their epistemic models by eliminating the states in their model which do not agree with the announcement, followed by the reducing the epistemic accessibility relation or the topology and the valuation with respect to the new, updated model. Therefore, the classical PAL does not *control* the inconsistencies, it completely eliminates them. Yet, in paraconsistent spaces, some contradictions need not be eliminated as they do not trivialize the theory. In short, the main problem caused by inconsistencies is that they trivialize the theory due to the choice of the underlying logic. Therefore, if there exists some contradictions that do not trivialize the theory (again, due to the choice of the underlying logical framework), there seems to be no need to eliminate them. This is our pivotal point for paraconsistent PAL. Also, notice that intuitionistic logic also admits explosion, thus suffers from the same problem as the classical logic.

Here, notice that we do not focus on inconsistent announcements or non-truthful announcements per se. Our framework reflects paraconsistent modal realism, and allows inconsistent possible worlds. Moreover, we also follow the standard "state elimination based" paradigm for PAL – with some differences which will be clarified in due time. Model theoretically, we can also eliminate the accessibility relation arrows or relativize only the topology and leaving the universe intact and keep the states. From modal logical perspective, there seems to be no model theoretical difference between these methods.

In paraconsistent spaces, public announcements obtain a broader meaning. Namely, when φ is announced in a paraconsistent space, it simply means "Keep the states that satisfy φ". It can very well be the case that some of the states that satisfy φ may also satisfy its negation $-\varphi$. Clearly, this stems from the fact that negation $-$ in paraconsistent PAL is not classical, thus the methods of "eliminating the states that do not satisfy the announcement" and "keeping the states that satisfy the announcement" are not identical, unlike in classical logic. This distinction surfaces very clearly in paraconsistent PAL, and is one of the most important contributions of paraconsistent public announcement logic.

Let us now give a precise meaning to the public announcements. First, we define the updated model M' after the announcement the same way. Let $M = (S, \sigma, v)$ be a topological model where (S, σ) is a closed set topology where every $K \in \sigma$ is a closed set. For a formula $[\varphi]$, we obtain an *updated model* $M'_\varphi = (S', \sigma', v')$ where

$S' = S \cap |\varphi|$, $\sigma' = \{K \cap S' : K \in \sigma\}$, and $v' = v \cap S'$. We will remove the subscript when it is clear from the context.

Notice that there could also exist some other ways to revise the given model after an announcement. In other words, one may wish to exclude the states that satisfy the negation of the announcement from the space. We define $M^-_\varphi := (S \setminus |-\varphi|, \sigma', v')$ as the model obtained after the announcement of $[\varphi]$. We will call M^-_φ as the *reduced model*. Clearly, in classical logic, $M^-_\varphi = M'_\varphi$ for all models M and all formulas φ. But, in paraconsistent PAL, the reduced model is a subset of the updated model.

Lemma 9.1. *In classical PAL, for a model M, updated model M', and reduced model M^- are identical. In paraconsistent PAL, $M^- \subseteq M'$.*

Proof. Follows immediately from the definitions. □

Let us now present the formal aspects of paraconsistent public announcement logic, which we will call ParaPAL in short. We define the syntax of ParaPAL as follows for a propositional variable p and a falsum symbol $\bot$.

$$\bot \mid p \mid -\varphi \mid \varphi \wedge \varphi \mid \mathsf{K}\varphi \mid [\varphi]\varphi$$

As expected, K is the knowledge operator, and $[\varphi]$ denotes the public announcement of φ. We define disjunction and implication in the usual way. The dual operator L is defined as expected: $\mathsf{L}p := -\mathsf{K}-p$. For a more detailed exposition of multi-agent PAL in topological setting, see Başkent (2012). For simplicity, both in notation and exposition, we will only discuss the single-agent ParaPAL in this paper as extending it to a multi-agent case is straight-forward (Başkent 2013).

Let us give the semantics of ParaPAL now. Note that in ParaPAL, we have $|-p| = \mathsf{Clo}(S \setminus |p|)$. Also, $\bot$ is true nowhere (even if $p \wedge -p$ can be true). The semantics for propositional variables and Booleans are as usual. Let us reinstate the semantics of the modal and dynamic operators.

$$\begin{array}{lll} M, s \models \mathsf{K}\varphi & \text{iff} & \exists O \in \sigma.(s \in O \wedge \forall s' \in O : s', M \models \varphi) \\ M, s \models [\varphi]\psi & \text{iff} & M, s \models \varphi \text{ implies } M', s \models \psi \end{array}$$

In ParaPAL, the fact that after an announcement, the updated model will keep the states that satisfy the announcement and also may satisfy the *negation* of the announcement reflects the basic dictum of paraconsistent logic: Paraconsistent logics distinguish (at least) two different types of *true*s and *false*s. The trues that are only true and the trues that are also false; and similarly falses that are only false and the falses that are also true (Priest 1979). Therefore, in ParaPAL, after an announcement of φ, the agent comes to know φ (i.e. $M, s \models \mathsf{K}\varphi$), but we may also consider $-\varphi$ possible at the same state (i.e. $M, s \models \mathsf{L}-\varphi$).

