Geometry of Epistemology

An Exposition

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Outlook of the Talk

- Topological Semantics
- Weak Structures: Subset Spaces
- A Brief Comparison

Topological Definitions

Definition (Topological Space)

A topological space $\mathcal{S} = \langle S, \sigma \rangle$ *is a structure with a set* S *and a collection* σ *of subsets of* S *satisfying the following axioms:*

1. *The empty set and* S *are in* σ.
2. *The union of any collection of sets in* σ *is also in* σ.
3. *The intersection of a finite collection of sets in* σ *is also in* σ.

Recall now that the topological interior operator $\mathbb{I}$ satisfies the following properties for each $X, Y \in \sigma$: (i) $\mathbb{I}(X) = X$, (ii) $\mathbb{I}(X \cap Y) = \mathbb{I}(X) \cap \mathbb{I}(Y)$, (iii) $\mathbb{I}(\mathbb{I}(X)) = \mathbb{I}(X)$ (McKinsey & Tarski, 1944)

Logical Definitions

A topological model $\mathcal{M}$ is a triple $\langle S, \sigma, v \rangle$ where $\mathcal{S} = \langle S, \sigma \rangle$ is a topological space, and v is a valuation function sending propositional letters to the subsets of S, i.e. $v : P \to \wp(S)$.

Definition (Topological Semantics)

$$
\begin{array}{lll}
\mathcal{M}, s \models p & \textit{iff} & s \in v(p) \textit{ for } p \in P \\
\mathcal{M}, s \models \neg\varphi & \textit{iff} & \textit{not } \mathcal{M}, s \models \varphi \\
\mathcal{M}, s \models \varphi \wedge \psi & \textit{iff} & \mathcal{M}, s \models \varphi \textit{ and } \mathcal{M}, s \models \psi \\
\mathcal{M}, s \models \mathsf{I}\varphi & \textit{iff} & \exists U \in \sigma (s \in U \wedge \forall t \in U, \mathcal{M}, t \models \varphi)
\end{array}
$$

The C operator can then be defined accordingly:

$$\mathcal{M}, s \models \mathsf{C}\varphi \quad \text{iff} \quad \forall U \in \sigma (s \in U \to \exists t \in U, \mathcal{M}, t \models \varphi)$$

Topological vs Kripkean Semantics

Topological

$\mathcal{M}, s \models \mathsf{I}\varphi$ iff $\exists U \in \sigma$ with $s \in U$ such that$(\forall t \in U)\mathcal{M}, t \models \varphi)$

Kripkean

$\mathcal{M}, s \models \Box\varphi$ iff $\forall t \in U(sRt \rightarrow \mathcal{M}, t \models \varphi)$

Complexity and Expressivity: Topological Semantics is Σ_2 as opposed to Π_1 Kripke Semantics.

Correspondence: Topological vs Kripke Frames

Every S4 Kripke frame $\langle S, R \rangle$ gives rise to a topological space $\langle S, \sigma_R \rangle$, where σ_R is the set of all upward closed subsets of the given frame. It is easy to see that the empty set and S are in σ_R, and furthermore arbitrary unions and finite intersections of upward closed sets are still upward closed. Hence, σ_R is a (Alexandroff) topology.

Alexandroff topologies are those in which each point has a *least* neighborhood (the least neighborhood of a point s is the set $\{t \in W : sRt\}$).

Note that Alexandroff spaces are those topological spaces in which intersection of any family of opens is again an open.

Vickers' Example

"My baby has green eyes."
The obvious question is, "Is this true or false?".
First, we may agree that her eyes really are green - we can *affirm* the assertion.
Second, we may agree that her eyes are some other colour, such as brown - we can *refute* the assertion.
Third, we may fail to agree; but perhaps if we hire a powerful enough colour analyser, that may decide us (Vickers, 1989).
etc...

Vickers' Example - Conclusion

What is crucial in Vickers' analysis is that statements are affirmable or refutable in a *finite* amount of time with spending *finite* amount of effort.

