Game Semantics for Paraconsistent Logics

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Outlook of the Talk

- Logic of Paradox
- First-Degree Entailment
- Relevant Logic
- Translation to S5 Modal Logic

Why Paraconsitency?

A logic is **paraconsistent** if contradictions do not entail everything.

In a paraconsistent logic, it is possible to have true contradictions.

How to understand paraconsistency from a game semantical point of view?

Why Paraconsitency?

- Philosophical Motivations: Motion, change, dialetheia, dialectics, identity, deontology....
- Semantic Motivations: Paradoxes, Pragmatics...
- Logical, Mathematical Motivations: Set theory, arithmetic, databases...
- Game Theoretical Motivations: Irrationality, bounded rationality, non-utilitarianism

Classical Game Semantics

During the game, the given formula is broken into subformulas by the players **step by step**, and the game **terminates** when it reaches the propositional atoms.

If we end up with a propositional atom which is true in the model in question, then Eloise the verifier wins the game. **Otherwise**, Abelard the falsifier wins. We associate **conjunction with Abelard**, **disjunction with Heloise**.

The major result of this approach states that Eloise has a winning strategy **if and only if** the given formula is true in the model.

Classical Games

Classical semantic games are

- Determined
- Sequential
- Zero-sum
- Winning strategies guarantee truth

Question How do these properties of semantical games depend on the underlying logical structure? How can we give game semantics for *deviant* logics?

Logic of Paradox and GTS

Consider Priest's Logic of Paradox (LP) (Priest, 1979).

LP introduces an additional truth value *P*, called *paradoxical*, that stands for both true and false.

	$\neg$
T	F
P	P
F	T

$\wedge$	T	P	F
T	T	P	F
P	P	P	F
F	F	F	F

$\vee$	T	P	F
T	T	T	T
P	T	P	P
F	T	P	F

Game Models

We define the verification game as a tuple $\Gamma = (\pi, \rho, \delta, \sigma)$ where

- π is the set of players,

- ρ is the set of well-defined game rules,

- δ is the set of designated truth values: the truth values preserved under validities: they determine the theorems of the logic.

- σ is the set of positions: subformula and player pairs.

It is possible to extend it to concurrent games as well.

Game Rules for LP

The introduction of the additional truth value *P* requires an additional player in the game, let us call him *Astrolabe* (after Abelard and Heloise's son).

Since we have three truth values in LP, we need three players that try to force the game to their win. If the game ends up in their truth set, then that player wins.

Then, how to associate moves with the connectives?

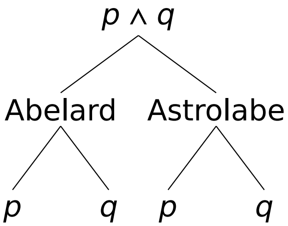
Game Rules for LP

Denote it with GTS^{LP}.

p	whoever has p in their extension, wins
$\neg F$	Abelard and Heloise switch roles
$F \wedge G$	Abelard and Astrolabe choose between F and G simultaneously
$F \vee G$	Eloise and Astrolabe choose between F and G simultaneously

An Example

Consider the conjunction. Take the formula $p \wedge q$ where p, q are P, F respectively. Then, $p \wedge q$ is F.



Abelard makes a move and chooses q which is false. This gives him a win. Interesting enough, Astrolabe chooses p giving him a win.

In this case both seem to have a winning strategy. Moreover, the win for Abelard does not entail a loss for Astrolabe.

Correctness

Theorem

In $\mathrm{GTS}^{\mathrm{LP}}$ verification game for φ,

- Eloise has a winning strategy if φ is true,
- Abelard has a winning strategy if φ is false,
- Astrolabe has a winning strategy if φ is paradoxical.

Correctness

Theorem

In a GTS^{LP} game for a formula φ in a LP model M,

- If Eloise has a winning strategy, but Astrolabe does not, then φ is true (and only true) in M,
- If Abelard has a winning strategy, but Astrolabe does not, then φ is false (and only false) in M,
- If Astrolabe has a winning strategy, then φ is paradoxical in in M.

First-Degree Entailment

Semantic evaluations are *functions* from formulas to truth values.

If we replace the valuation function with a valuation *relation*, we obtain *First-degree entailment* (FDE) which is due to Dunn (Dunn, 1976).

We use $\varphi\mathbf{r}1$ to denote the truth value of φ (which is 1 in this case).

Since, $\mathbf{r}$ is a relation, we therefore allow $\mathbf{r}(\varphi) = \emptyset$ or $\mathbf{r}(\varphi') = \{0, 1\}$ for some formula φ, φ'.

Thus, FDE is a paraconsistent (inconsistency-tolerant) and paracomplete (incompleteness-tolerant) logic.

