Geometry of Dynamic Epistemology

An Exposition

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Outlook of the Talk

- Topological Semantics
- Weak Structures: Subset Spaces
- Dynamic Epistemology

Topological Definitions

Definition (Topological Space)

A topological space $\mathcal{S} = \langle S, \sigma \rangle$ is a structure with a set S and a collection σ of subsets of S satisfying the following axioms:

1. *The empty set and S are in σ.*
2. *The union of any collection of sets in σ is also in σ.*
3. *The intersection of a finite collection of sets in σ is also in σ.*

Recall now that the topological interior operator $\mathbb{I}$ satisfies the following properties for each $X, Y \in \sigma$:
(i) $\mathbb{I}(X) = X$, (ii) $\mathbb{I}(X \cap Y) = \mathbb{I}(X) \cap \mathbb{I}(Y)$, (iii) $\mathbb{I}(\mathbb{I}(X)) = \mathbb{I}(X)$

Logical Definitions

A topological model $\mathcal{M}$ is a triple $\langle S, \sigma, v \rangle$ where $\mathcal{S} = \langle S, \sigma \rangle$ is a topological space, and v is a valuation function sending propositional letters to the subsets of S, i.e. $v : P \to \wp(S)$.

Definition (Topological Semantics)

$$
\begin{array}{lll}
\mathcal{M}, s \models p & \textit{iff} & s \in v(p) \textit{ for } p \in P \\
\mathcal{M}, s \models \neg\varphi & \textit{iff} & \textit{not } \mathcal{M}, s \models \varphi \\
\mathcal{M}, s \models \varphi \wedge \psi & \textit{iff} & \mathcal{M}, s \models \varphi \textit{ and } \mathcal{M}, s \models \psi \\
\mathcal{M}, s \models \mathsf{I}\varphi & \textit{iff} & \exists U \in \sigma(s \in U \wedge \forall t \in U, \mathcal{M}, t \models \varphi)
\end{array}
$$

The C operator can then be defined accordingly:

$$\mathcal{M}, s \models \mathsf{C}\varphi \quad \text{iff} \quad \forall U \in \sigma(s \in U \to \exists t \in U, \mathcal{M}, t \models \varphi)$$

Topological vs Kripkean Semantics

Topological

$\mathcal{M}, s \models \mathsf{I}\varphi$ iff $\exists U \in \sigma$ with $s \in U$ such that $(\forall t \in U), \mathcal{M}, t \models \varphi)$

Kripkean

$\mathcal{M}, s \models \Box\varphi$ iff $\forall t \in U(sRt \rightarrow \mathcal{M}, t \models \varphi)$

Complexity and Expressivity: Topological Semantics is Σ_2 as opposed to Π_1 Kripke Semantics.

Correspondence: Topological vs Kripke Frames

Every S4 Kripke frame $\langle S, R \rangle$ gives rise to a topological space $\langle S, \sigma_R \rangle$, where σ_R is the set of all upward closed subsets of the given frame. It is easy to see that the empty set and S are in σ_R, and furthermore arbitrary unions and finite intersections of upward closed sets are still upward closed. Hence, σ_R is a (Alexandroff) topology.

Alexandroff topologies are those in which each point has a *least* neighborhood (the least neighborhood of a point s is the set $\{t \in W : sRt\}$).

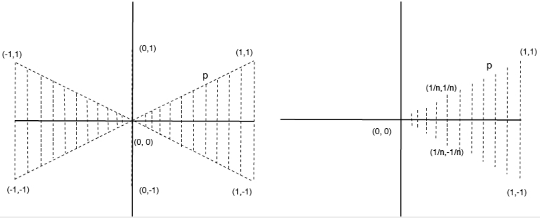
Note that Alexandroff spaces are those topological spaces in which intersection of any family of opens is again an open.

