What is Game Theoretical Negation?

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Outlook of the Talk

- Classical (but Extended) Game Theoretical Semantics for Negation
- Paraconsistent Game Theoretical Semantics for Negation

What is Hintikka's Game Theoretical Semantics? I

The *semantic verification game* is played by two players, traditionally called Abelard (after $\forall$) and Eloise (after $\exists$), and the rules are specified syntactically.

During the game, the given formula is broken into subformulas by the players step by step, and the game terminates when it reaches the propositional atoms.

If we end up with a propositional atom which is true in the model in question, then Eloise wins the game. Otherwise, Abelard wins. We associate conjunction with Abelard, disjunction with Heloise.

What is Hintikka's Game Theoretical Semantics? II

The major result of this approach states that Eloise has a winning strategy if and only if the given formula is true in the model.

When conjunction and disjunction are considered, game theoretical semantics is very appealing. In negated formulas, game theoretical semantics says that the players switch their roles. Abelard takes up Eloise's verifier role, and Eloise becomes the falsifier.

I think this is counter-intuitive - game theoretically.

An Example I

An Example

Two men want to marry a princess. The king says they have to race on a horceback. The slowest one wins, and can marry the princess. How can one win this game and marry the princess?

The answer: men need to swap their horses. Since the fastest lose, and players race with each other's horses, what they need to do is to become the fastest in the dual game. Fastest one in the switched horse, considered as the negation of the slowest in the dual game, wins the game.

An Example II

In this example, GTS for negation becomes evident. If the slowest one wins the game, then the fastest one wins the dual game.

There is certainly some sense of rationality here. Namely, the players consider it easier to switch horses and race in the dual game.

Namely, can we play chess in this way? Can we play football in this fashion? Is it always rational to play in the dual game with switched roles?

To switch to the easier dual game with switched roles is a **meta-game theoretical** move. This is not a strategy within the given game, it is a *strategy on the games and over the games*.

What is **Wrong** with Game Theoretical Semantics?

First, insistence on "negation normal form": For Hintikka, insisting on negation normal form is not restrictive since each formula can be effectively transformed into a formula in negation normal form (Hintikka, 1996). However, he fails to mention that in this case the game becomes a different one.

Second, it fails to address formula equivalence: compare $p \wedge (q \vee r)$ vs $(p \wedge q) \vee (p \wedge r)$ and their game trees.

What is game theoretical equivalence? (van Benthem *et al.*, 2011). Is it a strategy transformation? What about DeMorgan's Laws?

What is **Wrong** with Game Theoretical Semantics?

Third, it is not entirely clear how the semantics of negation agrees with the rationality of the players. Sure, in some games, it makes sense. But, is the idea strong enough to generalize?

In other words, what is the element of rationality in GTS when it comes to negation?

Extended Game Semantics for the Classical Case

We need to explicate the semantics of negation inductively for each case.

The ideas we will use resemble the tableau method.

Extended Game Semantics for the Classical Case

$\neg(F \wedge G)$	Eloise chooses between $\neg F$ and $\neg G$
$\neg(F \vee G)$	Abelard chooses between $\neg F$ and $\neg G$
$\neg(F \rightarrow G)$	Abelard chooses between F and $\neg G$
$\neg\neg F$	game continues with F
$\neg p$	Heloise wins if p is not true for her. Otherwise, Abelard wins.

It hints out how we can alter the GTS for the logics where DeMorgan's laws do not hold as well.

Moreover, it is also possible to extend it to multi-agent / multi-player cases (Olde Loohuis & Venema, 2010).

Correctness of the Extended Semantics

We denote the extended (classical) semantics we suggested as GTS*.

Theorem

For any formula φ and model M, we have

$M \models_{GTS} \varphi$ if and only if $M \models_{GTS^*} \varphi$ if and only if $M \models \varphi$.

It is also not difficult to see that in GTS*, Eloise has a winning strategy iff the formula in question is true.

Hintikka and Sandu on Non-classicity

Even if Hintikka and Sandu conservatively remarked that “it is difficult to see how else negation could be treated game-theoretically”, they later on discussed non-classicity in GTS (Hintikka & Sandu, 1997; Peitarinen & Sandu, 2000):

- When informational independence is allowed, the law of excluded middle fails.
- Constructivistic ideas are most naturally implemented by restricting the initial verifiers’ strategies in a semantical games to recursive ones.
- Games of inquiry involve an epistemic element.
- Nonclassical game rules can be given for propositional connectives, especially for conditional and negation.