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First-Degree Entailment

For formulas φ, ψ, we define $\mathbf{r}$ as follows.

$\neg\varphi\mathbf{r}1$	*iff*	$\varphi\mathbf{r}0$
$\neg\varphi\mathbf{r}0$	*iff*	$\varphi\mathbf{r}1$
$(\varphi \wedge \psi)\mathbf{r}1$	*iff*	$\varphi\mathbf{r}1$ *and* $\psi\mathbf{r}1$
$(\varphi \wedge \psi)\mathbf{r}0$	*iff*	$\varphi\mathbf{r}0$ *or* $\psi\mathbf{r}0$
$(\varphi \vee \psi)\mathbf{r}1$	*iff*	$\varphi\mathbf{r}1$ *or* $\psi\mathbf{r}1$
$(\varphi \vee \psi)\mathbf{r}0$	*iff*	$\varphi\mathbf{r}0$ *and* $\psi\mathbf{r}0$

Game Semantics for FDE

The truth values $\{0\}, \{1\}$ and $\{0, 1\}$ work exactly as the truth values F, T, P respectively in LP. In fact, LP can be obtained from FDE by introducing a restriction that no formula gets the truth value $\emptyset$.

Recall that for GTS^{LP}, we allowed parallel plays for selected players depending on the syntax of the formula: we associated conjunction with Abelard and Astrolabe, disjunction with Heloise and Astrolabe.

Game Semantics for FDE

For FDE, the idea is to allow each player play at each node.

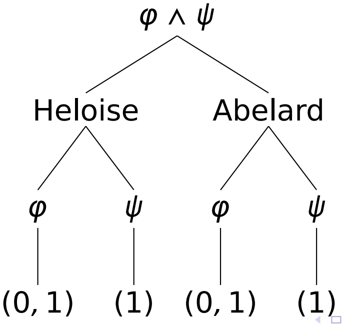
Therefore, it is possible that both players (or none) may have a winning strategy.

Also, notice that allowing each player play at each node does not necessarily mean that they will always be able to make a move. It simply means, it is allowed for them to move simultaneously.

An Example

Consider two formulas with the following relational semantics: $\varphi\mathbf{r}0$, $\varphi\mathbf{r}1$ and $\psi\mathbf{r}1$. In this case, we have $(\varphi \wedge \psi)\mathbf{r}1$ and $(\varphi \wedge \psi)\mathbf{r}0$.

We expect both Abelard and Heloise have winning strategies, and allow each player make a move at each node.



Game Rules for FDE

p	whoever has p in their extension, wins
$\neg F$	players switch roles
$F \wedge G$	Abelard and Heloise choose between F and G simultaneously
$F \vee G$	Abelard and Heloise choose between F and G simultaneously

Correctness

Theorem

In a GTS^{FDE} verification game for a formula φ, we have the following:

- Heloise has a winning strategy if $\varphi \mathbf{r} 1$
- Abelard has a winning strategy if $\varphi \mathbf{r} 0$
- No player has a winning strategy if $\varphi \mathbf{r} \emptyset$

Routleys' Relevant Logic

An interesting way to extend the relational semantics is to add possible worlds to the model. The idea is due to Routley and Routley (Routley & Routley, 1972).

A *Routley model* is a structure $(W, \#, \mathbf{r})$ where W is a set of possible worlds, $\#$ is a map from W to itself, and $\mathbf{r}$ is a valuation from $W \times \mathbf{P}$ to $\{0, 1\}$ assigning truth to propositional variables at each world.

Let us now give the semantics for this system.

$$\mathbf{r}(w, \varphi \wedge \psi) = 1 \text{ iff } \mathbf{r}(w, \varphi) = 1 \text{ and } \mathbf{r}(w, \psi) = 1$$

$$\mathbf{r}(w, \varphi \vee \psi) = 1 \text{ iff } \mathbf{r}(w, \varphi) = 1 \text{ or } \mathbf{r}(w, \psi) = 1$$

$$\mathbf{r}(w, \neg\varphi) = 1 \text{ iff } \mathbf{r}(\#w, \varphi) = 1$$

Game Rules for RR

(w, p)	whoever has p in their extension, wins
$(w, \neg F)$	switch the roles, continue with $(\# w, F)$
$(w, F \wedge G)$	Abelard chooses between (w, F) and (w, G)
$(w, F \vee G)$	Heloise chooses between (w, F) and (w, G)

Correctness

Theorem

For the evaluation games for a formula φ and a world w for Routleys' systems, we have the following:

1. Heloise has a winning strategy if $\varphi \mathbf{r} 1$.
2. Abelard has a winning strategy if $\varphi \mathbf{r} 0$.

