Geometry of Knowledge: An Exposition, Meditations and Reflections

Can BAŞKENT

Graduate Center, City University of New York

cbaskent@gc.cuny.edu www.canbaskent.net

January 2008

İstanbul University, İstanbul Kültür University, Bilkent University, METU

Outlook of the Talk

- Formalizing Knowledge
- Epistemic Logic
- Topological Models for Knowledge
- Non-Topological Models for Knowledge
- Interlude: Dynamic Epistemic Logic
- Applications
- Conclusion

Aims

- To understand the properties of "knowledge".
- To derive technical/mathematical results about "knowledge".
- Establish a *beautiful* mathematical theory in a la Platonic sense.
- Apply those ideas to linguistics, computer science, economics, cognitive science and applied philosophy.

Ancient Times

- Aristotle: Prior Analytics and Posterior Analytics. Sea Battle argument for temporal logic. Boethius' modal logical expositions of Aristotle's works.
- Duns Scotus and William of Ockham
- Ibn Khaldun: S5 postulates (cf. van Ditmarsch)

Modern Times: Hintikka

Initiated the modern study with his book *Knowledge and Belief* (1962) by extending extensively von Wright's works.

The mathematical tools are based on Lewis, McKinsey and Tarski.

The notions of knowledge and belief are distinguished formally.

Language of Epistemic Logic

Language of propositional logic extended with a modal operator K.

$$p \mid \top \mid \neg\varphi \mid \varphi \wedge \varphi \mid \mathsf{K}\varphi$$

We are in the realm of propositional modal logic: no quantifiers. $\top$ stands for "truth", its dual $\bot$ stands for contradiction. Thus, $\vee$ and $\rightarrow$ are shorthand notations.

Semantics of Epistemic Logic

A Relational Modal Model: $M = \langle S, R, V \rangle$,
where S: nonempty set, R: a binary relation, V: valuation function from the set of propositional variables to the power set of S.

$$
\begin{array}{llll}
M, s & \models p & \text{iff} & s \in V(p) \\
M, s & \models \top & & \text{always} \\
M, s & \models \neg\varphi & \text{iff} & M, s \not\models \varphi \\
M, s & \models \varphi \wedge \psi & \text{iff} & M, s \models \varphi \text{ and } M, s \models \psi \\
M, s & \models \mathsf{K}\varphi & \text{iff} & \forall t (sRt \rightarrow M, t \models \varphi)
\end{array}
$$

Possible World Semantics: Some Remarks

A point t is said to be accessible from a point s, if sRt.

The set of accessible states for a point s are called *possible worlds* or *compatible worlds* for s.

According to the semantics, the agent knows φ at the point s, whenever φ is the case in each possible world wrt. s.

Changing the current state/actual world, changes the knowledge!

K has a dual: L

$$M, s \models \mathrm{K}\varphi \text{ iff } \forall t(sRt \rightarrow M, t \models \varphi)$$

$$M, s \models \mathrm{L}\varphi \text{ iff } \exists t(sRt \wedge M, t \models \varphi)$$

Possible World Semantics: Further Philosophy

Leibniz: "God created the best of all possible worlds".

David Lewis (decd., Princeton): Possible worlds exist. Modal realism. Counterfactuals! See the paper "Possible Worlds".

Saul Kripke (CUNY): Rigid Designator. See "Naming and Necessity".

Robert Stalnaker (MIT): A very involved account. Two-semantics-dimensional modal logic: context dependent meaning and the normal intentional meaning. See the paper "Possible Worlds".

Properties of K

- K $[K(\varphi \rightarrow \psi)] \rightarrow (K\varphi \rightarrow K\psi)$
- T $K\varphi \rightarrow \varphi$
- 4 $K\varphi \rightarrow KK\varphi$
- 5 $\neg K\varphi \rightarrow K\neg K\varphi$

Hence R is an equivalence relation, and there are equivalence classes in the set. The logic is then called S5 (K + T + 4 + 5) for historical reasons due to C.I. Lewis.

One can deny Axiom 5 and hence get a S4 (K + T + 4) epistemic logic.