He defines: an assertion is *affirmative*, if and only if it is true precisely in the circumstances when it can be affirmed. Likewise, an assertion is *refutative* if and only if it is false precisely in the circumstances when it can be refuted.

A Dynamic Epistemology

"[N]otion of *effort* enters in topology. Thus if we are at some point at s and make a measurement, we will then discover that we are in some neighborhood U of s, but not know where. If we make my measurement finer, then U will shrink, say, to a smaller neighborhood V." (Parikh & Moss, 1992).

By spending some effort, we eliminate some of the possibilities, and obtain a smaller set of possibilities. The smaller the set of observation is, the larger the information we have.

Therefore, as it was also observed in the above example, to gain *knowledge*, we need to spend some *effort*.

SSL: Model and Language

A subset space model is a triple $\mathcal{S} = \langle S, \sigma, v \rangle$ where $\langle S, \sigma \rangle$ is a subset frame, $v : P \to \wp(S)$ is a valuation function for the countable set of propositional variables P

The language $\mathcal{L}_S$ of SSL is:
$p \mid \top \mid \neg\varphi \mid \varphi \wedge \psi \mid \mathsf{K}\varphi \mid \Box\varphi$

SSL: Semantics

$s, U \models p$	if and only if	$s \in v(p)$	
$s, U \models \varphi \wedge \psi$	if and only if	$s, U \models \varphi$	and $s, U \models \psi$
$s, U \models \neg\varphi$	if and only if	$s, U \not\models \varphi$	
$s, U \models \mathsf{K}\varphi$	if and only if	$t, U \models \varphi$	for all $t \in U$
$s, U \models \Box\varphi$	if and only if	$s, V \models \varphi$	for all $V \in \sigma$ such that $s \in V \subseteq U$

Axioms

1. All the substitutional instances of the tautologies of the classical propositional logic
2. $(A \to \Box A) \land (\neg A \to \Box \neg A)$ for atomic sentence A
3. $\mathsf{K}(\varphi \to \psi) \to (\mathsf{K}\varphi \to \mathsf{K}\psi)$
4. $\mathsf{K}\varphi \to (\varphi \land \mathsf{KK}\varphi)$
5. $\mathsf{L}\varphi \to \mathsf{KL}\varphi$ Euclidean
6. $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$
7. $\Box\varphi \to (\varphi \land \Box\Box\varphi)$
8. $\mathsf{K}\Box\varphi \to \Box\mathsf{K}\varphi$ Cross-Axiom

K is S5 and $\Box$ is S4 (Parikh *et al.*, 2007).

Completeness

SSL is strongly complete and decidable.

NOT trivial!
The reason for that is the fact that at the level of maximally consistent theories, there is no known way to define a corresponding subset space structure.

Decidability

Finite model property fails in SSL.
Consider $\Box(\Diamond\varphi \wedge \Diamond\neg\varphi)$ at (s, U) where U is the minimal open about s.

Decidability then can be shown on Cross Axiom models by filtration as Cross Axiom models has a finite model property.

Defining Properties

WDA	$\Diamond\Box\varphi \rightarrow \Box\Diamond\varphi$
	sound for weakly directed spaces
UA	$\Diamond\varphi \wedge \mathsf{L}\Diamond\psi \rightarrow \Diamond(\Diamond\varphi \wedge \mathsf{L}\Diamond\psi \wedge \mathsf{K}\Diamond\mathsf{L}(\varphi \vee \psi))$
	sound for subset spaces closed under binary unions
WUA	$\mathsf{L}\Diamond\varphi \wedge \mathsf{L}\Diamond\psi \rightarrow \mathsf{L}\Diamond(\mathsf{L}\Diamond\varphi \wedge \mathsf{L}\Diamond\psi \wedge \mathsf{K}\Diamond\mathsf{L}(\varphi \vee \psi))$
	weaker than UA
CI	$\Box\Diamond\varphi \rightarrow \Diamond\Box\varphi$
	sound for subset spaces closed under all intersections
M_n	$(\Box\mathsf{L}\Diamond\varphi \wedge \Diamond\mathsf{K}\psi_1 \wedge \cdots \wedge \psi_n)$
	$\rightarrow \mathsf{L}(\Diamond\varphi \wedge \Diamond\mathsf{K}\psi_1 \wedge \cdots \wedge \Diamond\mathsf{K}\psi_n)$
	WD and all M_n are complete for directed spaces