Topological Definability

Topological Goldblatt-Thomason theorem that states that the class **K** of topological spaces which is closed under formation of Alexandroff extensions is modally definable if and only if **K** is closed under taking open subspaces, interior images, topological sums and it reflects Alexandroff extensions (Cate *et al.*, 2009)

Products of Topological Spaces

Moreover, it is a very well known fact that the product logics in Kripke semantics validate the axioms COM and CHR where COM is the commutativity principle ($\Box_1\Box_2 p \equiv \Box_2\Box_1 p$) and CHR is the Church - Rosser property ($\Diamond_1\Box_2 p \rightarrow \Box_2\Diamond_1 p$) (Gabbay *et al.*, 2003). However, the topological products refute both COM and CHR (van Benthem *et al.*, 2006). The counterexample given is the product $\mathbb{R} \times \mathbb{R}$.



Dynamic Topological Logic: The Intuition

As Kremer and Mints put in their paper, dynamic topological logic "provides a context for studying the confluence of three research areas: the topological semantics of S4, topological dynamics, and temporal logic" (Kremer & Mints, 2005). The very core ideas of dynamic topological logic can be found in an earlier paper which presented several significant results (Artemov *et al.*, 1997).

Dynamic Topological Logic: The Idea

Dynamic topological logic is a trimodal logic with topological interior modality I, and two temporal modalities next $\bigcirc$ and henceforth $*$. We interpret them as follows.

- $s \in \bigcirc\varphi$ if and only if $fs \in \varphi$.
- $\bigcirc\varphi = f^{-1}\varphi$
- $*\varphi = \bigcap_{n \geq 0} f^{-n}\varphi$

Dynamic Topological Logic: Some Results

As continuity plays a key role in dynamic topological logic, one needs to axiomatize it. The following axiom

$$\bigcirc\Box p \rightarrow \Box \bigcirc p$$

works very well. The extension of the logic **S4** with the above axiom is called **S4C**. The next theorem establishes the expected connection (Artemov *et al.*, 1997).

Theorem

S4C *is sound and complete with respect to the class of dynamic topological logics where the underlying function is continuous.*

Spatial Proof Theory: Motivation

Read $\Box\varphi$ as "φ is provable". Yet another instance of $\exists$-sickness emerges here: What is the proof of φ then? *Logic of Proofs* addresses this issue by specifying the proof of the expression together with the expression itself.

A recent work investigated the relation between topological semantics and logic of proofs (Artemov & Nogina, 2008). The connection between the proof polynomials and the formulae in a topological setting is achieved by *test functions*. The test function $M(t, \varphi)$ of the proof polynomial t and the formula φ "represents a 'potentially accessible' region of S associated with t and φ".

Spatial Proof Theory: Results

The crucial point is to determine the extension $Ext(t : \varphi)$ of $t : \varphi$

1. $Ext(t : \varphi) = Ext(\varphi) \cap M(t, \varphi)$
2. $Ext(t : \varphi) = \mathbb{I}(Ext(\varphi)) \cap M(t, \varphi)$

The first schema expresses the cases when the outcome of t lies within φ. The second schema, however, expresses the cases when the outcome of t is in the interior of φ.

If we extend S4 with $t : \varphi \rightarrow \varphi$, we will need the first representation.

If we extend S4 with $t : \varphi \rightarrow \Box\varphi$, we will need the second representation.

Topology of First Order Modal Logic -1

A very recent paper on the subject introduced topological semantics for first-order modal logic using *sheaves* (Awodey & Kishida, 2008).

Definition

A sheaf over a topological space $\mathcal{S} = \langle S, \sigma \rangle$ consists of a topological space $\mathcal{T} = \langle T, \tau \rangle$ and a local homomorphism $h : T \to S$ in such a way that every point t in T has a neighborhood O with t such that $h(O)$ is open and restriction $h|O : h(O) \to O$ is a homomorphism as well. In this case, $\mathcal{T}$ is called total space, and h is called the projection from $\mathcal{T}$ to $\mathcal{S}$.

