What is Game Theoretical Negation?

Can BAŞKENT

Institut d'Histoire et de Philosophie des Sciences et des Techniques
can@canbaskent.net www.canbaskent.net/logic

Adam Mickiewicz University, Poznań
April 17-19, 2013

Outlook of the Talk

- Classical (but Extended) Game Theoretical Semantics for Negation
  - Inquiry as a paraconsistent dialogue
- Paraconsistent Game Theoretical Semantics for Negation

What is Hintikka's Game Theoretical Semantics? I

The *semantic verification game* is played by two players, traditionally called Abelard (after $\forall$) and Eloise (after $\exists$), and the rules are specified syntactically.

During the game, the given formula is broken into subformulas by the players step by step, and the game terminates when it reaches the propositional atoms.

If we end up with a propositional atom which is true in the model in question, then Eloise wins the game. Otherwise, Abelard wins. We associate conjunction with Abelard, disjunction with Heloise.

What is Hintikka's Game Theoretical Semantics? II

The major result of this approach states that Eloise has a winning strategy if and only if the given formula is true in the model.

When conjunction and disjunction are considered, game theoretical semantics (GTS, henceforth) is very appealing. However, when it comes to negation, aforementioned intuitiveness is lost. In negated formulas, game theoretical semantics dictates that the players switch their roles. Abelard takes up Eloise's verifier role, and Eloise becomes the falsifier.

Example

Two men want to marry a princess. The king says they have to race on a horceback. The slowest one wins, and can marry the princess. How can one win this game and marry the princess?

The answer simply entails that the men need to swap their horses. Since the fastest lose, and players race with each other's horses, what they need to do is to become the fastest in the dual game. Fastest one in the switched horse, considered as the negation of the slowest in the dual game, wins the game.

In this example, GTS for negation becomes evident. If the slowest one wins the game, then the fastest one wins the dual game.

There is certainly some sense of rationality here. Namely, the players consider it easier to switch horses and race in the dual game. Yet, this story and the idea are not strong enough to generalize.

Namely, can we play chess in this way? Can we play football in this fashion?

The trick, to switch to the easier dual game to win, is a **meta-game theoretical** move. This is not a strategy within the given game, it is a strategy on the games and over the games.

What is **Wrong** with Game Theoretical Semantics?

First, insistence on “negation normal form”: For Hintikka, insisting on negation normal form is not restrictive since each formula can be effectively transformed into a formula in negation normal form (Hintikka, 1996). However, he fails to mention that in this case the game becomes a different one.

Second, it fails to address formula equivalence: compare $p \wedge (q \vee r)$ vs $(p \wedge q) \vee (p \wedge r)$ and their game trees.

What is game theoretical equivalence? (van Benthem *et al.*, 2011). Is it a strategy transformation? What about DeMorgan’s Laws?

What is **Wrong** with Game Theoretical Semantics?

Third, it is not entirely clear how the semantics of negation agrees with rationality of the players. Namely, would be even rational to play chess this way: switch the roles, and try to lose in your new set?

In other words, what is the element of rationality in GTS?

Extended Game Semantics for the Classical Case

We need to explicate the semantics of negation inductively for each case.

The ideas we will use will resemble tableaus.

Extended Game Semantics for the Classical Case

$\neg(F \wedge G)$	Eloise chooses between $\neg F$ and $\neg G$
$\neg(F \vee G)$	Abelard chooses between $\neg F$ and $\neg G$
$\neg(F \rightarrow G)$	Abelard chooses between F and $\neg G$
$\neg\neg F$	game continues with F
$\neg p$	Heloise wins if p is not true for her. Otherwise, Abelard wins.

It hints out how we can alter the GTS for the logics where DeMorgan's laws do not hold as well.

Correctness of the Extended Semantics

We denote the extended (classical) semantics we suggested as GTS*.

Theorem

For any formula φ and model M, we have

$M \models_{GTS} \varphi$ if and only if $M \models_{GTS*} \varphi$ if and only if $M \models \varphi$.

It is also not difficult to see that in GTS*, Eloise has a winning strategy if the formula in question is true.

Hintikkan Inquiry

Hintikka's interrogative inquiry is a well-known example of a dynamic epistemic game procedure which can result in an increase in knowledge.

In a nutshell, in an interrogative inquiry, the inquirer is given a theory and a question. He then tries to answer the question based on the theory by posing some questions to nature or an oracle.

Bracketing to Maintain Consistency

Hintikka introduced bracketing as a tool to omit irrelevant or uncertain answers during an interrogation.

Hintikka on Bracketing I

“An important aspect of this general applicability of the interrogative model is its ability to handle uncertain answers - that is, answers that may be false. The model can be extended to this case simply by allowing the inquirer to tentatively disregard (“bracket”) answers that are dubious. The decision as to when the inquirer should do so is understood as a strategic problem, not as a part of the definition of the questioning game. Of course, all the subsequent answers that depend on the bracketed one must then also be bracketed, together with their logical consequences.

