Game Semantics for Paraconsistent Logics

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January 26-29, 2016

Frontiers of Non-Classicality: Logic, Mathematics, Philosophy - Auckland, New Zealand

Slogan: Paraconsistency for Game Theory!

Paraconsistency helps us understand game theoretical agency better.

Outlook of the Talk

- Logic of Paradox
- First-Degree Entailment
- Relevant Logic
- Connexive Logic
- Belnap's 4-valued Logic
- Translation to S5 Modal Logic

Why Paraconsitency? Why Games?

- How to understand paraconsistency from a game semantical point of view?
- How to understand game semantics from a paraconsistent point of view?
- ▶ How to relate games and paraconsistency from a rationalistic point of view?

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- ▶ ▶ How to relate games and paraconsistency from a rationalistic point of view?

Classical Game Semantics

During the semantic verification game, the given formula is broken into subformulas by **two** players (Abelard and Heloise) **step by step**, and the game **terminates** when it reaches the propositional atoms.

If we end up with a propositional atom which is true, then Eloise the verifier wins the game. **Otherwise**, Abelard the falsifier wins. We associate **conjunction with Abelard**, **disjunction with Heloise**.

A win for the verifier is when the game terminates with a true statement. The verifier is said to have a winning strategy if she can force the game to her win, regardless of how her opponent plays.

Classical Game Semantics

Just because the game may end with a true/false atom does not necessarily suggest the truth/falsity of the given formula in general.

In classical logic, however, the major result of game theoretical semantics states that the verifier has a winning strategy **if and only if** the given formula is true in the model.

Classical Games

Classical semantic games are

- Two-player,
- Determined,
- Sequential,
- Zero-sum,
- Complete: winning strategies necessarily and sufficiently guarantee the truth value.

Question How do these attributes of semantical games depend on the underlying logical structure? How can we give game semantics for *deviant* logics?

Logic of Paradox

Logic of Paradox and GTS

Consider Priest's Logic of Paradox (LP) (Priest, 1979).

LP introduces an additional truth value P, called *paradoxical*, that stands for both true and false.

	$\neg$
T	F
P	P
F	T

$\wedge$	T	P	F
T	T	P	F
P	P	P	F
F	F	F	F

$\vee$	T	P	F
T	T	T	T
P	T	P	P
F	T	P	F

Game Models

We define the verification game as a tuple $\Gamma = (\pi, \rho, \delta, \sigma)$ where

- π is the set of players,

- ρ is the set of well-defined game rules,

- δ is the set of designated truth values: the truth values preserved under validities: they determine the theorems of the logic.

- σ is the set of positions: subformula and player pairs.

It is possible to extend it to concurrent games as well.

Game Rules for LP

The introduction of the additional truth value P requires an additional player in the game, let us call him *Astrolabe* (after Abelard and Heloise's son).

Since we have three truth values in LP, we need three players forcing the game to their win. If the game ends up in their truth set, then that player wins.

Then, how to associate moves with the connectives?

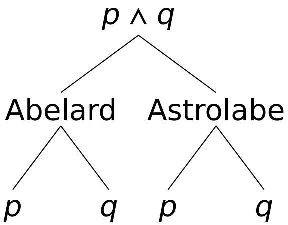
Game Rules for LP

Denote this system with GTS^{LP}.

p	whoever has p in their extension, wins
$\neg F$	Abelard and Heloise switch roles
$F \wedge G$	Abelard and Astrolabe choose between F and G simultaneously
$F \vee G$	Eloise and Astrolabe choose between F and G simultaneously

An Example

Consider the conjunction. Take the formula $p \wedge q$ where p, q are P, F respectively. Then, $p \wedge q$ is F.



Abelard makes a move and chooses q which is false. This gives him a win. Interesting enough, Astrolabe chooses p giving him a win.

In this case both seem to have a winning strategy. Moreover, the win for Abelard does not entail a loss for Astrolabe.

Correctness

Theorem

In $\mathrm{GTS}^{\mathrm{LP}}$ verification game for φ,

- Eloise has a winning strategy if φ is true,
- Abelard has a winning strategy if φ is false,
- Astrolabe has a winning strategy if φ is paradoxical.