Topology of First Order Modal Logic - 2

Sheaves are equivalent to *functors* in the category theory.
The interpretation of quantified formulae in topological spaces can be given as follows.

$$v(\exists y.\varphi) = h(\varphi) \subseteq S$$

where y is the only free variable in φ (which may or may not appear in the actual formula).

Vickers' Example

"My baby has green eyes."
The obvious question is, "Is this true or false?".
First, we may agree that her eyes really are green - we can *affirm* the assertion.
Second, we may agree that her eyes are some other colour, such as brown - we can *refute* the assertion.
Third, we may fail to agree; but perhaps if we hire a powerful enough colour analyser, that may decide us (Vickers, 1989).
etc...

Vickers' Example - Conclusion

What is crucial in Vickers' analysis is that statements are affirmable or refutable in a *finite* amount of time with spending *finite* amount of effort.

He defines: an assertion is *affirmative*, if and only if it is true precisely in the circumstances when it can be affirmed. Likewise, an assertion is *refutative* if and only if it is false precisely in the circumstances when it can be refuted.

A Dynamic Epistemology

"[N]otion of *effort* enters in topology. Thus if we are at some point at s and make a measurement, we will then discover that we are in some neighborhood U of s, but not know where. If we make my measurement finer, then U will shrink, say, to a smaller neighborhood V." (Moss & Parikh, 1992).

By spending some effort, we eliminate some of the possibilities, and obtain a smaller set of possibilities. The smaller the set of observation is, the larger the information we have.

Therefore, as it was also observed in the above example, to gain *knowledge*, we need to spend some *effort*.

SSL: Model and Language

A subset space model is a triple $\mathcal{S} = \langle S, \sigma, v \rangle$ where $\langle S, \sigma \rangle$ is a subset frame, $v : P \to \wp(S)$ is a valuation function for the countable set of propositional variables P

The language $\mathcal{L}_S$ of SSL is:
$p \mid \top \mid \neg\varphi \mid \varphi \wedge \psi \mid \mathsf{K}\varphi \mid \Box\varphi$

SSL: Semantics

$s, U \models p$	if and only if	$s \in v(p)$	
$s, U \models \varphi \wedge \psi$	if and only if	$s, U \models \varphi$	and $s, U \models \psi$
$s, U \models \neg\varphi$	if and only if	$s, U \not\models \varphi$	
$s, U \models \mathrm{K}\varphi$	if and only if	$t, U \models \varphi$	for all $t \in U$
$s, U \models \Box\varphi$	if and only if	$s, V \models \varphi$	for all $V \in \sigma$ such that $s \in V \subseteq U$

Axioms

1. All the substitutional instances of the tautologies of the classical propositional logic
2. $(A \to \Box A) \land (\neg A \to \Box \neg A)$ for atomic sentence A
3. $\mathsf{K}(\varphi \to \psi) \to (\mathsf{K}\varphi \to \mathsf{K}\psi)$
4. $\mathsf{K}\varphi \to (\varphi \land \mathsf{KK}\varphi)$
5. $\mathsf{L}\varphi \to \mathsf{KL}\varphi$ Euclidean
6. $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$
7. $\Box\varphi \to (\varphi \land \Box\Box\varphi)$
8. $\mathsf{K}\Box\varphi \to \Box\mathsf{K}\varphi$ Cross-Axiom

K is S5 and $\Box$ is S4.
SSL is strongly complete and decidable.

Decidability

Finite model property fails in SSL.
Consider $\Box(\Diamond\varphi \wedge \Diamond\neg\varphi)$ at (s, U) where U is the minimal open about s.

Decidability then can be shown on Cross Axiom models by filtration as Cross Axiom models has a finite model property.