Properties of K

- K $[K(\varphi \rightarrow \psi)] \rightarrow (K\varphi \rightarrow K\psi)$
- T $K\varphi \rightarrow \varphi$
- 4 $K\varphi \rightarrow KK\varphi$
- 5 $\neg K\varphi \rightarrow K\neg K\varphi$

Hence R is an equivalence relation, and there are equivalence classes in the set. The logic is then called S5 (K + T + 4 + 5) for historical reasons due to C.I. Lewis.
One can deny Axiom 5 and hence get a S4 (K + T + 4) epistemic logic.

Properties of the Relation R

- K $[\mathsf{K}(\varphi \to \psi)] \to (\mathsf{K}\varphi \to \mathsf{K}\psi)$
- T $\mathsf{K}\varphi \to \varphi$
- 4 $\mathsf{K}\varphi \to \mathsf{K}\mathsf{K}\varphi$
- 5 $\neg\mathsf{K}\varphi \to \mathsf{K}\neg\mathsf{K}\varphi$

- T K is reflexive, i.e. sRs holds for each s.
- 4 K is transitive, i.e. sRt and tRu imply sRu for each s, t, u.
- 5 K is symmetric, i.e. sRt implies tRs.

Hence R is an equivalence relation in our setting, and there are equivalence classes in the set.

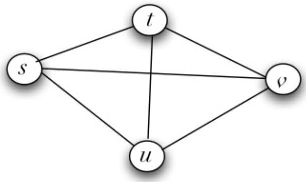
Properties of the Relation R

- K $[\mathsf{K}(\varphi \to \psi)] \to (\mathsf{K}\varphi \to \mathsf{K}\psi)$
- T $\mathsf{K}\varphi \to \varphi$
- 4 $\mathsf{K}\varphi \to \mathsf{KK}\varphi$
- 5 $\neg\mathsf{K}\varphi \to \mathsf{K}\neg\mathsf{K}\varphi$

- T K is reflexive, i.e. sRs holds for each s.
- 4 K is transitive, i.e. sRt and tRu imply sRu for each s, t, u.
- 5 K is symmetric, i.e. sRt implies tRs.

Hence R is an equivalence relation in our setting, and there are equivalence classes in the set.

An Example Model for Knowledge



$$(\varphi) = \{s, t, u, v\}$$
$$(\psi) = \{u, v\}$$
$$(\eta) = \{\}$$

$$M, s \models K\varphi$$

Completeness and Decidability

(S5) Epistemic logic is complete with respect to reflexive, transitive and symmetric frames (i.e. structures without valuation). Proof is an application of Henkin completeness proof.

Epistemic Logic has finite model property (i.e. there exists a finite model for each model).
Hence, it is decidable. Proof is by filtrations.

Language of Topological Epistemic Logic

A topological space $\mathcal{S}$ is a pair $\langle S, \sigma \rangle$ where S is a set of points and the set of opens $\sigma \subseteq \wp(S)$ contains $\emptyset, S$, and is closed under finite intersection and arbitrary unions.
We have the following language:

$$p \mid \top \mid \neg\varphi \mid \varphi \wedge \varphi \mid \mathsf{Int}(\varphi)$$

Semantics of Topological Epistemic Logic

We assume the epistemic logic is S4 (K + T + 4) in order to comply with subset relation. (what happens if we don't!)

The boolean and propositional cases are as before.

Int has a dual: Clo.

$$
\begin{array}{lll}
M, s \models \mathrm{Int}(\varphi) & \text{iff} & \exists O \in \sigma(s \in O \wedge \forall t \in U(M, t \models \varphi)) \\
M, s \models \mathrm{Clo}(\varphi) & \text{iff} & \forall O \in \sigma(s \in O \rightarrow \exists t \in U(M, t \models \varphi))
\end{array}
$$

Completeness Results

Theorem S4 is a complete axiomatization of K interpreted over arbitrary topological spaces.

Theorem S4 is a complete axiomatization of modal K interpreted over any metric space which is dense-in-itself (eg. $\mathbb{R}, \mathbb{Q}$).

What about Cantor spaces?

Why Complete?

Propostions are sets!

Topological Space	**Logic S4**
$Int(O) \subseteq O$	$K\varphi \rightarrow \varphi$
$Int(O) \subseteq Int(Int(O))$	$K\varphi \rightarrow KK\varphi$
$S \in \sigma$ and $\emptyset \in \sigma$	$M \models \top$ and $M \not\models \bot$
$O \subseteq O'$ implies $Int(O) \subseteq Int(O')$	$[K(\varphi \rightarrow \psi)] \rightarrow (K\varphi \rightarrow K\psi)$

(McKinsey and Tarski)

Alexandroff Topology

Alexandroff space is the topological space in which the arbitrary intersection of opens is still an open.
In Alexandroff spaces every point has a smallest neighborhood.