An interesting observation here is that in ParaPAL, since the extension of *each* propositional variable is a closed set, we have $\mathsf{L}p \leftrightarrow p$. This observation follows from the topological fact that the closure of a closed set is already itself. That is, if the extension of each formula is a closed set already, its extension under the epistemic modal operator L will be the closure of the extension of the given formula. But, the closure of a closed set is already itself by definition, therefore, the modal operator will not change the extension of a given formula yielding the logical equivalence $\mathsf{L}\varphi \leftrightarrow \varphi$ (Başkent 2013). Nevertheless, for expressivity purposes, we will keep the epistemic modal operator. This is a design decision similar to the classical PAL where the public announcement operator is not more expressive, yet provides succinctness (Kooi 2007). For convenience, we call the static fragment of ParaPAL (without the public announcement operator, but with the modal epistemic operator) as PTL after paraconsistent topological logic.

Before proceeding further, we need to make sure that the updated topology in ParaPAL is indeed a topology.

Lemma 9.2. *Given a closed set topology* (S, σ). *Then, for any p with a closed set extension, the updated space* (S', σ') *where* $S' = S \cap |p|$ *and* $\sigma' = \{K \cap S' : K \in \sigma\}$ *is also a topological space.*

Proof. The topology (S', σ') is indeed a well-known topology and called an induced topology. See Başkent (2012), for example, for a direct proof. □

The above lemma ensures that the semantics of public announcements in ParaPAL is well-defined.

9.3.2 *Further Observations*

9.3.2.1 Epistemic Modal Operator Is Redundant

An interesting result of PTL is that the epistemic operator is redundant. Nevertheless, for succinctness reasons, we keep the epistemic operator, as we already argued.

Lemma 9.3. *ParaPAL and PTL are equi-expressible.*

We will focus more on the reduction of ParaPAL to PTL in the next part.

Lemma 9.4. *ParaPAL is more expressive than PAL.*

In ParaPAL, we can have true statements such as $[p]\mathsf{K}(q \wedge -q)$. It would not be wrong to think that the introduction of impossible worlds to the model provides expressive richness for ParaPAL.

9.3.2.2 Reduction Axioms

Let us see whether the standard reduction axioms of classical PAL (which we gave in the Introduction) works in ParaPAL.

Consider the axiom $[\varphi]p \leftrightarrow (\varphi \rightarrow p)$ on a ParaPAL model $M = (S, \sigma, v)$ where $w \in S$, and p is a propositional variable. Suppose further that $M, w \models \varphi$.

$$\begin{array}{lll} M, w \models [\varphi]p & \text{iff} & M', w \models p \\ & \text{iff} & M, w \models p \\ & \text{iff} & M, w \models (\varphi \to p) \end{array}$$

Notice that the above result simply depends on the fact that the valuation of the propositional variables are independent from the topology.

ParaPAL presents a new negation. Thus, it is more important now to consider the reduction axiom for negation: $[\varphi]-\psi \leftrightarrow (\varphi \to -[\varphi]\psi)$. Similarly, take a ParaPAL model $M = (S, \sigma, v)$ where $w \in S$, and p is a propositional variable. Suppose further that $M, w \models \varphi$.

$$\begin{array}{lll} M, w \models [\varphi]-\psi & \text{iff} & M', w \models -\psi \\ & \text{iff} & w \in \mathsf{Clo}(S' \setminus |\psi|) \\ & \text{iff} & w \in \mathsf{Clo}((S \cap |\varphi|) \setminus |\psi|) \\ & & \text{as } w \in |\varphi| \text{ is assumed} \\ & \text{iff} & w \in \mathsf{Clo}(S \setminus (|\varphi| \cap |\psi|)) \\ & \text{iff} & w, M \models -[\varphi]\psi \end{array}$$

As we already pointed out, the reduction axioms for the epistemic modal operator holds vacuously. Thus, we obtain the following result.

