Game Semantics and Paraconsistency

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Why Paraconsitency?

- Philosophical Motivations: Motion, change, dialetheia, dialectics, identity, deontology....
- Semantic Motivations: Paradoxes, Pragmatics...
- Logical, Mathematical Motivations: Set theory, arithmetic, databases...
- Game Theoretical Motivations: Irrationality, bounded rationality, non-utilitarianism

Outlook of the Talk

- Logic of Paradox and Game Semantics
- First-Degree Entailment and Game Semantics
- Relevant Logic and Game Semantics
- Translation to S5 Modal Logic

What is Hintikka's Game Theoretical Semantics?

During the game, the given formula is broken into subformulas by the players **step by step**, and the game **terminates** when it reaches the propositional atoms.

If we end up with a propositional atom which is true in the model in question, then Eloise wins the game. **Otherwise**, Abelard wins. We associate **conjunction with Abelard**, **disjunction with Heloise**.

The major result of this approach states that Eloise has a winning strategy **if and only if** the given formula is true in the model.

Non-classical Games

We consider the following five non-classical / non-zero sum possibilities:

1. Abelard and Eloise both win.
2. Abelard and Eloise both lose.
3. Eloise wins, Abelard does not lose.
4. Abelard wins, Eloise does not lose.
5. There is a tie.

Non-classical Games

Some propositions can belong to both player: that is, both the proposition and its negation can be true.

Some propositions can belong to the neither: that is, neither the proposition nor its negation can be true.

Some propositions may not belong to one player without the negation belonging to the opponent: that is, the proposition can be true, but its negation may not be false.

Logic of Paradox and GTS

Consider Priest's Logic of Paradox (LP) (Priest, 1979). LP introduces an additional truth value P, called *paradoxical*, that stands for both true and false.

	$\neg$
T	F
P	P
F	T

$\wedge$	T	P	F
T	T	P	F
P	P	P	F
F	F	F	F

$\vee$	T	P	F
T	T	T	T
P	T	P	P
F	T	P	F

Game Rules for LP

The introduction of the additional truth value *P* requires an additional player in the game, let us call him *Astrolabe* (after Abelard and Heloise's son).

Since we have three truth values in LP, we need three players that try to force the game to their win. If the game ends up in their truth set, then that player wins.

Then, how to associate moves with the connectives?

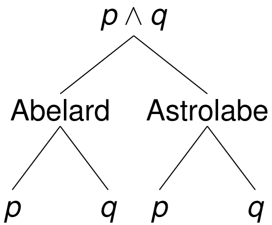
Game Rules for the stronger version

Denote it with GTS^{LP}.

p	whoever has p in their extension, wins
$\neg F$	Abelard and Heloise switch roles
$F \wedge G$	Abelard and Astrolabe choose between F and G simultaneously
$F \vee G$	Eloise and Astrolabe choose between F and G simultaneously

Game Theoretical Semantics for LP

Consider the conjunction. Take the formula $p \wedge q$ where p, q are P, F respectively.



Abelard makes a move and chooses q which is false. This gives him a win. Interesting enough, Astrolabe chooses p giving him a win.

In this case both seem to have a winning strategy. Moreover, the win for Abelard does not entail a loss for Astrolabe.

Correctness

Theorem

In GTS^{LP} verification game for φ,

- Eloise has a winning strategy if φ is true
- Abelard has a winning strategy if φ is false
- Astrolabe has a winning strategy if φ is paradoxical

Correctness

Theorem

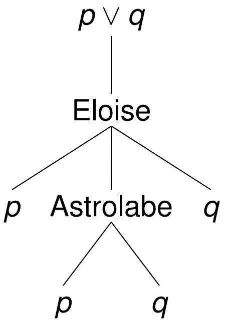
In a GTS^{LP} game for a formula φ in a LP model M,

- If Eloise has a winning strategy, but Astrolabe does not, then φ is true (and only true) in M
- If Abelard has a winning strategy, but Astrolabe does not, then φ is false (and only false) in M
- If Astrolabe has a winning strategy, then φ is paradoxical in in M

Weakening

For a biconditional correctness result, we need to introduce *priorities* in play to the game. For example:

Consider $p \vee q$ where p, q has truth values P, F respectively. So, $p \vee q$ has truth value P.
In this case, Eloise cannot force a win because neither p nor q has the truth value T.
On the other hand, Astrolabe has a winning strategy as the truth value of p is P when it is his turn to play. Thus, he chooses p yielding the truth value P for the given formula $p \vee q$.



First-Degree Entailment

Semantic evaluations are *functions* from formulas to truth values.

If we replace the valuation function with a valuation *relation*, we obtain *First-degree entailment* (FDE) which is due to Dunn (Dunn, 1976).

We use $\varphi \mathbf{r} 1$ to denote the truth value of φ (which is 1 in this case).

Since, **r** is a relation, we therefore allow $\mathbf{r}(\varphi) = \emptyset$ or $\mathbf{r}(\varphi') = \{0, 1\}$ for some formula φ, φ'.