Correctness

Theorem

In a GTS$^{\mathrm{LP}}$ game for a formula φ in a LP model M,

- If Eloise has a winning strategy, but Astrolabe does not, then φ is true (and only true) in M,
- If Abelard has a winning strategy, but Astrolabe does not, then φ is false (and only false) in M,
- If Astrolabe has a winning strategy, then φ is paradoxical in in M.

First-Degree Entailment

First-Degree Entailment

Semantic valuations are *functions* from formulas to truth values.

If we replace the valuation function with a valuation *relation*, we obtain *First-degree entailment* (FDE) which is due to Dunn (Dunn, 1976).

We use $\varphi \mathbf{r} 1$ to denote the truth value of φ (which is 1 in this case).

Since, $\mathbf{r}$ is a relation, we allow $\varphi \mathbf{r} \emptyset$ or $\varphi \mathbf{r} \{0, 1\}$.

Thus, FDE is a paraconsistent (inconsistency-tolerant) and paracomplete (incompleteness-tolerant) logic.

First-Degree Entailment

For formulas φ, ψ, we define $\mathbf{r}$ as follows.

$$
\begin{array}{lll}
\neg\varphi\mathbf{r}1 & \textit{iff} & \varphi\mathbf{r}0 \\
\neg\varphi\mathbf{r}0 & \textit{iff} & \varphi\mathbf{r}1 \\
(\varphi \wedge \psi)\mathbf{r}1 & \textit{iff} & \varphi\mathbf{r}1 \textit{ and } \psi\mathbf{r}1 \\
(\varphi \wedge \psi)\mathbf{r}0 & \textit{iff} & \varphi\mathbf{r}0 \textit{ or } \psi\mathbf{r}0 \\
(\varphi \vee \psi)\mathbf{r}1 & \textit{iff} & \varphi\mathbf{r}1 \textit{ or } \psi\mathbf{r}1 \\
(\varphi \vee \psi)\mathbf{r}0 & \textit{iff} & \varphi\mathbf{r}0 \textit{ and } \psi\mathbf{r}0
\end{array}
$$

Game Semantics for FDE

The truth values $\{0\}, \{1\}$ and $\{0, 1\}$ work exactly as the truth values F, T, P respectively in LP. In fact, LP can be obtained from FDE by introducing a restriction that no formula gets the truth value $\emptyset$.

Recall that for GTS^{LP}, we allowed parallel plays for selected players depending on the syntax of the formula: we associated conjunction with Abelard and Astrolabe, disjunction with Heloise and Astrolabe.

Game Semantics for FDE

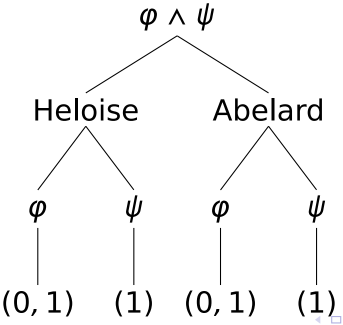
For FDE, the idea is to allow each player play at each node.

Therefore, it is possible that both players (or none) may have a winning strategy.

An Example

Consider two formulas with the following relational semantics: $\varphi\mathbf{r}0$, $\varphi\mathbf{r}1$ and $\psi\mathbf{r}1$. In this case, we have $(\varphi \wedge \psi)\mathbf{r}1$ and $(\varphi \wedge \psi)\mathbf{r}0$.

We expect both Abelard and Heloise have winning strategies, and allow each player make a move at each node.



Game Rules for FDE

p	whoever has p in their extension, wins
$\neg F$	players switch roles
$F \wedge G$	Abelard and Heloise choose between F and G simultaneously
$F \vee G$	Abelard and Heloise choose between F and G simultaneously

Correctness

Theorem

In a GTS^{FDE} verification game for a formula φ, we have the following:

- Heloise has a winning strategy if $\varphi \mathbf{r} 1$
- Abelard has a winning strategy if $\varphi \mathbf{r} 0$
- No player has a winning strategy if $\varphi \mathbf{r} \emptyset$

Relevant Logic

Routleys' Relevant Logic

An interesting way to extend the relational semantics is to add possible worlds to the model for negation. The idea is due to Routley and Routley (Routley & Routley, 1972). We call this system RR.