Theorem 9.4. *ParaPAL reduces to PTL by the following reduction axioms:*

- $[\varphi]p \leftrightarrow (\varphi \to p)$
- $[\varphi]-\psi \leftrightarrow (\varphi \to -[\varphi]\psi)$
- $[\varphi]\psi \wedge \chi \leftrightarrow [\varphi]\psi \wedge [\varphi]\chi$
- $[\varphi]\mathsf{K}\psi \leftrightarrow (\varphi \to \mathsf{K}[\varphi]\psi)$

Proof. We already showed the soundness of the first two axioms. The third one on conjunction follows immediately, and the fourth one on the epistemic modality follows almost trivially as in ParaPAL and PTL the epistemic modality becomes redundant due to the properties of the closure operator (Başkent 2013). □

9.3.2.3 Topological Results

The most important advantage of adopting a topological background theory to express dynamic epistemic matters in a paraconsistent logic is to have the ability to make use of the topological properties of the model in understanding dynamic epistemic reasoning. In this section, we will consider various relevant topological concepts, and observe how they relate to expressing dynamic epistemologies.

Definition 9.3. A set X is called connected if $A \cap B \neq \emptyset$ whenever A, B are closed non-empty subsets and $X = A \cup B$. It is called totally disconnected if all of its subsets with more than one element are disconnected.

An interesting result for PTL models is the following.

Theorem 9.5 (Başkent 2012). *A PTL model with totally disconnected topology cannot be inconsistent.*

This theorem suggests a way to make the space consistent. It is also useful for our purposes in this paper. In other words, if the public announcement *disconnects* a space, then we can reduce the inconsistency to consistency by means of public announcements. The following theorem establishes the connection between inconsistent and consistent public announcement models via topological operations.

Theorem 9.6. *Let $M = (S, \sigma, v)$ be ParaPAL model where (S, σ) is an arbitrary topological space. Then if there exists a formula φ such that the topological space (S', σ') obtained after the announcement is totally disconnected, then $M'_\varphi = (S', \sigma', v')$ cannot be inconsistent.*

Proof. Given a ParaPAL model $M = (S, \sigma, v)$, call the updated model M'_φ. By reduction axioms, M'_φ reduces to a PTL model without changing the topology. Thus, if M'_φ, as a PTL model, is disconnected, by Theorem 9.5 it cannot be inconsistent. □

However, we should not over-read the above theorem. The existence of the public announcement φ that can turn arbitrary topological spaces to totally disconnected topological spaces is not guaranteed in each and every model.

A similar connection can be built between the static PTL and the dynamic ParaPAL.

Theorem 9.7 (Başkent 2012). *Let X be a connected topological space of closed sets with a PTL model on it. Then, the only subtheory that is not inconsistent is the empty theory.*

We can improve the above result within the context of ParaPAL as follows.

Theorem 9.8. *Let $M = (S, \sigma, v)$ be a ParaPAL model where (S, σ) is a connected topological space of closed sets. Then, the announcement of $\bot$ produces an updated model of M that has consistent theories.*

Proof. Let $M = (S, \sigma, v)$ be a ParaPAL model. We know that it is also a PTL model with the same topological structure. By Theorem 9.7, we know that the only theory that is consistent is the empty theory. In public announcement setting, we obtain this by announcing $\bot$ which is true nowhere. □

The above theorem is interesting. It reminds us that $\bot$ is nowhere true in paraconsistent spaces whereas some contradictions (in the form of $\varphi \wedge -\varphi$ for some φ) can be true somewhere. Additionally, it shows that the boundary points, the points that satisfy contradictions, are crucial to controls the inconsistencies. Concepts such as connectedness, as they relate to the boundary points, therefore play an essential role capturing inconsistent epistemologies in a dynamic setting.

An interesting aspect of topological PAL is whether/how the announcements stabilize the model, and how we can reach the limit models.

Definition 9.4. For a model M and a formula φ, define the announcement limit $\lim(M, \varphi)$ as the first model reached by successive announcements of φ that no longer changes after the announcement is made.

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With static, ground Boolean formulas, the limit models are reached immediately after the first announcement. Moreover, in topological models for classical PAL, it is known that stabilization can take more than ω steps (Başkent 2012). This can also be seen as one of the strengths of topological models within the context of infinitary models. Then, the natural question is whether this property remains true in ParaPAL.

Theorem 9.9. *Model stabilization for ParaPAL models cannot take more than ω steps.*

Proof. The key point here is to observe that different definitions of common knowledge coincide in ParaPAL. This is usually the standard way to prove this statement (van Benthem and Sarenac 2004). As widely known, an announcement becomes a common knowledge after it is announced. Therefore a way to see how long the stabilization takes is to observe whether different definitions of common knowledge agree in ParaPAL.

Consider the following two definitions of common knowledge in Kripke models which we will only give in words, and refer the reader to van Benthem and Sarenac (2004) for a more detailed discussion.

– The reflexive and transitive closure of accessibility relations
– The fixed-point of the epistemic operator

In Kripkean models, these two definitions coincide as the knowledge modalities distribute over any arbitrary conjunctions. However, in PAL with classical topological semantics, these definitions do not coincide (van Benthem and Sarenac 2004; Başkent 2012).