Is $\mathbb{R}$ with usual topology an Alexandroff space?
Is $\mathbb{Q}$ with usual topology an Alexandroff space?

Can We Go Back and Forth Between Two Models?

Given an Alexandroff space, define $sRt \equiv s \in Clo(t)$. Easy to verify that R is reflexive and transitive and hence S4.

Given a Kripke structure, define a topology σ whose opens are upsets of the tree structure. As an exercise, verify that σ is an Alexandroff topology.

Can We Go Back and Forth Between Two Models?

Given an Alexandroff space, define $sRt \equiv s \in Clo(t)$. Easy to verify that R is reflexive and transitive and hence S4.

Given a Kripke structure, define a topology σ whose opens are upsets of the tree structure. As an exercise, verify that σ is an Alexandroff topology.

Arbitrary Collections

What happens if $\langle S, \sigma \rangle$ is not a topology?
Let S: a nonempty set, σ: an arbitrary collection of subsets of S.

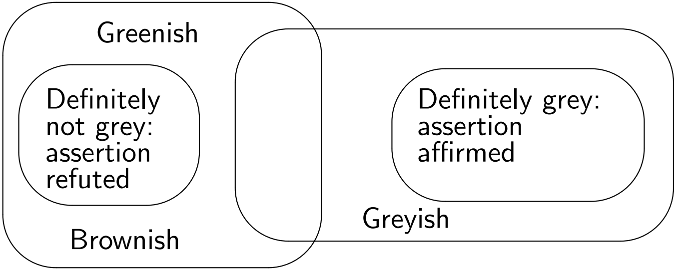
Vickers' Example

"My baby has green eyes."
The obvious question is, "Is this true or false?".
First, we may agree that her eyes really are green - we can *affirm* the assertion.
Second, we may agree that her eyes are some other colour, such as brown - we can *refute* the assertion.
Third, we may fail to agree; but perhaps if we hire a powerful enough colour analyser, that may decide us.
etc...

(Vickers, Topology via Logic)

Vicker's Example

One can come up with the following diagram (Vickers).



Vickers' Example - Conclusion

What is crucial in Vickers' analysis is that statements are affirmable or refutable in a *finite* amount of time with spending *finite* amount of effort.

He defines: an assertion is *affirmative*, if and only if it is true precisely in the circumstances when it can be affirmed. Likewise, an assertion is *refutative* if and only if it is false precisely in the circumstances when it can be refuted.

Vickers' Example - Conlusion

"[N]otion of *effort* enters in topology. Thus if we are at some point at s and make a measurement, we will then discover that we are in some neighborhood U of s, but not know where. If we make my measurement finer, then U will shrink, say, to a smaller neighborhood V." [Moss and Parikh]

Therefore, by spending some effort, we eliminate some of the possibilities, and finally obtain a smaller set of possibilities. The smaller the set of observation is, the larger the information we have.

Therefore, as it was also observed in the above example, to gain *knowledge*, we need to spend some *effort*.

Language

A subset space model is a triple $\mathcal{S} = \langle S, \sigma, V \rangle$ where $\langle S, \sigma \rangle$ is a subset frame, $V : P \to \wp(S)$ is a valuation function for the countable set of propositional variables P

The language of Subset Space Logic is:

$$p \mid \top \mid \neg\varphi \mid \varphi \wedge \varphi \mid \mathsf{K}\varphi \mid \Box\varphi$$

Language

A subset space model is a triple $\mathcal{S} = \langle S, \sigma, V \rangle$ where $\langle S, \sigma \rangle$ is a subset frame, $V : P \to \wp(S)$ is a valuation function for the countable set of propositional variables P

The language of Subset Space Logic is:

$$p \mid \top \mid \neg\varphi \mid \varphi \wedge \varphi \mid \mathsf{K}\varphi \mid \Box\varphi$$