A *Routley model* is a structure $(W, \#, V)$ where W is a set of possible worlds, $\#$ is a map from W to itself, and V is a valuation defined in the standard way.

The semantics for RR is as follows.

$$
\begin{aligned}
V(w, \varphi \wedge \psi) = 1 &\quad \text{iff} \quad V(w, \varphi) = 1 \text{ and } V(w, \psi) = 1 \\
V(w, \varphi \vee \psi) = 1 &\quad \text{iff} \quad V(w, \varphi) = 1 \text{ or } V(w, \psi) = 1 \\
V(w, \neg\varphi) = 1 &\quad \text{iff} \quad V(\#w, \varphi) = 1
\end{aligned}
$$

Game Rules for RR

The game semantics for RR is given as follows.

(w, p)	whoever has p in their extension, wins
$(w, \neg F)$	switch roles, continue with $(\#w, F)$
$(w, F \wedge G)$	Abelard chooses between (w, F) and (w, G)
$(w, F \vee G)$	Heloise chooses between (w, F) and (w, G)

Correctness

Theorem

For the evaluation games for a formula φ and a world w for Routleys' systems, we have the following:

- Heloise has a winning strategy if $\varphi \mathbf{r} 1$.
- Abelard has a winning strategy if $\varphi \mathbf{r} 0$.

Connexive Logic

McCall's Connexive Logic

Connexive logic is a "comparatively little-known and to some extent neglected branch of non-classical logic" (Wansing, 2015). Even if it is under-studied, its philosophical roots can be traced back to Aristotle and Boethius.

Connexive logic is defined as a system which satisfies the following two schemes of conditionals:

- **Aristotle's Theses**: $\neg(\neg\varphi \to \varphi)$
- **Boethius' Theses**: $(\varphi \to \neg\psi) \to \neg(\varphi \to \psi)$

In this work, we discuss one of the earliest examples of connexive logics CC, which is due to McCall (McCall, 1966).

McCall's Connexive Logic

CC is axiomatized by adding the scheme $(\varphi \to \varphi) \to \neg(\varphi \to \neg\varphi)$ to the propositional logic. The rules of inference for CC is modus ponens and adjunction, which is given as $\vdash \varphi, \vdash \psi \therefore \vdash \varphi \wedge \psi$.

The semantics for CC is given with 4 truth values: T, t, f and F which can be viewed as "logical necessity", "contingent truth", "contingent falsehood", and "logical impossibility" respectively (Routley & Montgomery, 1968).

In CC, the designated truth values are T and t.

McCall's Connexive Logic

	$\neg$
T	F
t	f
f	t
F	T

$\wedge$	T	t	f	F
T	T	t	f	F
t	t	T	F	f
f	f	F	f	F
F	F	f	F	f

$\vee$	T	t	f	F
T	t	T	t	T
t	T	t	T	t
f	t	T	F	f
F	T	t	f	F

First, we introduce 4 players for 4 truth values: T is forced by Heloise, F by Abelard, t by Aristotle and f by Boethius.

Game Rules for CC

As the *trues* and *falses* are closed under the binary operations respectively, we suggest the following coalitions.

Truth-maker Coalition:

Heloise (T) and Aristotle (t)

False-maker Coalition:

Abelard (F) and Boethius (f)

Game Rules for CC

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Truth-maker Coalition:

Heloise (T) and Aristotle (t)

False-maker Coalition:

Abelard (F) and Boethius (f)

Game Rules for CC

p	whoever has p in their extension, wins
$\neg F$	switch the roles: Heloise assumes Abelard's role, Aristotle assumes Boethius' role, Boethius assumes Aristotle's role, Abelard assumes Heloise's role, and the game continues with F
$F \wedge G$	false-makers coalition chooses between F and G
$F \vee G$	truth-makers coalition chooses between F and G

Correctness

Theorem

For the evaluation games for a formula φ in McCall's Connexive logic, we have the following:

- truth-makers have a winning strategy if and only if φ has the truth value t or T in M,
- false-makers have a winning strategy if and only if φ has the truth value f or F in M.