Translation to S5

The translation of LP to S5 is built on the following observation: "In an S5-model there are three mutually exclusive and jointly exhaustive possibilities for each atomic formula *p*: either *p* is true in all possible worlds, or *p* is true in some possible worlds and false in others, or *p* is false in all possible worlds" (Kooi & Tamminga, 2013).

Translation

Given the propositional language $\mathcal{L}$, we extend it with the modal symbols $\Box$ and $\Diamond$ and close it under the standard rules to obtain the modal language $\mathcal{L}_M$. Then, the translations $\mathrm{Tr}_{LP} : \mathcal{L} \mapsto \mathcal{L}_M$ and $\mathrm{Tr}_{K3} : \mathcal{L} \mapsto \mathcal{L}_M$ for LP and K3 respectively are given inductively as follows where p is a propositional variable (Kooi & Tamminga, 2013).

$$\mathrm{Tr}_{LP}(p) = \Diamond p \qquad \mathrm{Tr}_{LP}(\varphi \wedge \psi) = \mathrm{Tr}_{LP}(\varphi) \wedge \mathrm{Tr}_{LP}(\psi)$$
$$\mathrm{Tr}_{K3}(p) = \Box p \qquad \mathrm{Tr}_{K3}(\varphi \wedge \psi) = \mathrm{Tr}_{K3}(\varphi) \wedge \mathrm{Tr}_{K3}(\psi)$$
$$\mathrm{Tr}_{LP}(\neg\varphi) = \neg\mathrm{Tr}_{K3}(\varphi) \qquad \mathrm{Tr}_{LP}(\varphi \vee \psi) = \mathrm{Tr}_{LP}(\varphi) \vee \mathrm{Tr}_{LP}(\psi)$$
$$\mathrm{Tr}_{K3}(\neg\varphi) = \neg\mathrm{Tr}_{LP}(\varphi) \qquad \mathrm{Tr}_{K3}(\varphi \vee \psi) = \mathrm{Tr}_{K3}(\varphi) \vee \mathrm{Tr}_{K3}(\psi)$$

Results

Theorem

Let $\Gamma_{LP}(M, \varphi)$ be given. Then,

- if Heloise has a winning strategy in $\Gamma_{LP}(M, \varphi)$, then she has a winning strategy in $\Gamma_{S5}(M, \mathrm{Tr}_{LP}(\varphi))$,
- if Abelard has a winning strategy in $\Gamma_{LP}(M, \varphi)$, then he has a winning strategy in $\Gamma_{S5}(M, \mathrm{Tr}_{LP}(\varphi))$,
- if Astrolabe has a winning strategy in $\Gamma_{LP}(M, \varphi)$, then both Abelard and Heloise have a winning strategy in $\Gamma_{S5}(M, \mathrm{Tr}_{LP}(\varphi))$.

Results

Theorem

Let M be an S5 model, $\varphi \in \mathcal{L}$ with an associated verification game $\Gamma_{S5}(M, \varphi)$. Then, there exists an LP model M' and a game $\Gamma_{LP}(M', \varphi)$ where,

- if Heloise (resp. Abelard) has a winning strategy for $\Gamma_{S5}(M, \varphi)$ at each point in M, then Heloise (resp. Abelard) has a winning strategy in $\Gamma_{LP}(M', \varphi)$,
- if Heloise or Abelard has a winning strategy for $\Gamma_{S5}(M, \varphi)$ at some points but not all in M, then Astrolabe has a winning strategy in $\Gamma_{LP}(M', \varphi)$,

What Have We Observed?

- Failure of the biconditional correctness
- Multiplayer semantic games in a nontrivial way
- Non-sequential / paralel / concurrent plays
- Variable sum games

If winning strategies are proofs, game semantics for paraconsistent logics present a constructive way to give proofs for inconsistencies.

Some Other Deviant Logics

Connexive Logics validate $\neg(\neg\varphi \to \varphi)$ and $(\varphi \to \psi) \to \neg(\varphi \to \neg\psi)$. The semantic games for them generate *coalitions*.

Belnap's 4-valued Logic has the truth conditions $B \wedge N = F$ and $B \vee N = T$. Their games seem to call for dependance games: winning strategy of a player depends on the winning strategies of some others.

Conclusion

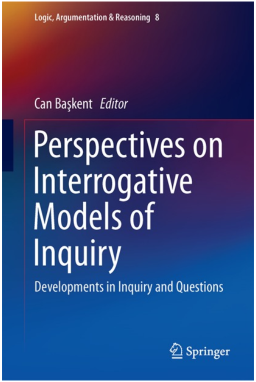
I consider this work as a first-step towards paraconsistent / non-classical game theory.

Our long term goal is to give a broader theory of (non-classical, non-utilitarian) rationality via games and logic.

Thank you for your attention!

Talk slides and the paper are available at:

`www.CanBaskent.net/Logic`



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