On the other hand, in ParaPAL, since we have a closed set topology, and arbitrary intersections of closed set is still a closed set, we observe that the two definitions of common knowledge coincide, and they stabilize less than ω step. This can also be seen by the fact that the ParaPAL reduces to PTL losing is dynamic and epistemic modalities which make the stabilization *faster*. □

Another interesting direction is to observe how public announcements behave in some special inconsistent topological models. Now, we can turn into a well-known topological space, and observe how it affects the ParaPAL models. In Hausdorff spaces where distinct points have disjoint neighborhoods, we obtain the following results. Also note that, as a fact, in Hausdorff spaces compact sets are always closed.

Theorem 9.10. *Let $M = (S, \sigma, v)$ be a ParaPAL model where (S, σ) is a compact Hausdorff space. The stabilization for M takes less than ω steps.*

Proof. Let $M = (S, \sigma, v)$ be a ParaPAL model where (S, σ) is a compact Hausdorff space. Then, it is a closed set topology (thus, we do not need to impose it additionally). Since it is compact every arbitrary cover has a finite sub-cover. Thus, the stabilization, even if it takes more than ω-step can be converted into a stabilization with finitely many steps. □

Then, the next question is whether the PAL updates employ a continuous transformation in the model. Namely, given a ParaPAL model M and an arbitrary formula φ, what is the connection between M and M'_φ in terms of continuous transformations? For this question, we use the functional representation of announcements, which we defined earlier. The following theorem holds immediately.

Theorem 9.11. *Every announcement is functionally representable in ParaPAL.*

Notice that, similar to the classical case of topological PAL, this result does not entail that f as above is truth preserving.

Now, we take one step further and consider the separation axiom $\mathbf{T}_6$ or perfectly normal spaces.

Definition 9.5 (Perfectly normal spaces). Given arbitrary closed sets K_1 and K_2 in a topology (S, σ). If there exists a continuous function $f : S \mapsto [0, 1]$ that separates K_1 and K_2 such that $f^{-1}(0) = K_1$ and $f^{-1}(1) = K_2$, then (S, σ) is called a perfectly normal topological space.

We then have the following theorem.

Theorem 9.12. *Let $M = (S, \sigma, v)$ be a ParaPAL model where (S, σ) is a perfectly normal topological space. If for two formulas φ and ψ, $M \not\models \varphi \wedge \psi$, then there exists a continuous transformation between M'_φ and M'_ψ.*

Proof. Let $M = (S, \sigma, v)$ be a ParaPAL model where (S, σ) is a perfectly normal topological space. Denote the extension of $|\varphi|^M = K_1$ and $|\psi|^M = K_2$. Then, as $M \not\models \varphi \wedge \psi$, we have $|\varphi \wedge \psi|^M = \emptyset$. Then, there exists a continuous function $f : S \mapsto [0, 1]$ that separates K_1 and K_2 such that $f^{-1}(0) = K_1$ and $f^{-1}(1) = K_2$ by definition.

Now, consider M'_φ and M'_ψ. In this case, observe that the carrier sets of M'_φ and M'_ψ are K_1 and K_2 respectively, again by definition. Thus, the transformation t from M'_φ to M'_ψ is given as follows:

$$t(x) = f^{-1}(f(x) + 1), \forall x \in K_1$$

The transformation t' from M'_ψ to M'_φ can also be defined similarly:

$$t'(y) = f^{-1}(f(y) - 1), \forall y \in K_2$$

By definition, t and t' are continuous. However, notice that, the transformation t is not truth preserving, nor a bisimulation. Therefore, the continuous transformation is, semantically, a renaming. □

Continuous transformation between two updated models mean that the topological (thus model theoretical) qualities of the two models are the same. Yet, since they may not have the same propositional valuation, these two models may not be bisimular.

In this section, we consider some topological concepts that are relevant to our discussion of paraconsistent public announcement logic. The field of topology is virtually unbounded, and it is possible to consider many other topological spaces and notions, and their impact on paraconsistent epistemologies.

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9.4 Conclusion

Public announcement logic is an interesting playground to observe how epistemic reasoning based on paraconsistency works dynamically. Agents in ParaPAL can reason soundly in a world of inconsistencies. Our system is based on an inconsistent universe, yet takes announcements as honest and truthful epistemic operations.

The field is rich, and there can be considered a variety of future work possibilities including the algebraic connection between paraconsistency and public announcements, and paradoxical announcements. We leave it to future work.

Another interesting direction is the relation between mereology and public announcements. Mereology is the research area that studies the connection between parts and wholes, and exhibits intriguing algebraic qualities. Therefore, the question of how the relation between parts and wholes change after a public announcement is yet another interesting research direction to pursue.

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