Semantics

$s, U \models p$	if	$s \in v(p)$	
$s, U \models \varphi \wedge \psi$	iff	$s, U \models \varphi$	and $s, U \models \psi$
$s, U \models \neg\varphi$	iff	$s, U \not\models \varphi$	
$s, U \models \mathrm{K}\varphi$	iff	$t, U \models \varphi$	$\forall t \in U$
$s, U \models \Box\varphi$	iff	$s, V \models \varphi$	$\forall V \in \sigma$ such that $s \in V \subseteq U$
$s, U \models \mathrm{L}\varphi$	iff	$t, U \models \varphi$	for some $t \in U$
$s, U \models \Diamond\varphi$	iff	$s, V \models \varphi$	for some $V \in \sigma$ such that $s \in V \subseteq U$

(s, U) is called a neighborhood situation.

This logic is not substitutive. Why?

Topological Modal Logic vs SSL

SSL is more expressive. (Why?)

Observe: $\mathsf{Int}(\varphi) \equiv \Diamond \mathsf{K}\varphi$. (Why?)

Axioms

1. All the substitutional instances of the tautologies of the classical propositional logic
2. $(A \rightarrow \Box A) \wedge (\neg A \rightarrow \Box \neg A)$ for atomic sentence A
3. $\mathsf{K}(\varphi \rightarrow \psi) \rightarrow (\mathsf{K}\varphi \rightarrow \mathsf{K}\psi)$ — K
4. $\mathsf{K}\varphi \rightarrow (\varphi \wedge \mathsf{KK}\varphi)$ — T and 4
5. $\mathsf{L}\varphi \rightarrow \mathsf{KL}\varphi$ — Euclidean
6. $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$ — K
7. $\Box\varphi \rightarrow (\varphi \wedge \Box\Box\varphi)$ — T and 4
8. $\mathsf{K}\Box\varphi \rightarrow \Box\mathsf{K}\varphi$ — Cross-Axiom

K is S5 and $\Box$ is S4.

Completeness

SSL is strongly complete and decidable.

NOT trivial!

The reason for that is the fact that at the level of maximally consistent theories, there is no known way to define a corresponding subset space structure.

Semantics in Kripke Structures

Let $M = \langle S, R, V \rangle$ be an epistemic model. For atomic propositions, negations and conjunction the definition is as usual. For modal operators, we have the following semantics:

$M, s \models \mathrm{K}\varphi$ iff $M, t \models \varphi$ for each t such that $(s, t) \in R$
$M, s \models [\varphi]\psi$ iff $M, s \models \varphi$ implies $M|\varphi, s \models \psi$

Here the updated model $M|\varphi = \langle S', R', V' \rangle$ is defined by restricting M to those states where φ holds.

Reduction Axioms in Kripke Structures

The proof system of public announcement logic is the proof system of multi-modal S5 epistemic logic with the following additional axioms.

Atoms	$[\varphi]p \leftrightarrow (\varphi \rightarrow p)$
Partial Functionality	$[\varphi]\neg\psi \leftrightarrow (\varphi \rightarrow \neg[\varphi]\psi)$
Distribution	$[\varphi](\psi \wedge \chi) \leftrightarrow ([\varphi]\psi \wedge [\varphi]\chi)$
Knowledge Announcement	$[\varphi]\mathsf{K}\psi \leftrightarrow (\varphi \rightarrow \mathsf{K}[\varphi]\psi)$

The rule of inference for $[*]$ is called the *announcement generalization* and is described as follows.

From $\vdash \psi$, *derive* $\vdash [\varphi]\psi$.

Semantics in Subset Space Logic

The semantics for topologic PAL differs only on public announcement operator whose semantics is given as follows:

$$s, U \models [\varphi]\psi \quad \textit{if and only if} \quad s, U \models \varphi \textit{ implies } s, U_\varphi \models \psi$$

EXERCISE: How to define U_φ?
Hint: SSL was not substitutive!

Compare: $M, s \models [\varphi]\psi$ iff $M, t \models \varphi$ implies $M|\varphi, t \models \psi$

Axioms in Subset Space Logic

Therefore, it is easy to see that the following axiomatize the topologic-PAL:

Atoms	$[\varphi]p \leftrightarrow (\varphi \rightarrow p)$
Partial Functionality	$[\varphi]\neg\psi \leftrightarrow (\varphi \rightarrow \neg[\varphi]\psi)$
Distribution	$[\varphi](\psi \wedge \chi) \leftrightarrow ([\varphi]\psi \wedge [\varphi]\chi)$
Knowledge Announcement	$[\varphi]K\psi \leftrightarrow (\varphi \rightarrow K[\varphi]\psi)$
Shrinking Reduction	$[\varphi]\Box\psi \leftrightarrow (\varphi \rightarrow \Box[\varphi]\psi)$

Completeness

Theorem (Completeness of Topologic PAL)

Topologic PAL is complete with respect to the axiom system given above.