(...)

Hintikka on Bracketing II

Equally obviously, further inquiry might lead the inquirer to reinstate (“unbracket”) a previously bracketed answer. This means thinking of interrogative inquiry as a self-corrective process. It likewise means considering discovery and justification as aspects of one and the same process. This is certainly in keeping with scientific and epistemological practice. There is no reason to think that the interrogative model does not offer a framework also for the study of this self-correcting character of inquiry.”

(Hintikka, 2007, p. 3)

and

Hintikka on Bracketing III

"In a typical application of interrogative inquiry - for instance in the cross-examination of a witness in a court of law - the inquirer cannot simply accept all answers at their face value. They can be false. Hence we must have rules allowing the rejection or, as I will call it, the "bracketing of an answer", and rules governing such bracketing."

(Hintikka, 2007, p. 223)

Problems with Bracketing

I maintain that bracketing is an overkill, and suffers from various problems.

I categorize them as epistemic, game theoretical, and heuristic problems.

Epistemic Problems I

In an inquiry or a dialogue game, how can we know which answers to ignore? How can we know what to reject or accept? This epistemic problem empties the notion of bracketing.

In other words, if inquiry is a procedure during which we want to acquire and learn some information, this implies that we *did not* have that information before.

We cannot discard some responses in favor of or against some questions or propositions - simply because we do not know the answer.

Epistemic Problems II

The epistemic problem appears to be connected to the issue of derivation in an inquiry. Rules of the IMI game allow us to use the previous answers we obtained during our inquiry. But this does not necessarily mean that we need to incorporate *all* the answers we have received into the inquiry. Some answers may be helpful, some may not. This procedure calls for a choice mechanism. In an investigative deduction, how can we know which propositions and answers to use?

Game Theoretical Problems I

It can be said that in an inquiry, we simply choose the assumptions and responses that help us *win* the game. If we can win the game with a particular set of assumptions, then we adopt these assumptions for a win. If we fail to win the game with that set of assumptions and previous answers, we simply select another set of assumptions and answers, and keep playing.

Game Theoretical Problems II

This objection bluntly undermines the *agency* of the players. In a game theoretical setting, each player follows a *strategy*, and employ a method to choose their moves, and usually the strategy is predetermined based on some understanding of rationality and players' priors. Players decide how they will play before they start playing the game. If we allow them to exercise their choice of moves based on their *a posteriori* success, that means that they did not have an *a priori* strategy before the game-play.

Game Theoretical Problems III

Additionally, bracketing poses another game theoretical problem as it seems to ignore the element of rationality in the game. In an inquiry game, all parties have an intrinsic prior commitment to play the game to win and to engage in the dialogue.

Questions and answers should be assumed to be somehow relevant in a dialogue - otherwise, the dialogue would turn into two parallel simultaneous monologues which are not semantically associated to each other in any way. Suggesting the use of bracketing for such a trivial purpose is unnecessary as it ignores the rational commitment of the players involved in the inquiry. Putting it game theoretically, irrelevant answers may be signals or part of a strategy.

Heuristic Problems I

How can we then learn from our mistakes if we bracket them out?

What about improvement and learning?

Recall the Lakatosian notion of proofs that do not prove (Lakatos, 2005).

Isn't contradictory information an essential element of a dialogue?

Then Why Bracketing?

Then, the only reasonable motivation for bracketing is to maintain consistency.

But,

the reason as to why Jaśkowski's discussive logics are not explosive applies to our discussion here as well (Jaśkowski, 1999). In an inquiry, assume that the inquirer received two answers p and $\neg p$ at different times during the inquiry.

And, it is posssible that there exists a q which is nowhere true in the model. Thus, q may not be deducible under the presence of a contradiction - concluding that inquiries are not explosive.

Consistency thus is Not a Requirement in an Inquiry

Hintikkan inquiry, taken as a game, shows that it does not have to be consistent as I argue that bracketing is not a very sensible idea.

Thus, we can now be more encouraged to argue about non-classicity in game semantics and games.

Hintikka and Sandu on Non-classicity

Even if Hintikka and Sandu conservatively remarked that “it is difficult to see how else negation could be treated game-theoretically”, they later on discussed non-classicity in GTS without offering much insight on non-classical negation (Hintikka & Sandu, 1997; Peitarinen & Sandu, 2000).

- When informational independence is allowed, the law of excluded middle fails.
- Constructivistic ideas are most naturally implemented by restricting the initial verifiers’ strategies in a semantical games to recursive ones.
- Games of inquiry involve an epistemic element.
- Nonclassical game rules can be given for propositional connectives, especially for conditional and negation.