Defining Properties

WDA $\Diamond\Box\varphi \rightarrow \Box\Diamond\varphi$
sound for weakly directed spaces

UA $\Diamond\varphi \wedge L\Diamond\psi \rightarrow \Diamond(\Diamond\varphi \wedge L\Diamond\psi \wedge K\Diamond L(\varphi \vee \psi))$
sound for subset spaces closed under binary unions

WUA $L\Diamond\varphi \wedge L\Diamond\psi \rightarrow L\Diamond(L\Diamond\varphi \wedge L\Diamond\psi \wedge K\Diamond L(\varphi \vee \psi))$
weaker than UA

CI $\Box\Diamond\varphi \rightarrow \Diamond\Box\varphi$
sound for subset spaces closed under all intersections

M_n $(\Box L\Diamond\varphi \wedge \Diamond K\psi_1 \wedge \cdots \wedge \psi_n)$
$\rightarrow L(\Diamond\varphi \wedge \Diamond K\psi_1 \wedge \cdots \wedge \Diamond K\psi_n)$
WD and all M_n are complete for directed spaces

(Georgatos, 1997), (Weiss & Parikh, 2002)

Some Basic Topological Properties in SSL

Proposition

φ is open if and only if $\varphi \rightarrow \Diamond \mathrm{K}\varphi$ is valid.

Proposition

Dually, φ is closed if and only if $\Box \mathrm{L}\varphi \rightarrow \varphi$.

Proposition

$v(p)$ is dense if and only if $\Box \mathrm{L}p$ holds. Similarly, $v(p)$ is nowhere dense if and only if $\Diamond \mathrm{L}\neg p$ is valid.

Overlap Modality

$$s, U \models O\varphi \quad \textit{iff} \quad \forall U' \in \sigma : (s \in U' \rightarrow s, U' \models \varphi)$$

$\Box$ is a special case of O.

Overlap operator was designed to enable us to quantify "not only downwards, but also diagonally" among the set of observations (Heinemann, 2006).

Disjoint Unions

Definition

Two subset space models are disjoint if their domain contains no common element. For disjoint subset space models $\mathcal{S}_i = \langle S_i, \sigma_i, v_i \rangle$, for $i \in I$ their disjoint union is the structure $\mathcal{S} = \biguplus_{i \in I} \mathcal{S}_i = \langle S, \sigma, v \rangle$ where $S = \bigcup_{i \in I} S_i$, $\sigma = \bigcup_{i \in I} \sigma_i$ and $v(p) = \bigcup_{i \in I} v_i(p)$.

Theorem

For disjoint subset space models $\mathcal{S}_i$ for $i \in I$ and for each neighborhood situation (s, U) in $\mathcal{S}_i$, we have $s, U \models_{\mathcal{S}} \varphi$ if and only if $s, U \models_{\mathcal{S}_i} \varphi$, for each formula φ in the language of subset space logic $\mathcal{L}_S$.

Generated Subset Spaces

We can throw away the points at which we do not have any observations.

Proposition

For $\mathcal{S} = \langle S, \sigma, v \rangle$, let $S' = S - \{s : s \notin \cup\sigma\}$ and $v'(p) = v(p) \cap S'$. Then $\mathcal{S}' = \langle S', \sigma, v' \rangle$ and $\mathcal{S} = \langle S, \sigma, v \rangle$ satisfy the same formulae.

Generated Subset Spaces

Definition

Let $\mathcal{S} = \langle S, \sigma, v \rangle$ be a subset space model. Let (s, U) be the designated neighborhood situation. Then we obtain the generated subset space $\mathcal{S}' = \langle S', \sigma', v' \rangle$ of $\mathcal{S}$ as follows.

- $\sigma' := \sigma - \{V \in \sigma : V \not\subseteq U\}$
- $S' := S - \cup\sigma'$
- $v'(p) := v(p) \cap S'$ *for each propositional letter* p*.*

Proposition

For each $s \in S'$, we have $s, U \models_{\mathcal{S}} \varphi$ if and only if $s, U \models_{\mathcal{S}'} \varphi$.