Proof.

By reduction axioms we can reduce each formula in the language of topologic PAL to a formula in the language of (basic) topologic. As topologic is complete, so is topologic PAL. □

A Motivation from Philosophy of Science: Lakatos

Proofs and Refutations gives a rationally reconstructed account of the methodological evaluation of Euler's formula for polyhedra: $V - E + F = 2$.
Starting from a collection of observations (or assertions) about some peculiar properties of polyhedron, the arguments proceed by reducing these observations (or assertions) by some mathematical *thought experiments* as Lakatos himself called.

A Motivation from Philosophy of Science: Lakatos

Let us see an example.
Let us assume $(polyhedron, U) \models V_{torus} - E_{torus} + F_{torus} = 2$ where U is the collection of observed polyhedral objects. Some may be genuine polyhedra, some not.
Clearly, $E(torus) = 0$. Contradiction.
Then we need to get rid of some objects in U we previously thought of genuine polyhedra. For example, we need to get rid of torus, Klein bottle, Mobiüs strip etc. to get $U' \subset U$.
The formal way of achieving that is to introduce the Euler characteristic function for both oriented and non-oriented objects.

Philosophy of Science: Lakatos

The effort in this context corresponds to some mathematical calculations or suggesting a counter example or even refuting a counterexample.
For example, if we establish that the Euler formula holds for simply connected polyhedra, then, we will discard some observations about the polyhedra which are not simply connected - such as torus. Hence, without changing our point of view, we changed our neighborhood situation by considering some smaller set around the reference point we are occupying.

Semantics of Controlled Shrinking

Let $\mathcal{F}$ be a collection of functions from S to S, and further let $F \subseteq \mathcal{F}$. Take two subset spaces $\mathcal{S} = \langle S, \sigma, v \rangle$ and $\mathcal{S}_F = \langle S, \sigma_F, v \rangle$. Here, σ_F is the image of each $U \in \sigma$ under each function $f \in F$. In other words, $\sigma_F := \{fU : f \in F, U \in \sigma\}$. We will call $\mathcal{S}_F$ *the image space of* $\mathcal{S}$ *under* F.

Each function $f \in F$ are contracting mappings intended to represent the increase in the information. Hence, $fU \subseteq U$ should hold for each function f and for each observation set U

Semantics of Controlled Shrinking

$s, U \models_{\mathcal{S}} [F]\varphi$ iff $s, fU \models_{\mathcal{S}_F} \varphi$ for each $f \in F$

The dual of $[F]$ will be defined as follows:

$s, U \models_{\mathcal{S}} \langle F \rangle\varphi$ iff $s, fU \models_{\mathcal{S}_F} \varphi$ for some $f \in F$

What is Common Knowledge

Meaningful when there are more than one agent.

There is common knowledge of p in a group of agents G when all the agents in G know p, they all know that they know p, they all know that they all know that they know p, and so on ad infinitum. (Wikipedia defintion)

Introduced by D. Lewis in *Convention* when discussing the epistemological status of some certain games, i.e. coordination games (eg. prisoner's dilemma).

An event takes place. Agent 1 and 2 see that event and they further see that the other saw the event. Then, that event becomes common knowledge.

Common Knowledge: Logical Approach

$$M, s \models \mathrm{C}_{1,2}\varphi \text{ iff } \forall t \text{ with } s(R_1 \cup R_2)^* t, M, t \models \varphi$$

$(R_1 \cup R_2)^*$: reflexive and transitive closure of R_1 and R_2.
R_i: the epistemic accessibility relation of the agent i.
(Aumann)

$$M, s \models \mathrm{C}_{1,2}\varphi \text{ iff } M, s \models \varphi \wedge \mathrm{K}_1 \wedge \mathrm{K}_2\varphi \wedge \mathrm{K}_1\mathrm{K}_1\varphi \wedge \mathrm{K}_1\mathrm{K}_2\varphi \wedge \mathrm{K}_2\mathrm{K}_1\varphi \ldots$$

(D. Lewis)

Common Knowledge: Mathematical Approach

- Iterate Approach
- Fixed Point Approach
- Shared Environment Approach

(J. Barwise, Three Views of Common Knowledge)

Fixed Point: $C_{1,2}\varphi$ is the fixed point of the epistemic operator

$$\mu X.\varphi \wedge K_1 X \wedge K_2 X$$

What is modal-mu-calculus?