Thus, FDE is a paraconsistent (inconsistency-tolerant) and paracomplete (incompleteness-tolerant) logic.

First-Degree Entailment

For formulas φ, ψ, we define **r** as follows.

$$
\begin{array}{lll}
\neg\varphi\mathbf{r}1 & \textit{iff} & \varphi\mathbf{r}0 \\
\neg\varphi\mathbf{r}0 & \textit{iff} & \varphi\mathbf{r}1 \\
(\varphi \wedge \psi)\mathbf{r}1 & \textit{iff} & \varphi\mathbf{r}1 \textit{ and } \psi\mathbf{r}1 \\
(\varphi \wedge \psi)\mathbf{r}0 & \textit{iff} & \varphi\mathbf{r}0 \textit{ or } \psi\mathbf{r}0 \\
(\varphi \vee \psi)\mathbf{r}1 & \textit{iff} & \varphi\mathbf{r}1 \textit{ or } \psi\mathbf{r}1 \\
(\varphi \vee \psi)\mathbf{r}0 & \textit{iff} & \varphi\mathbf{r}0 \textit{ and } \psi\mathbf{r}0
\end{array}
$$

First-Degree Entailment and GTS

The truth values $\{0\}, \{1\}$ and $\{0, 1\}$ work exactly as the truth values F, T, P respectively in LP. In fact, LP can be obtained from FDE by introducing a restriction that no formula gets the truth value $\emptyset$.

Recall that for GTS^{LP}, we allowed parallel plays for selected players depending on the syntax of the formula: we associated conjunction with Abelard and Astrolabe, disjunction with Heloise and Astrolabe.

First-Degree Entailment and GTS

For FDE, the idea is to allow each player play at each node.

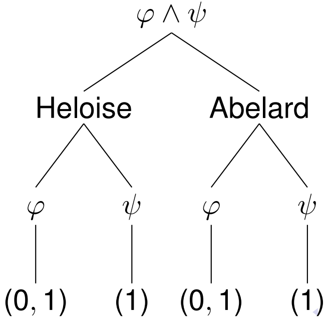
Therefore, it is possible that both players (or none) may have a winning strategy.

Also, notice that allowing each player play at each node does not necessarily mean that they will always be able to make a move. It simply means, it is allowed for them to move simultaneously.

First-Degree Entailment and GTS: An Example

Consider two formulas with the following relational semantics: $\varphi\mathbf{r}0$, $\varphi\mathbf{r}1$ and $\psi\mathbf{r}1$. In this case, we have $(\varphi \wedge \psi)\mathbf{r}1$ and $(\varphi \wedge \psi)\mathbf{r}0$.

We expect both Abelard and Heloise have winning strategies, and allow each player make a move at each node.



First-Degree Entailment and GTS: Game rules

p	whoever has p in their extension, wins
$\neg F$	players switch roles
$F \wedge G$	Abelard and Heloise choose between F and G simultaneously
$F \vee G$	Abelard and Heloise choose between F and G simultaneously

First-Degree Entailment and GTS: Correctness

Theorem

In a GTS^{FDE} verification game for a formula φ, we have the following:

- Heloise has a winning strategy if $\varphi \mathbf{r} 1$
- Abelard has a winning strategy if $\varphi \mathbf{r} 0$
- No player has a winning strategy if $\varphi \mathbf{r} \emptyset$

Routleys' Relevant Logic

An interesting way to extend the relational semantics is to add possible worlds to the model. The idea is due to Routley and Routley (Routley & Routley, 1972).

A *Routley model* is a structure $(W, \#, \mathbf{r})$ where W is a set of possible worlds, $\#$ is a map from W to itself, and $\mathbf{r}$ is a valuation from $W \times \mathbf{P}$ to $\{0, 1\}$ assigning truth to propositional variables at each world.

Let us now give the semantics for this system.

$$\mathbf{r}(w, \varphi \wedge \psi) = 1 \text{ iff } \mathbf{r}(w, \varphi) = 1 \text{ and } \mathbf{r}(w, \psi) = 1$$

$$\mathbf{r}(w, \varphi \vee \psi) = 1 \text{ iff } \mathbf{r}(w, \varphi) = 1 \text{ or } \mathbf{r}(w, \psi) = 1$$

$$\mathbf{r}(w, \neg\varphi) = 1 \text{ iff } \mathbf{r}(\#w, \varphi) = 1$$

Relevant Logic and GTS: rules

(w, p)	whoever has p in their extension, wins
$(w, \neg F)$	switch the roles, continue with $(\#w, F)$
$(w, F \wedge G)$	Abelard chooses between (w, F) and (w, G)
$(w, F \vee G)$	Heloise chooses between (w, F) and (w, G)

Relevant Logic and GTS: Correctness

Theorem

For the evaluation games for a formula φ and a world *w* for Routleys' systems, we have the following:

1. Heloise has a winning strategy if $\varphi\mathbf{r}1$.
2. Abelard has a winning strategy if $\varphi\mathbf{r}0$.

Belnap's Four Valued Logic

Belnap's four valued logic introduces two additional truth values: The truth value P represents the over-valuation, and N represents the under-valuation.