Bisimulation

For, $\mathcal{S} = \langle S, \sigma, u \rangle$ and $\mathcal{T} = \langle T, \tau, v \rangle$, if $(s, U) \rightleftarrows (t, V)$, then:

1. **Base Condition**

   $s \in u(p)$ if and only if $t \in v(p)$ for each p

2. **Back Conditions**

   2.1 $\forall t' \in V$ there exists $s' \in U$ with $(s', U) \rightleftarrows (t', V)$.

   2.2 $\forall V' \subseteq V$ such that $t \in V'$, there is $U' \subseteq U$ with $s \in U'$ such that $(s, U') \rightleftarrows (t, V')$

3. **Forth Conditions**

   3.1 $\forall s' \in U$ there exists $t' \in V$ with $(s', U) \rightleftarrows (t', V)$.

   3.2 $\forall U' \subseteq U$ such that $s \in U'$, there is $V' \subseteq V$ with $t \in V'$ such that $(s, U') \rightleftarrows (t, V')$.

Bisimulation Invariance

Theorem (Bisimulation Invariance for Subset Spaces)
If $(s, U) \rightleftarrows (t, V)$ then they satisfy the same formulae.

Converse is true only under the special conditions.

Theorem
Let $\mathcal{S} = \langle S, \sigma, u \rangle$ and $\mathcal{T} = \langle T, \tau, v \rangle$ be two finite subset space. Then for each neighborhood situations (s, U) in $S \times \sigma$ and (t, V) in $T \times \tau$; we have $(s, U) \rightleftarrows (t, V)$ if and only if $(s, U) \leftrightsquigarrow (t, V)$.

Evaluation and Bisimulation Games

Position	Player	Admissible Moves
$(\bot, (s, U))$	$\exists$	$\emptyset$
$(\top, (s, U))$	$\forall$	$\emptyset$
$(p, (s, U))$ with $s \in v(p)$	$\forall$	$\emptyset$
$(p, (s, U))$ with $s \notin v(p)$	$\exists$	$\emptyset$
$(\psi_1 \wedge \psi_2, (s, U))$	$\forall$	$\{(\psi_1, (s, U)), (\psi_2, (s, U))\}$
$(\psi_1 \vee \psi_2, (s, U))$	$\exists$	$\{(\psi_1, (s, U)), (\psi_2, (s, U))\}$
$(\mathsf{L}\psi, (s, U))$	$\exists$	$\{(\psi, (t, U)) : t \in U\}$
$(\mathsf{K}\psi, (s, U))$	$\forall$	$\{(\psi, (t, U)) : t \in U\}$
$(\Diamond\psi, (s, U))$	$\exists$	$\{(\psi, (s, V)) : s \in V \subseteq U\}$
$(\Box\psi, (s, U))$	$\forall$	$\{(\psi, (s, V)) : s \in V \subseteq U\}$

Adequacy Theorems for Evaluation and Bisimulation games follow.

Semantics

Definition

Let $\mathcal{M} = \langle W, R, V \rangle$ be a model and i be an agent. For atomic propositions, negations and conjunction the definition is as usual. For modal operators, we have the following semantics:

$$\mathcal{M}, w \models \mathsf{K}_i\varphi \quad \text{iff} \quad \mathcal{M}, v \models \varphi \text{ for each } v \text{ such that } (w, v) \in R_i$$
$$\mathcal{M}, w \models [\varphi]\psi \quad \text{iff} \quad \mathcal{M}, w \models \varphi \text{ implies } \mathcal{M}|\varphi, w \models \psi$$

Here the updated model $\mathcal{M}|\varphi = \langle W', R', V' \rangle$ is defined by restricting $\mathcal{M}$ to those states where φ holds.
(Plaza, 1989)

Reduction Axioms

The proof system of public announcement logic is the proof system of multi-modal S5 epistemic logic with the following additional axioms.