Common Knowledge: A Game Theoretical Approach

Plenty of examples from game theory. Philosophers are interested in the knowledge dynamics in games, eg. Muddy Children, Clumsy Waiter.
If two agents have common prior probability over a certain event, and further if the posterior probabilities are common knowledge, then those posterior probabilities are equal. (Aumann, Agreeing to Disagree)

Nash Equilibrium vs Common Knowledge!
(Aumann and Brandenburger)

Common Knowledge in SSL

Simplest definition:

$$\mathsf{C}\varphi \equiv \varphi \wedge \Diamond\mathsf{K}\varphi \wedge \Diamond\mathsf{K}\Diamond\mathsf{K}\varphi \ldots$$

$s, U \models \mathsf{C}\varphi :=$
$\forall n \in \mathbb{N}$ and $t \in S$, we then have:
if $U_0, U_1, \ldots, U_n \in \sigma$ satisfy $U_0 = U$ and $U_i \cap U_{i+1} \neq \emptyset$
for $i = 0, \ldots, n-1$ and, $t \in U_n$, then $t, U_n \models \varphi$

The following is the iteration definition of common knowledge.

$$s, U \models \mathsf{C}\varphi \equiv s, U \models \underbrace{\mathsf{KO} \ldots \mathsf{KO}}_{n-\text{times}} \varphi$$

Ehrenfeucht - Fraïßé Games

Adequacy and Bisimultions games can easily be defined both in topological and non-topological logics.

Multi Agent Systems

- How to process information when more than one agent is present
- How to acquire information when more than one agent is present
- How to run non-deterministic algorithms when an additional information is acquired
- Automata Theory

Meaning and Reference

- Natural Language - Machine Language interaction
- Context and Content depended knowledge
- Interaction between the agent and utterance
- Theories of Meaning

A Tradition

- Completeness proofs
- Products of logics
- Closure and Definability Properties
- CoAlgebraic / Category Theoretic approach
- Geometric, Topologic and Metric properties
- Game Theory
- Proof Theory: Parikh sentences

Epistemology

- What is knowledge?
- What is belief?
- What is justified belief?
- Natural language vs Machine language?
- Philosophical implications of completeness.
- Paradoxes of Epistemology: Moore paradox etc.

Mind!

- What is knowledge acquisition?
- Geometric models of learning
- Belief revision
- Logic of Communication
- Learning theories
- Heuristics
- Machine proofs

Divide the Resources

- What is the role of knowledge in games?
- Fair Division (Brams and Taylor)
- Social Software (Parikh)
- Strategic decision making
- Politics of knowledge

Thinking Computers

- How can a machine learn?
- How to update a machine state?
- How to revise knowledge?
- How to revise beliefs?
- Decision making
- Machine proofs
- Nondeterministic and (maybe) quantum algorithms

How to Formalize Heuristics

Recall Lakatos example.

Key words: growth of information and the geometry of it.

You know more when there are less possible worlds

You gain information by spending some effort, i.e. by shrinking your set of accessible states.

Learning is reducing the set of possibilities.

Geometry of Belief

Not as smooth as epistemic logic.

Onions, spheres, brocollis have been suggested

Some belief sets (i.e. stable belief sets) behave nice but does not provide much.

Results

We observed:

- Formal epistemological properties of knowledge
- Geometries of knowledge (topological and non-topological)
- Geometry of knowledge interaction
- Games

Some Problems

- Logic of Chu spaces.
- Topological and topologic exposition of common knowledge in multi-agent systems.
- Logic of Compact Spaces (unbelievebly hard!!).
- Multiagent SSL
- Modal-mu topological logic
- First Order Topological Modal Logic
- Completeness via Topology: Exceptions!
- Geometry of Belief Revision

Questions or Comments?

Talk slides and the thesis are available at:

www.canbaskent.net