More on Non-classicity

These points are rather self-evident, and seem to include most of the concerns about the classicity of GTS.

I believe, in the above list, Hintikka and Sandu had intuitionism, more specifically the law of excluded middle, in mind when they discussed non-classicity.

However, another alternative to classical logic is also possible. Dual-intuitionistic logic, or paraconsistent logics in general, poses influential approaches to classical problems of logic.

Non-classical Games

It is not difficult to perceive and thus introduce additional outcomes for GTS. We introduce the following five non-classical possibilities:

1. Abelard and Eloise both win.
2. Abelard and Eloise both lose.
3. Eloise wins, Abelard does not lose.
4. Abelard wins, Eloise does not lose.
5. There is a tie.

What are the Non-classical Games?

Some propositions can belong to both player: namely, both the proposition and its negation can be true.

Some propositions can belong to the neither: namely, neither the proposition nor its negation can be true.

Some propositions may not belong to one player without the negation belonging to the opponent: namely, the proposition can be true, but its negation may not be false.

In short, the game does not have to be a zero-sum game. One's win may not imply the other's loss.

Logic of Paradox and GTS

The formalism we adopt here is Graham Priest's Logic of Paradox (Priest, 1979). The logic of paradox (LP, for short) introduces an additional truth value P, called *paradoxical*, that stands for both true and false.

	$\neg$
T	F
F	T
P	P

$\wedge$	T	P	F
T	T	P	F
P	P	P	F
F	F	F	F

$\vee$	T	P	F
T	T	T	T
P	T	P	P
F	T	P	F

Game Rules for LP

The introduction of the additional truth value *P* requires an additional player in the game, let us call him *Astrolabe* (after Abelard and Heloise's son).

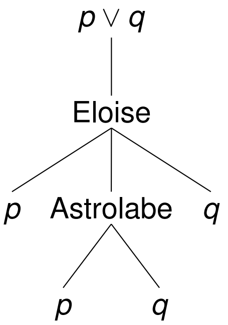
The reason is quite obvious. Since we have three truth values in LP, we need three players that try to force the game to their win. If the game ends up in their truth set, then that player wins.

Examples

Consider the formula $p \vee q$ where p, q are propositional variables with truth values P, F respectively. Therefore, the truth value of $p \vee q$ is also P.

In this case, Eloise cannot force a win because neither p nor q has the truth value T.

On the other hand, Astrolabe has a winning strategy as the truth value of p is P when it is his turn to play. Thus, he chooses p yielding the truth value P for the given formula $p \vee q$.



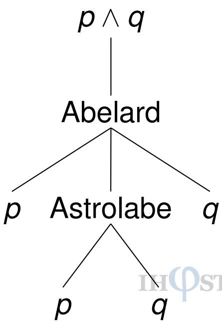
Examples

Let us now consider the conjunction. Take the formula $p \wedge q$ where p, q are propositional variables with truth values P, F respectively.

In this case, Abelard first makes a move, and as the falsifier, he can choose q which is false. This gives him a win.

Therefore, Astrolabe does not get a chance to make a move. However, interesting enough, if he had a chance to play, he would go for p which has a truth value of P, and this would him Astrolabe his win.

Remember, first the parents make a move, then Astrolabe.



Remarks

1. Disjunction belongs to Eloise (and Astrolabe) and conjunction belongs to Abelard (and Astrolabe).
2. First parents make a move, if they have a winning strategy in the subgame they choose at the connective, the game proceeds.
3. Otherwise, if they do not have a winning strategy when it is their turn, then Astrolabe plays.

Game Theoretical Semantics for LP

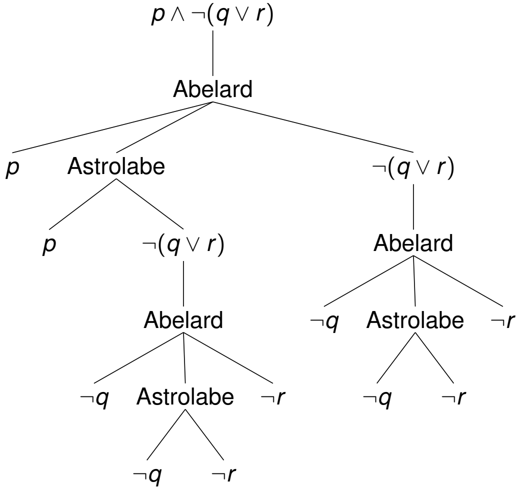
p (or $\neg p$)	whoever has p (or $\neg p$) in their extension, wins
$F \wedge G$	First Abelard, then Astrolabe chooses between F and G
$F \vee G$	First Eloise, then Astrolabe chooses between F and G
$\neg(F \wedge G)$	First Eloise, then Astrolabe chooses between $\neg F$ and $\neg G$
$\neg(F \vee G)$	First Abelard, then Astrolabe chooses between $\neg F$ and $\neg G$

Another Example

Let us now consider a bit complicated formula $p \wedge \neg(q \vee r)$ where the truth values of p, q and r are T, P and F respectively. According to the LP truth table, the given formula has the truth value of P. Thus, we expect Astrolabe to have a winning strategy.