Atoms	$[\varphi]p \leftrightarrow (\varphi \rightarrow p)$
Partial Functionality	$[\varphi]\neg\psi \leftrightarrow (\varphi \rightarrow \neg[\varphi]\psi)$
Distribution	$[\varphi](\psi \wedge \chi) \leftrightarrow ([\varphi]\psi \wedge [\varphi]\chi)$
Knowledge Announcement	$[\varphi]\mathsf{K}_i\psi \leftrightarrow (\varphi \rightarrow \mathsf{K}_i[\varphi]\psi)$

The rule of inference for $[*]$ is called the *announcement generalization* and is described as follows.

From $\vdash \psi$, *derive* $\vdash [\varphi]\psi$.

Problems

State elimination does not perfectly correspond to the intuitive idea of learning / knowledge update.

Relation is restrictied after the state elimination - which requires an additional computational effort.

Semantics

We ignore the interaction between the agents. As the announcement is external, we will focus on the knowledge update of one agent.
The semantics for topologic PAL differs only on public announcement operator whose semantics is given as follows:

$$s, U \models [\varphi]\psi \quad \textit{if and only if} \quad s, U \models \varphi \textit{ implies } s, U_\varphi \models \psi$$

where $U_\varphi = U \cap (\varphi)$

Compare: $\mathcal{M}, w \models [\varphi]\psi$ iff $\mathcal{M}, w \models \varphi$ implies $\mathcal{M}|\varphi, w \models \psi$

Axioms

Therefore, it is easy to see that the following axiomatize the topologic-PAL:

Atoms	$[\varphi]p \leftrightarrow (\varphi \rightarrow p)$
Partial Functionality	$[\varphi]\neg\psi \leftrightarrow (\varphi \rightarrow \neg[\varphi]\psi)$
Distribution	$[\varphi](\psi \wedge \chi) \leftrightarrow ([\varphi]\psi \wedge [\varphi]\chi)$
Knowledge Announcement	$[\varphi]\mathsf{K}\psi \leftrightarrow (\varphi \rightarrow \mathsf{K}[\varphi]\psi)$
Shrinking Reduction	$[\varphi]\Box\psi \leftrightarrow (\varphi \rightarrow \Box[\varphi]\psi)$

Completeness

Theorem (Completeness of Topologic PAL)

Topologic PAL is complete with respect to the axiom system given above.

Proof.

By reduction axioms we can reduce each formula in the language of topologic PAL to a formula in the language of (basic) topologic. As topologic is complete, so is topologic PAL. □

(Başkent, 2007)

Philosophy of Science: Lakatos

Proofs and Refutations gives a rationally reconstructed account of the methodological evaluation of Euler's formula for polyhedra: $V - E + F = 2$.

Improvements of the conjecture, proof and the theorem are of utmost importance in Lakatosian context. We need a tool for the general framework of Lakatosian heuristics.

(Lakatos, 2005), (Başkent & Bağçe, 2009)

Semantics

Let $\mathcal{F}$ be a collection of functions from S to S, and further let $F \subseteq \mathcal{F}$. Take two subset spaces $\mathcal{S} = \langle S, \sigma, v \rangle$ and $\mathcal{S}_F = \langle S, \sigma_F, v \rangle$. Here, σ_F is the image of each $U \in \sigma$ under each function $f \in F$. In other words, $\sigma_F := \{fU : f \in F, U \in \sigma\}$. We will call $\mathcal{S}_F$ *the image space of* $\mathcal{S}$ *under* F.