See (Georgatos, 1997), (Weiss & Parikh, 2002)

Some Basic Topological Properties in SSL

Proposition

φ is open if and only if $\varphi \rightarrow \Diamond \mathsf{K}\varphi$ is valid.

Proposition

Dually, φ is closed if and only if $\Box \mathsf{L}\varphi \rightarrow \varphi$.

Proposition

$v(p)$ is dense if and only if $\Box \mathsf{L}p$ holds. Similarly, $v(p)$ is nowhere dense if and only if $\Diamond \mathsf{L}\neg p$ is valid.

Overlap Modality

$$s, U \models \mathsf{O}\varphi \quad \textit{iff} \quad \forall U' \in \sigma : (s \in U' \rightarrow s, U' \models \varphi)$$

$\Box$ is a special case of O.

Overlap operator was designed to enable us to quantify "not only downwards, but also diagonally" among the set of observations (Heinemann, 2006).

Disjoint Unions

Definition

Two subset space models are disjoint if their domain contains no common element. For disjoint subset space models $\mathcal{S}_i = \langle S_i, \sigma_i, v_i \rangle$, for $i \in I$ their disjoint union is the structure $\mathcal{S} = \biguplus_{i \in I} \mathcal{S}_i = \langle S, \sigma, v \rangle$ where $S = \bigcup_{i \in I} S_i$, $\sigma = \bigcup_{i \in I} \sigma_i$ and $v(p) = \bigcup_{i \in I} v_i(p)$.

Theorem

For disjoint subset space models $\mathcal{S}_i$ for $i \in I$ and for each neighborhood situation (s, U) in $\mathcal{S}_i$, we have $s, U \models_{\mathcal{S}} \varphi$ if and only if $s, U \models_{\mathcal{S}_i} \varphi$, for each formula φ in the language of subset space logic $\mathcal{L}_S$.

Generated Subset Spaces

We can throw away the points at which we do not have any observations.

Proposition

For $\mathcal{S} = \langle S, \sigma, v \rangle$, *let* $S' = S - \{s : s \notin \cup\sigma\}$ *and* $v'(p) = v(p) \cap S'$. *Then* $\mathcal{S}' = \langle S', \sigma, v' \rangle$ *and* $\mathcal{S} = \langle S, \sigma, v \rangle$ *satisfy the same formulae.*

See (Başkent, 2007)

Generated Subset Spaces

Definition

Let $\mathcal{S} = \langle S, \sigma, v \rangle$ be a subset space model. Let (s, U) be the designated neighborhood situation. Then we obtain the generated subset space $\mathcal{S}' = \langle S', \sigma', v' \rangle$ of $\mathcal{S}$ as follows.

- $\sigma' := \sigma - \{V \in \sigma : V \not\subseteq U\}$
- $S' := S - \cup\sigma'$
- $v'(p) := v(p) \cap S'$ *for each propositional letter* p.

Proposition

For each $s \in S'$, we have $s, U \models_{\mathcal{S}} \varphi$ if and only if $s, U \models_{\mathcal{S}'} \varphi$.

Bisimulation

For, $\mathcal{S} = \langle S, \sigma, u \rangle$ and $\mathcal{T} = \langle T, \tau, v \rangle$, if $(s, U) \rightleftarrows (t, V)$, then:

1. **Base Condition**

   $s \in u(p)$ if and only if $t \in v(p)$ for each p

2. **Back Conditions**

   2.1 $\forall t' \in V$ there exists $s' \in U$ with $(s', U) \rightleftarrows (t', V)$.