Based on the given truth values for the propositional variables, what we expect is to see that Astrolabe can forse and $\neg r$ (or r) output in the game. The game tree below explicates how Astrolabe wins the game based on the game rules.

Another Example



Observations

Similar to Priest's early theorem on LP, we have the following.

Theorem

For any formula φ and model M, we have $M \models_{GTS} \varphi$ if and only if $M \models_{GTS^p} \varphi$.

Correctness

Theorem

In GTS^p verification game for φ,

- Eloise has a winning strategy if φ is true
- Abelard has a winning strategy if φ is false
- Astrolabe has a winning strategy if φ is paradoxical

Dominating Strategies

Note that, in the parallel play, we simply eliminated the dominated strategies, and iterate the procedure.

Thus, it can be seen as an iterated elimination of dominated strategies - which is not visible in the classical case, but clearer in the paraconsistent case - due to the truth table of LP.

Conclusion I

In this work, we do not aim at giving a full picture of game theoretical semantics of negation in all non-classical logics. The literature on non-classical logics (which include intuitionistic, paraconsistent and relative logics amongst many others) is vast, and all of those logics are not transformable to each other making it almost impossible to give a unifying theme for GTS.

Conclusion II

In a recent paper, Priest alludes to similar concepts (Priest, 2013). We can add some further points by noting that our approach here can be a case for the plurality of logic. The well-known classical GTS is essentially a very narrow, limited case with many additional and auxiliary game theoretical assumptions. Clearly, once those assumptions are removed for various reasons, the basic (and *pure*) GTS turns out to be expressive enough for various non-classical logics.

Similarly, Dialogical Logic can initially be taught of providing a good approach to negation. However, a closer inspection reveals that in dialogical logical cases, the role switching idea is maintained and even taken to a higher level creating more schizophrenic players (Rahman & Tulenheimo, 2009).

Conclusion III

Behavioral economics and the charming examples that it provides (for example (Ariely, 2008; Ariely, 2010; Harford, 2009)) constitutes an interesting playground for the ideas we have developed here.

And we hope that our contribution will help the field to formalize a more realistic and down to earth game theory.

Thanks for your attention!

Talk slides and the papers are available at

`www.CanBaskent.net/Logic`

References I

ARIELY, DAN. 2008.

Predictably Irrational: The Hidden Forces That Shape Our Decisions.

New York, NY: HarperCollins.

ARIELY, DAN. 2010.

The Upside of Irrationality.

Harper.

HARFORD, TIM. 2009.

Logic of Life.

Random House.

HINTIKKA, JAAKKO. 1996.

The Principles of Mathematics Revisited.

Cambridge University Press.

References II

HINTIKKA, JAAKKO. 2007.

Socratic Epistemology.

Cambridge University Press.

HINTIKKA, JAAKKO, & SANDU, GABRIEL. 1997.

Game-theoretical semantics.

Pages 361–410 of: VAN BENTHEM, JOHAN, & TER MEULEN, ALICE (eds), *Handbook of Logic and Language*.

Elsevier.

JAŚKOWSKI, STANISŁAW. 1999.

A Propositional Calculus for Inconsistent Deductive Systems.

Logic and Logical Philosophy, **7**(1), 35–56.

LAKATOS, IMRE. 2005.

Proofs and Refutations.

Cambridge University Press.

References III

PEITARINEN, AHTI, & SANDU, GABRIEL. 2000.

Games in Philosophical Logic.

Notdic Journal of Philosophical Logic, **4**(2), 143–173.

PRIEST, GRAHAM. 1979.

The Logic of Paradox.

Journal of Philosophical Logic, **8**, 219–241.

PRIEST, GRAHAM. 2013.

Mathematical Pluralism.

Logic Journal of the IGPL, **21**(1), 4–13.

RAHMAN, SHAHID, & TULENHEIMO, TERO. 2009.

From Games to Dialogues and Back.

Pages 153–208 of: MAHER, ONDREJ, PEITARINEN, AHTI, & TULENHEIMO, TERO (eds), *Games: Unifying Logic, Language and Philosophy.*

Springer.

References IV

VAN BENTHEM, JOHAN, PACUIT, ERIC, & ROY, OLIVIER. 2011.

Toward A Theory of Play: A Logical Perspective on Games and Interaction.

Games, **2**(1), 52–86.