Each function $f \in F$ are contracting mappings intended to represent the increase in the information. Hence, $fU \subseteq U$ should hold for each function f and for each observation set U (Başkent, 2007)

Semantics

$s, U \models_{\mathcal{S}} [F]\varphi$ iff $s, fU \models_{\mathcal{S}_F} \varphi$ for each $f \in F$

The dual of $[F]$ will be defined as follows:

$s, U \models_{\mathcal{S}} \langle F \rangle\varphi$ iff $s, fU \models_{\mathcal{S}_F} \varphi$ for some $f \in F$

Some Observations

1. $[F](\varphi \rightarrow \psi) \rightarrow ([F]\varphi \rightarrow [F]\psi)$
   It is easy to see that $[F]$ modality realizes the **K** axiom
2. $[F][F]\varphi \rightarrow [F]\varphi$
   This axiom is valid if F is closed under function decomposition.
3. $[F]\varphi \rightarrow [F][F]\varphi$
   This axiom is valid if F is closed under function composition.
4. $[F]\varphi \rightarrow \varphi$
   This axiom is valid if the identity function id_F is in F.
5. $\Box\varphi \rightarrow [F]\varphi$
6. $\mathrm{K}[F]\varphi \rightarrow [F]\mathrm{K}\varphi$
   This is the cross axiom for $[F]$ and K

Common Knowledge in SSL

Simplest definition:

$$C\varphi \equiv \varphi \wedge \Diamond K\varphi \wedge \Diamond K \Diamond K\varphi \dots$$

$s, U \models C\varphi :=$
$\forall n \in \mathbb{N}$ and $t \in S$, we then have:
if $U_0, U_1, \dots, U_n \in \sigma$ satisfy $U_0 = U$ and $U_i \cap U_{i+1} \neq \emptyset$ for $i = 0, \dots, n-1$ and, $t \in U_n$, then $t, U_n \models \varphi$
The following is the iteration definition of common knowledge.

$$s, U \models C\varphi \equiv s, U \models \underbrace{KO \dots KO}_{n-\text{times}} \varphi, \text{ for all } n \in \mathbb{N}$$

(Parikh *et al.*, 2007)

Future Work: Learning

Epistemic logics, in my opinion, should have an intuitive basis. The motto "to act to learn" has both dynamic and epistemic connotations. Therefore, starting from subset space logics, I will suggest formal ideas to represent learning in spatial setting. There are several considerations one should meditate thoroughly. One is the dynamic epistemological research which employs learning with communications. The second is belief revisions and updates. Therefore, what we need to come up with should work in such situations as well. In other words, learning, and its conceptual converse *unlearning* includes situations where the new information becomes *known* or *not-known* or even maybe *unknown*. In computer science, this is heavily related to history based processes and communication.

Strategies

The idea of heuristics implicitly suggest the use of games, *i.e.* strategy based decision protocols. What distinguishes heuristics from a mere learning process is the strategic methodology attached to it. For instance, the order of the information received, the propositional content of the information influence the heuristic learning. In order to illustrate all (and more) of such considerations, one can investigate the history of mathematics carefully. Lakatos's approach to the subject, as always, is quite notable (Lakatos, 2005). Therefore, the very first formal epistemic step is to come with a *heuristic subset space logic* which is expressive enough for heuristics.

Qualitative DEL

The second step is to reconsider the setting of DEL. One of the *sickness* of DEL is the "update by state elimination" paradigm. The reason why I think that it is a sickness is the mere fact that it cannot express the cases when one changes his mind while keeping the states as it is. Therefore, "learning new things" that might go well with the preexisting things is quite unusual for DEL approach. Similarly, unspecified and perhaps unknown belief revision paradigms suffers from the similar problems (cf. (Gärdenfors, 1985)). Third step is to formalize forgetting and information loose. The current trend in epistemic logic mostly focuses on information retrieval, and ignores the cases when the agents do not want more information, and wants to *clear their memories*.

Constructive SSL

Akin to the *knowledge structures* (Fagin *et al.*, 1991), a constructive account of multi-agent (also by definition, multi-modal) version of SSL is to be given.

Application of SSL to methodology of science: real case studies of Lakatosian heuristics is to be given.

Furthermore, following the research program of *social software* (Parikh, 2002), application of SSL to political science is to be given: *political science*. Moreover, application of SSL to deontic / ethical theories seems feasible (Pacuit *et al.*, 2006).

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Thanks for your attention!

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