Belnap's 4-Valued Logic

Belnap's 4-Valued Logic

Belnap's 4-Valued system, call it B4, introduces two non-classical truth values. Traditionally, P stands for both truth values and N stands for neither of the truth values.

	$\neg$
T	F
P	P
N	N
F	T

$\wedge$	T	P	N	F
T	T	P	N	F
P	P	P	F	F
N	N	F	N	F
F	F	F	F	F

$\vee$	T	P	N	F
T	T	T	T	T
P	T	P	T	P
N	T	T	N	N
F	T	P	N	F

Notice that P and N are the fixed-points under negation.

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$\vee$	T	P	N	F
T	T	T	T	T
P	T	P	T	P
N	T	T	N	N
F	T	P	N	F

Notice that P and N are the fixed-points under negation.

Game Rules for B4

From a game-semantics perspective, the problems with B4 include

- Two fixed-points for negation
- Non-monotonicity: two truth values may produce a third truth value under binary connectives

In particular, we have $P \wedge N = F$ and $P \vee N = T$.

Game Rules for B4

Let us have 4 players for 4 truth values:

The truth value T is forced by Heloise, F by Abelard, P by Astrolabe and N by Bernard[1].

Two negation-fixed-points suggest that Astrolabe and Bernard both will be the concurrent players.

[1] After Abelard's rival Bernard of Clairvaux.

Game Rules for B4

p	whoever has p in their extension, wins
$\neg F$	Heloise assumes Abelard's role, Abelard assumes Heloise's role, Astrolabe and Bernard keep their previous roles, and the game continues with F,
$F \wedge G$	if Bernard has a winning strategy for F and Astrolabe has a winning strategy for G, then Abelard wins,
$F \wedge G$	otherwise Abelard, Astrolabe and Bernard choose simultaneously between F and G,
$F \vee G$	if Bernard has a winning strategy for F and Astrolabe has a winning strategy for G, then Heloise wins,
$F \vee G$	otherwise Heloise, Astrolabe and Bernard choose simultaneously between F and G.

Correctness

Theorem

For the evaluation games for a formula φ in Belnap's 4-valued logic, we have the following:

- Heloise the verifier has a winning strategy if φ evaluates to T,
- Abelard the falsifier has a winning strategy if φ evaluates to F,
- Astrolabe the paradoxifier has a winning strategy if φ evaluates to P,
- Bernard the nullifier has a winning strategy if φ evaluates to N.

Translation to Classical S5

Translation to S5

The translation of LP to S5 is built on the following observation:

> *“In an S5-model there are three mutually exclusive and jointly exhaustive possibilities for each atomic formula p: either p is true in all possible worlds, or p is true in some possible worlds and false in others, or p is false in all possible worlds" (Kooi & Tamminga, 2013).*

Translation

Given the propositional language $\mathcal{L}$, we extend it with the modal symbols $\Box$ and $\Diamond$ and close it under the standard rules to obtain the modal language $\mathcal{L}_M$. Then, the translations $\mathrm{Tr}_{LP} : \mathcal{L} \mapsto \mathcal{L}_M$ and $\mathrm{Tr}_{K3} : \mathcal{L} \mapsto \mathcal{L}_M$ for LP and K3 respectively are given inductively as follows where p is a propositional variable (Kooi & Tamminga, 2013).

$$
\begin{array}{ll}
\mathrm{Tr}_{LP}(p) = \Diamond p & \mathrm{Tr}_{LP}(\varphi \wedge \psi) = \mathrm{Tr}_{LP}(\varphi) \wedge \mathrm{Tr}_{LP}(\psi) \\
\mathrm{Tr}_{K3}(p) = \Box p & \mathrm{Tr}_{K3}(\varphi \wedge \psi) = \mathrm{Tr}_{K3}(\varphi) \wedge \mathrm{Tr}_{K3}(\psi) \\
\mathrm{Tr}_{LP}(\neg\varphi) = \neg \mathrm{Tr}_{K3}(\varphi) & \mathrm{Tr}_{LP}(\varphi \vee \psi) = \mathrm{Tr}_{LP}(\varphi) \vee \mathrm{Tr}_{LP}(\psi) \\
\mathrm{Tr}_{K3}(\neg\varphi) = \neg \mathrm{Tr}_{LP}(\varphi) & \mathrm{Tr}_{K3}(\varphi \vee \psi) = \mathrm{Tr}_{K3}(\varphi) \vee \mathrm{Tr}_{K3}(\psi)
\end{array}
$$