More on Non-classicity

I believe, in the above list, Hintikka and Sandu had intuitionism and IF-logic, more specifically the law of excluded middle, in mind when they discussed non-classicity.

However, another alternative to classical logic is also possible. Dual-intuitionistic logic, or paraconsistent logics in general, poses influential approaches to classical problems of logic.

Non-classical Games

It is not difficult to introduce additional outcomes for GTS. We introduce the following five non-classical / non-zero sum possibilities:

1. Abelard and Eloise both win.
2. Abelard and Eloise both lose.
3. Eloise wins, Abelard does not lose.
4. Abelard wins, Eloise does not lose.
5. There is a tie.

What are the Non-classical Games?

Some propositions can belong to both player: namely, both the proposition and its negation can be true.

Some propositions can belong to the neither: namely, neither the proposition nor its negation can be true.

Some propositions may not belong to one player without the negation belonging to the opponent: namely, the proposition can be true, but its negation may not be false.

In short, the game does not have to be a zero-sum game. One's win may not imply the other's loss.

Logic of Paradox and GTS

The formalism we adopt here is Graham Priest's Logic of Paradox (Priest, 1979). The logic of paradox (LP, for short) introduces an additional truth value P, called *paradoxical*, that stands for both true and false.

	$\neg$
T	F
P	P
F	T

$\wedge$	T	P	F
T	T	P	F
P	P	P	F
F	F	F	F

$\vee$	T	P	F
T	T	T	T
P	T	P	P
F	T	P	F

go to example

go to strategies

Game Rules for LP: weaker version

The introduction of the additional truth value *P* requires an additional player in the game, let us call him *Astrolabe* (after Abelard and Heloise's son).

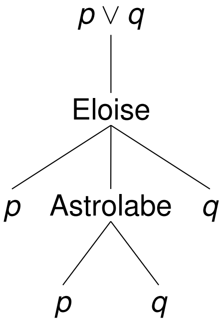
Since we have three truth values in LP, we need three players that try to force the game to their win. If the game ends up in their truth set, then that player wins.

Examples

Consider the formula $p \vee q$ where p, q are propositional variables with truth values P, F respectively. Therefore, the truth value of $p \vee q$ is also P.

In this case, Eloise cannot force a win because neither p nor q has the truth value T.

On the other hand, Astrolabe has a winning strategy as the truth value of p is P when it is his turn to play. Thus, he chooses p yielding the truth value P for the given formula $p \vee q$.



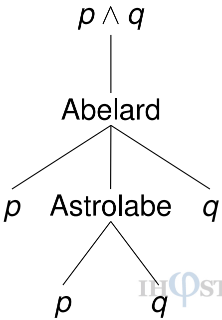
Examples

Let us now consider the conjunction. Take the formula $p \wedge q$ where p, q are propositional variables with truth values P, F respectively.

In this case, Abelard first makes a move, and as the falsifier, he can choose q which is false. This gives him a win.

Therefore, Astrolabe does not get a chance to make a move. However, interesting enough, if he had a chance to play, he would go for p which has a truth value of P, and this would him Astrolabe his win.

Remember, first the parents make a move, then Astrolabe.



Remarks for weaker version

1. Disjunction belongs to Eloise (and Astrolabe) and conjunction belongs to Abelard (and Astrolabe).
2. First parents make a move, if they have a winning strategy in the subgame they choose at the connective, the game proceeds.
3. Otherwise, if the parents do not have a winning strategy when it is their turn, then Astrolabe plays.

The problem here is that we include the existence of winning strategies in the game rules.

Game Theoretical Semantics for LP: weaker version

Denote it with GTS^p.

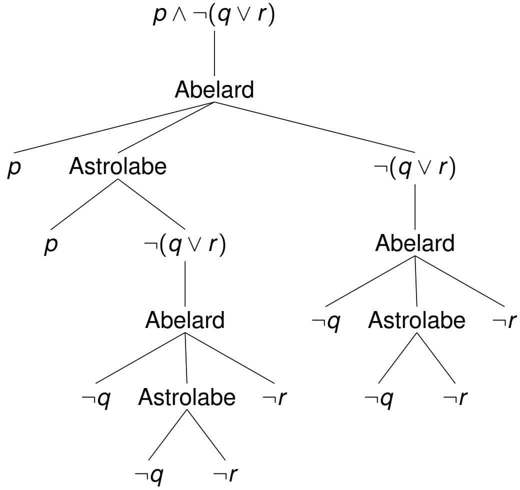
p (or $\neg p$)	whoever has p (or $\neg p$) in their extension, wins
$F \wedge G$	First Abelard, then Astrolabe chooses between F and G
$F \vee G$	First Eloise, then Astrolabe chooses between F and G
$\neg(F \wedge G)$	First Eloise, then Astrolabe chooses between $\neg F$ and $\neg G$
$\neg(F \vee G)$	First Abelard, then Astrolabe chooses between $\neg F$ and $\neg G$

Another Example

Let us now consider a bit complicated formula $p \wedge \neg(q \vee r)$ where the truth values of p, q and r are T, P and F respectively. According to the LP truth table, the given formula has the truth value of P. Thus, we expect Astrolabe to have a winning strategy.