	$\neg$
T	F
P	P
N	N
F	T

$\wedge$	T	P	N	F
T	T	P	N	F
P	P	P	F	F
N	N	F	N	F
F	F	F	F	F

$\vee$	T	P	N	F
T	T	T	T	T
P	T	P	T	P
N	T	T	N	N
F	T	P	N	F

Hereditary Condition

BL truth table looks rather *different*. For instance, $P \vee N$ yields T, and $P \wedge N$ yields F.

Definition

Let L be a n-valued logic where $n \geq 2$, $\{V_i\}_{i \leq n}$ the set of truth-values, and $\{C_j\}_{j \in J}$ be set of binary logical connectives for some index set J. Then, L is said to have the hereditary condition if for all $i, i' \leq n, j \in J$, $C_j(V_i, V_{i'})$ evaluates to either V_i or $V_{i'}$. In short, logical connectives cannot produce a truth value different than those of the input values. This definition can easily be extended to k-ary logical connectives.

Classical, intuitionistic logics, and LP, RR, FDE all possess the hereditary condition. BL does not have the hereditary condition.

Translation to S5

The translation of LP to S5 is built on the following observation: “In an S5-model there are three mutually exclusive and jointly exhaustive possibilities for each atomic formula p: either p is true in all possible worlds, or p is true in some possible worlds and false in others, or p is false in all possible worlds” (Kooi & Tamminga, 2013).

Translation

Given the propositional language $\mathcal{L}$, we extend it with the modal symbols $\Box$ and $\Diamond$ and close it under the standard rules to obtain the modal language $\mathcal{L}_M$. Then, the translations $\mathrm{Tr}_{LP} : \mathcal{L} \mapsto \mathcal{L}_M$ and $\mathrm{Tr}_{K3} : \mathcal{L} \mapsto \mathcal{L}_M$ for LP and K3 respectively are given inductively as follows where p is a propositional variable (Kooi & Tamminga, 2013).

$$\mathrm{Tr}_{LP}(p) = \Diamond p \qquad \mathrm{Tr}_{LP}(\varphi \wedge \psi) = \mathrm{Tr}_{LP}(\varphi) \wedge \mathrm{Tr}_{LP}(\psi)$$
$$\mathrm{Tr}_{K3}(p) = \Box p \qquad \mathrm{Tr}_{K3}(\varphi \wedge \psi) = \mathrm{Tr}_{K3}(\varphi) \wedge \mathrm{Tr}_{K3}(\psi)$$
$$\mathrm{Tr}_{LP}(\neg\varphi) = \neg\mathrm{Tr}_{K3}(\varphi) \qquad \mathrm{Tr}_{LP}(\varphi \vee \psi) = \mathrm{Tr}_{LP}(\varphi) \vee \mathrm{Tr}_{LP}(\psi)$$
$$\mathrm{Tr}_{K3}(\neg\varphi) = \neg\mathrm{Tr}_{LP}(\varphi) \qquad \mathrm{Tr}_{K3}(\varphi \vee \psi) = \mathrm{Tr}_{K3}(\varphi) \vee \mathrm{Tr}_{K3}(\psi)$$

Results

Theorem

Let $\Gamma_{LP}(M, \varphi)$ be given. Then,

- if Heloise has a winning strategy in $\Gamma_{LP}(M, \varphi)$, then she has a winning strategy in $\Gamma_{S5}(M, \text{Tr}_{LP}(\varphi))$,
- if Abelard has a winning strategy in $\Gamma_{LP}(M, \varphi)$, then he has a winning strategy in $\Gamma_{S5}(M, \text{Tr}_{LP}(\varphi))$,
- if Astrolabe has a winning strategy in $\Gamma_{LP}(M, \varphi)$, then both Abelard and Heloise have a winning strategy in $\Gamma_{S5}(M, \text{Tr}_{LP}(\varphi))$.

Results

Theorem

Let M be an S5 model, $\varphi \in \mathcal{L}$ with an associated verification game $\Gamma_{S5}(M, \varphi)$. Then, there exists an LP model M' and a game $\Gamma_{LP}(M', \varphi)$ where,

- if Heloise (resp. Abelard) has a winning strategy for $\Gamma_{S5}(M, \varphi)$ at each point in M, then Heloise (resp. Abelard) has a winning strategy in $\Gamma_{LP}(M', \varphi)$,
- if Heloise or Abelard has a winning strategy for $\Gamma_{S5}(M, \varphi)$ at some points but not all in M, then Astrolabe has a winning strategy in $\Gamma_{LP}(M', \varphi)$,

Conclusion

I consider this work as a first-step towards paraconsistent / non-classical game theory.

Our long term goal is to give a broader theory of (non-classical, non-utilitarian) rationality via games and logic.

Thank you for your attention!

Talk slides and the paper are available at:

`www.CanBaskent.net/Logic`

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