   2.2 $\forall V' \subseteq V$ such that $t \in V'$, there is $U' \subseteq U$ with $s \in U'$ such that $(s, U') \rightleftarrows (t, V')$

3. **Forth Conditions**

   3.1 $\forall s' \in U$ there exists $t' \in V$ with $(s', U) \rightleftarrows (t', V)$.

   3.2 $\forall U' \subseteq U$ such that $s \in U'$, there is $V' \subseteq V$ with $t \in V'$ such that $(s, U') \rightleftarrows (t, V')$.

Bisimulation Invariance

Theorem (Bisimulation Invariance for Subset Spaces)

If $(s, U) \rightleftarrows (t, V)$ then they satisfy the same formulae.

Converse is true only under the special conditions.

Theorem

Let $\mathcal{S} = \langle S, \sigma, u \rangle$ and $\mathcal{T} = \langle T, \tau, v \rangle$ be two finite subset space. Then for each neighborhood situations (s, U) in $S \times \sigma$ and (t, V) in $T \times \tau$; we have $(s, U) \rightleftarrows (t, V)$ if and only if $(s, U) \leftrightsquigarrow (t, V)$.

Evaluation and Bisimulation Games

Position	Player	Admissible Moves
$(\bot, (s, U))$	$\exists$	$\emptyset$
$(\top, (s, U))$	$\forall$	$\emptyset$
$(p, (s, U))$ with $s \in v(p)$	$\forall$	$\emptyset$
$(p, (s, U))$ with $s \notin v(p)$	$\exists$	$\emptyset$
$(\psi_1 \wedge \psi_2, (s, U))$	$\forall$	$\{(\psi_1, (s, U)), (\psi_2, (s, U))\}$
$(\psi_1 \vee \psi_2, (s, U))$	$\exists$	$\{(\psi_1, (s, U)), (\psi_2, (s, U))\}$
$(\mathsf{L}\psi, (s, U))$	$\exists$	$\{(\psi, (t, U)) : t \in U\}$
$(\mathsf{K}\psi, (s, U))$	$\forall$	$\{(\psi, (t, U)) : t \in U\}$
$(\Diamond\psi, (s, U))$	$\exists$	$\{(\psi, (s, V)) : s \in V \subseteq U\}$
$(\Box\psi, (s, U))$	$\forall$	$\{(\psi, (s, V)) : s \in V \subseteq U\}$

Adequacy Theorems for Evaluation and Bisimulation games follow (Başkent, 2007).

Historical

Kripke semantics is the latest but the most primitive semantics for modal logic.

Earlier semantics of modal logics were developed by Tsao-Chen Tang, C. I. Lewis, McKinsey and Tarski starting from 1930s by using topological semantics.

Algebraic semantics of modal logic and the correspondence results were established by Tarski and Jonnson in 1950s.

Computational

Topological (not topologic) semantics is arithmetically more complex, and requires more computation: First find the open set/neighborhood; then check the modality. Thus, it is a stronger approach to the modal epistemology.
Furthermore, the algebraic structures are more powerful and gives more abstract and general results.

Social

Topological products are weaker. In multi-agent case, they do not inherit the counterintuitive axioms from the single agent case. Kripke structures are essentially graphs and to distinguish the knowledge of agents, you just label the *arrows* differently. That's it. Subset Spaces are difficult for multi-agent case (Başkent, 2007)

Heuristics

The weaker the logic, the more it tells. The weaker notions of heuristics can be formalized in Subset Spaces.
Subset Space logic can be used to formalize communication, heuristics (Başkent & Bağçe, 2009), (Pacuit & Parikh, 2007).
Kripkean attempts cannot go much beyond the state-elimination based dynamic epistemic logics.

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Thanks for your attention!

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