Based on the given truth values for the propositional variables, what we expect is to see that Astrolabe can force and $\neg r$ (or r) output in the game. The game tree below explicates how Astrolabe wins the game based on the game rules.

Another Example



p is T, q is P and r is F

LP truth table

Observations

Similar to Priest's early theorem on LP, we have the following.

Theorem

For any formula φ and model M, we have $M \models_{GTS} \varphi$ if and only if $M \models_{GTS^p} \varphi$.

Correctness of the weaker version

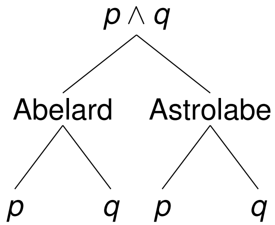
Theorem

In GTS^{p} verification game for φ,

- Eloise has a winning strategy iff φ is true
- Abelard has a winning strategy iff φ is false
- Astrolabe has a winning strategy iff φ is paradoxical

Game Theoretical Semantics for LP: stronger version

Consider the conjunction again. Take the formula $p \wedge q$ where p, q are P, F respectively.



Abelard makes a move, and as the falsifier, he chooses q which is false. This gives him a win. Interesting enough, Astrolabe also makes a move and, chooses p giving him a win. In this case both have a winning strategy. Moreover, the win for Abelard does not automatically entail that it is a loss for Astrolabe.

Game Rules for the stronger version

Denote it with GTS^{pp}.

p	whoever has p in their extension, wins
$F \wedge G$	Abelard and Astrolabe chooses between F and G simultaneously
$F \vee G$	Eloise and Astrolabe chooses between F and G simultaneously
$\neg(F \wedge G)$	Eloise and Astrolabe chooses between $\neg F$ and $\neg G$ simultaneously
$\neg(F \vee G)$	Abelard and Astrolabe chooses between $\neg F$ and $\neg G$ simultaneously

Correctness of the stronger version

Theorem

In $\mathrm{GTS}^{\mathrm{pp}}$ verification game for φ,

- Eloise has a winning strategy if φ is true
- Abelard has a winning strategy if φ is false
- Astrolabe has a winning strategy if φ is paradoxical

Correctness of the stronger version

Theorem

In a GTS^{pp} game for a formula φ in a LP model M,

- If Eloise has a winning strategy, but Astrolabe does not, then φ is true (and only true) in M
- If Abelard has a winning strategy, but Astrolabe does not, then φ is false (and only false) in M
- If Astrolabe has a winning strategy, then φ is paradoxical in in M

Dominating Strategies

Note that in the weaker version, we simply eliminate the dominated strategies (by embedding the players' rationality in the semantics), and iterate the procedure.

Thus, it can be seen as an iterated elimination of dominated strategies - which is not visible in the classical case, but clearer in the paraconsistent case - due to the truth table of LP.

LP truth table

Conclusion I

In this work, we do not aim at giving a full picture of game theoretical semantics of negation in all non-classical logics. The literature on non-classical logics (which include intuitionistic, paraconsistent and relative logics amongst many others) is vast, and all of those logics are not transformable to each other making it almost impossible to give a unifying theme for GTS.

Yet, the very same intuition can easily be applied to other non-classical logics, and their winning conditions can be examined.

Conclusion II

In a recent paper, Priest alludes to similar concepts (Priest, 2013). We can add some further points by noting that our approach here can be a case for the plurality of logic.

Similarly, Dialogical Logic can initially be taught of providing a good approach to negation. However, a closer inspection reveals that in dialogical logical cases, the role switching idea is maintained and even taken to a higher level creating more schizophrenic players (Rahman & Tulenheimo, 2009).

Conclusion III

Behavioral economics and the charming examples that it provides (for example (Ariely, 2008; Ariely, 2010; Harford, 2009)) constitutes an interesting playground for the ideas we have developed here.

Thanks for your attention!

Talk slides and the papers are available at

`www.CanBaskent.net/Logic`

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