Distributed Knowledge and Announcements

a geometry and announcement based approach

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Public Announcements

Concept and Syntax

Public announcements update the models by state elimination. After a truthful announcement, the states which do not conform with the announcement are eliminated. This brings along the restriction of the accessibility relation, too.

The language of public announcement logic is that of basic modal logic extended with the formulae of the form $[\varphi]\psi$ with the intended meaning that *after the public announcement of* φ, ψ *holds*. The important restriction is the fact that both φ and ψ should be basic modal formulae, i.e. an announcement cannot be announced.

Public Announcements

Semantics

Definition

Let $\mathcal{M} = \langle W, \{R_i\}_{i \in I}, V \rangle$ *be a model.*

$$\mathcal{M}, w \models \mathsf{K}_i \varphi \quad \textit{iff} \quad \mathcal{M}, v \models \varphi \textit{ for each } v \textit{ such that } (w, v) \in R_i$$
$$\mathcal{M}, w \models [\varphi]\psi \quad \textit{iff} \quad \mathcal{M}, w \models \varphi \textit{ implies } \mathcal{M}|\varphi, w \models \psi$$

Here, the model $\mathcal{M}|\varphi = \langle W', \{R'_i\}_{i \in I}, V' \rangle$ *we obtain after the update is defined by restricting* $\mathcal{M}$ *to those states where* φ *holds. Define* $(\varphi)^{\mathcal{M}} = \{v \in W : \mathcal{M}, v \models \varphi\}$*. Hence,* $W' = \{w \in W : w \models \varphi\}$*, i.e.* $W' = W \cap (\varphi)^{\mathcal{M}}$*;* $R'_i = R_i \cap (W' \times W')$ *and finally* $V'(p) = V(p) \cap W'$*.*

Public Announcements

Proof System

The proof system of public announcement logic is the proof system of multi-modal **S5** epistemic logic with the following additional axioms.

Atoms	$[\varphi]p \leftrightarrow (\varphi \to p)$
Partial Functionality	$[\varphi]\neg\psi \leftrightarrow (\varphi \to \neg[\varphi]\psi)$
Distribution	$[\varphi](\psi \wedge \chi) \leftrightarrow ([\varphi]\psi \wedge [\varphi]\chi)$
Knowledge Announcement	$[\varphi]\mathsf{K}_i\psi \leftrightarrow (\varphi \to \mathsf{K}_i[\varphi]\psi)$

The rule of inference is called the *announcement generalization* and is described as follows.

From $\vdash \psi$, *derive* $\vdash [\varphi]\psi$.

Multi-agent case

It is S4

We will utilize multi-agent epistemic logic with reflexive and transitive accessibility relation R_i for each agent i.

Therefore our multi-agent logic will be $\underbrace{\mathbf{S4} \oplus \mathbf{S4} \oplus \cdots \oplus \mathbf{S4}}_{k-\text{times}}$, where k is the number of agents, i.e. $I = \{i_1, \cdots, i_k\}$

Bisimulation

Multi-Agent Process Equivalence

A bisimulation is an equivalence relation between two modal models establishing the process equivalence of the models in question. The precise definition is as follows.

Let $\mathcal{M} = \langle W, \{R_i\}_i, V \rangle$ and $\mathcal{M}' = \langle W', \{R'_i\}_i, V' \rangle$ be two models. A nonempty binary relation $\sim$ is a bisimulation between $\mathcal{M}$ and $\mathcal{M}'$ if:

1. If $w \sim w'$, then both w and w' satisfy the same propositional letters.
2. If $w \sim w'$ and wR_iv, then there is v' in W' such that $v \sim v'$ and $w'R'_iv'$ for all i.
3. If $w \sim w'$ and $w'R'_iv'$, then there is v in W such that $v \sim v'$ and wR_iv for all i.

Distributed Knowledge

Definition

We say a group of agents I has distributed knowledge of φ if the "combined" knowledge of the agents in I implies φ. This expression corresponds to the following formula.

$$M, w \models D_I \varphi \text{ iff } M, v \models \varphi \text{ for all } v \text{ such that } (w, v) \in \cap_{i \in I} R_i.$$

Distributed Knowledge vs Bisimulation

not an invariance

As opposed to common knowledge and universal knowledge (*everyone knows*), distributed knowledge is **not** invariant under bisimulation.