Translation: Toy Examples

Contradictions are possible in LP, but not in K3.

$$
\begin{aligned}
\mathrm{Tr}_{LP}(p \wedge \neg p) &= \mathrm{Tr}_{LP}(p) \wedge \mathrm{Tr}_{LP}(\neg p) \\
&= \Diamond p \wedge \neg \mathrm{Tr}_{K3}(p) \\
&= \Diamond p \wedge \neg \Box p \\
&= \Diamond p \wedge \Diamond \neg p.
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Tr}_{K3}(p \wedge \neg p) &= \mathrm{Tr}_{K3}(p) \wedge \mathrm{Tr}_{K3}(\neg p) \\
&= \Box p \wedge \neg \mathrm{Tr}_{LP}(p) \\
&= \Box p \wedge \neg \Diamond p \\
&= \Box p \wedge \Box \neg p.
\end{aligned}
$$

Results

Theorem

Let $\Gamma_{LP}(M, \varphi)$ be given. Then,

- if Heloise has a winning strategy in $\Gamma_{LP}(M, \varphi)$, then she has a winning strategy in $\Gamma_{S5}(M, \mathrm{Tr}_{LP}(\varphi))$,
- if Abelard has a winning strategy in $\Gamma_{LP}(M, \varphi)$, then he has a winning strategy in $\Gamma_{S5}(M, \mathrm{Tr}_{LP}(\varphi))$,
- if Astrolabe has a winning strategy in $\Gamma_{LP}(M, \varphi)$, then both Abelard and Heloise have a winning strategy in $\Gamma_{S5}(M, \mathrm{Tr}_{LP}(\varphi))$.

Results

Theorem

Let M be an S5 model, $\varphi \in \mathcal{L}$ with an associated verification game $\Gamma_{S5}(M, \varphi)$. Then, there exists an LP model M' and a game $\Gamma_{LP}(M', \varphi)$ where,

- if Heloise (resp. Abelard) has a winning strategy for $\Gamma_{S5}(M, \varphi)$ at each point in M, then Heloise (resp. Abelard) has a winning strategy in $\Gamma_{LP}(M', \varphi)$,
- if Heloise or Abelard has a winning strategy for $\Gamma_{S5}(M, \varphi)$ at some points but not all in M, then Astrolabe has a winning strategy in $\Gamma_{LP}(M', \varphi)$,

Conclusion

What Have We Observed?

- Failure of the biconditional correctness
- Multiplayer semantic games in a nontrivial way
- Non-sequential / paralel / concurrent plays
- Variable sum games
- Coalitions

If winning strategies are proofs, game semantics for paraconsistent logics present a constructive way to give proofs for inconsistencies.

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- Failure of the biconditional correctness
- Multiplayer semantic games in a nontrivial way
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If winning strategies are proofs, game semantics for paraconsistent logics present a constructive way to give proofs for inconsistencies.

Difficult Logics

- Da Costa systems, Logics of Formal Inconsistency
- Preservationism
- First-order paraconsistent logics
- Infinitary, fixed-point non-classical logics

Generalized Game Semantics

Based on our observations here, by focusing on the truth values which are

- negation-fixed-points (parallel plays) and
- closed under binary operations (coalitions),

it could be possible to give game semantics for any finite truth table.

Conclusion

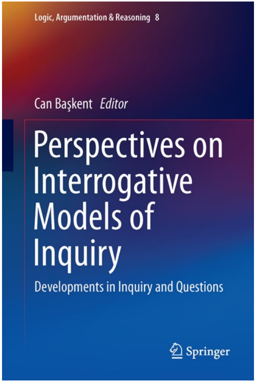
I consider this work as a very first step towards paraconsistent / non-classical game theory.

Our long term goal is to give a broader theory of (non-classical, non-utilitarian) rationality via games and logic.

Thank you!

Talk slides and the paper are available at:

`www.CanBaskent.net/Logic`



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