Underlying reason for this observation is the fact that the bisimulations cannot distinguish (or count) the splitting of accessibility arrows although this is essential in the process of obtaining the distributed knowledge.

Distributed Knowledge as a S4 Modality

D as a basic modality

Lemma

For the distributed knowledge operator D for the group of agents I, the following holds:

- $[D\varphi \wedge D(\varphi \wedge \psi)] \rightarrow D\psi$
- $D\varphi \rightarrow \varphi$
- $D\varphi \rightarrow DD\varphi$

Proof.

Trivial. However observe that, the first property is the **K** axiom, and the second property corresponds to the reflexivity and finally the last one corresponds to transitivity.

Topology of Distributed Knowledge

This is why we kept it S4

The oldest semantics of modal logics is topological semantics: 1944.

Topological space

Topological space $\mathcal{X}$ is a pair (X, T) where X is a set of points and T is the collection of subsets of X such that, the empty set and the whole set lie in T and it is closed under finite intersection and arbitrary unions.

Topological Semantics of Distributed Knowledge

A Σ^0_2 semantics for modalities

The semantic of topological interpretation for modal logic presents two new constructions: one for open sets and one for closed sets. They read as follows:

$M, w \models \Box\varphi$ iff $\exists U \in T$ such that $w \in U$ and $\forall v \in U$ we have $M, v \models \varphi$.

Dually, $M, w \models \Diamond\varphi$ iff $\forall U \in T$ such that $w \in U \rightarrow \exists v \in U$ and $M, v \models \varphi$.

Intersection Topology

A framework for Distributed Knowledge

Definition

$(X, T_1 \cap T_2), x \models D_{\{1,2\}}\varphi$ iff $\exists U \in T_1 \cap T_2$ such that $x \in U$ and for all $y \in U$ we then have $(X, T_1 \cap T_2), y \models \varphi$.

Lemma

For the given topological models (X, T_i) defined on the fixed set X, we then have

$$(X, \cap_i T_i), x \models D\varphi \text{ if and only if } (X, T'), x \models \boxdot\varphi,$$

where T' is the intersection topology and $\boxdot$ is the corresponding interior operator for T'. Furthermore, (X, T') is **S4** *with the interior operator $\boxdot$.*

Product Topology

Yet Another Framework for Distributed Knowledge

Given two topologies $\langle X_1, T_1\rangle$ and $\langle X_2, T_2\rangle$, we have the product topology $\langle X_1 \times X_2, T_1, T_2\rangle$. We define the $\Box_i$ operators as follows for given $(x_1, x_2) \in X_1 \times X_2$. $(x_1, x_2) \models \Box_1\varphi$ if and only if $\exists U_1 \in T_1$ such that $x_1 \in U_1$ and $\forall u \in U_1$ we then have $(u, x_2) \models \varphi$.
Likewise, for $\Box_2$.

Lemma

$(X \times X, T_1, T_2), (x_1, x_2) \models D\varphi$ if and only if $\exists U_1 \in T_1$ and $\exists U_2 \in T_2$ such that $x_1 \in T_1$ and $x_2 \in T_2$, and $\forall y_1 \in U_1, \forall y_2 \in U_2$ we then have $(X \times X, T_1, T_2), (y_1, y_2) \models \varphi$.
It is also easy to see that D in product spaces also satisfies **S4** *axioms.*

Fusion Topology

Yet One Another Framework for Distributed Knowledge

Put two models together without no further restrictions to get their fusion.

Exercise

How to approach distributed knowledge in fusion spaces.

Internal Announcements

Prometheus' Announcement

We will call the public announcements made by an external agent *external announcements*. Consequently, an announcement by an agent within the group will be called an *internal announcements*.

$$\mathcal{M}, w \models [\varphi]_i \psi \text{ for } i \in I \quad \text{iff} \quad \mathcal{M}, w \models \mathsf{K}_i \varphi \text{ implies } \mathcal{M}|\varphi, w \models \psi.$$

How to Combine Geometry and Announcements

Contraction Mappings

They formalize information updates in a continuous fashion.

Results

Last Remarks

Last Remarks

Future Research

A Selected Mini-bibliography - 1

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Thanks!

Questions or Comments?

Talk slides is available at:

www.canbaskent.net