Preferences and Equilibria in History Based Models

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Today's Plan

Slogan: It's All About the Past

A new dynamic model to

reason about histories and their changes!

Motivation

History Based Models

History based structures, first proposed by Parikh and Ramanujam (Parikh & Ramanujam, 2003), suggest a formal framework that lies between process models and propositional dynamic logic.

Epistemic and temporal reasoning in such models depends on sequences of events, called *histories*.

History based structures have successfully been used to model epistemic messages and communication between agents using a rather dynamic approach, and deontic obligations (Pacuit, 2007; Pacuit *et al.*, 2006; Parikh & Ramanujam, 2003).

Furthermore, they are technically similar to interpreted systems (Fagin *et al.*, 1995; Pacuit, 2007).

The Model

Technical Details

History based structures are constructed by using a fix set of events E and agents A. For each agent i, $\mathsf{E}_i \subseteq \mathsf{E}$ is the set of events which are "seen" or "accessible" by the agent i. A finite sequence of events h drawn from a set of events E is called a *history* over E.

A sequence h is a local history for agent i, if it is a finite history over the local event set E_i. A word H is a global history, if it is a (possibly infinite) history over the global event set E.

Let $\mathcal{H}_{\mathsf{E}}^{\text{fin}}$ be the set of all finite histories for a set of events E. For any set of histories $\mathcal{H}$, the set $\text{FinPre}(\mathcal{H})$ denotes the set of finite prefixes of the histories in $\mathcal{H}$.

Technical Details

Let i be an agent, and $\mathcal{H}$ be a set of histories. A function $\lambda_i : \text{FinPre}(\mathcal{H}) \rightarrow \mathcal{H}^{\text{fin}}_{\mathsf{E}_i}$ is an epistemic locality function for agent i, in history $\mathcal{H}$.

Let i be an agent, and let λ_i be its locality function. Histories h and h' are indistinguishable for agent i, written $h \sim_i h'$, if and only if h and h' are finite histories, and $\lambda_i(h) = \lambda_i(h')$.

Given a set of propositional variables P, we define the syntax of history based structures in the Backus - Naur form as follows.

$$\varphi := \; p \mid \neg\varphi \mid \varphi \wedge \varphi \mid K_i\varphi \mid \bigcirc\varphi \mid \varphi U \varphi$$

where $p \in \mathsf{P}$, $i \in \mathsf{A}$. The knowledge operator for agent i is denoted by K_i and the temporal next-time operator is denoted by $\bigcirc$. We call U the *until operator*.

A tuple $M = \{\mathsf{E}, \mathcal{H}, \mathsf{A}, \mathsf{E}_1, \ldots, \mathsf{E}_n, \lambda_1, \ldots, \lambda_n, V\}$ is a history-based model where E is a global set of events, $\mathcal{H} \subseteq \mathcal{H}_{\mathsf{E}}$ is a protocol, A is a set of agents, for each agent $i \in \mathsf{A}$, E_i and λ_i are i's local event set and locality function, and V is a valuation function.

We give the semantics as follows.

$$
\begin{array}{lll}
H, t \models_M p & \textit{iff} & H_t \in V(p), \\
H, t \models_M \neg\varphi & \textit{iff} & H, t \not\models_M \varphi, \\
H, t \models_M \varphi \wedge \psi & \textit{iff} & H, t \models_M \varphi \textit{ and } H, t \models_M \psi, \\
H, t \models_M \bigcirc\varphi & \textit{iff} & H, t+1 \models_M \varphi, \\
H, t \models_M K_i\varphi & \textit{iff} & \textit{for all } H' \in \mathcal{H}, H_t{\sim_i}H'_t \textit{ implies } H', t \models_M \varphi, \\
H, t \models_M \varphi U \psi & \textit{iff} & \textit{there exists } t \leq k \textit{ such that } H, k \models_M \psi \textit{ and,} \\
 & & \textit{for all } l, t \leq l < k \textit{ implies } H, l \models_M \varphi.
\end{array}
$$

Axioms

The axioms for history based models are given as follows.

- All tautologies of propositional logic,
- $K_i(\varphi \to \psi) \to (K_i\varphi \to K_i\psi)$,
- $K_i\varphi \to \varphi \wedge K_iK_i\varphi$,
- $\neg K_i\varphi \to K_i\neg K_i\varphi$,
- $\bigcirc(\varphi \to \psi) \to (\bigcirc\varphi \to \bigcirc\psi)$,
- $\bigcirc\neg\varphi \leftrightarrow \neg \bigcirc \varphi$,
- $\varphi U\psi \leftrightarrow \psi \vee (\varphi \wedge \bigcirc(\varphi U\psi))$.

The rules of inference are modus ponens, and normalization for all the modalities:

- $\vdash \varphi, \varphi \to \psi \therefore \vdash \psi$,
- $\vdash \varphi \therefore \vdash K_i\varphi$,
- $\vdash \varphi \therefore \vdash \bigcirc\varphi$,
- $\vdash \varphi \to (\neg\psi \wedge \bigcirc\varphi) \therefore \vdash \varphi \to \neg(\varphi' U\psi)$.

Introducing Preferences

Preferences

We amend the syntax of the logic of history based models with the modal operator $\Diamond_i\varphi$ which denotes that there is a history which is at least as good as the current one and satisfies φ for agent *i*. We specify the semantics of this new modality as follows.

$$H, t \models \Diamond_i\varphi \quad \textit{iff} \quad \exists H'.H \preceq_i H' \textit{ and } H', t \models \varphi$$

where the expression $H \preceq_i H'$ denotes that "the agent *i* (weakly) prefers H' to H".

Axioms

We take the preference modality as **S4**, and give the axiomatization of history based preference logic as follows.

- All tautologies of propositional logic,
- $K_i(\varphi \to \psi) \to (K_i\varphi \to K_i\psi)$,
- $K_i\varphi \to \varphi \wedge K_iK_i\varphi$,
- $\neg K_i\varphi \to K_i\neg K_i\varphi$,
- $\Box_i(\varphi \to \psi) \to (\Box_i\varphi \to \Box_i\psi)$,
- $\Box_i\varphi \to \varphi$,
- $\Box_i\varphi \to \Box_i\Box_i\varphi$,
- $\bigcirc(\varphi \to \psi) \to (\bigcirc\varphi \to \bigcirc\psi)$,
- $\bigcirc\neg\varphi \leftrightarrow \neg \bigcirc \varphi$,
- $\varphi U\psi \leftrightarrow \psi \vee (\varphi \wedge \bigcirc(\varphi U\psi))$.

The rules of inference are modus ponens, and normalization for all three modalities:

- $\vdash \varphi, \varphi \to \psi \therefore \vdash \psi$,
- $\vdash \varphi \therefore \vdash K_i\varphi$,
- $\vdash \varphi \therefore \vdash \Box_i\varphi$
- $\vdash \varphi \therefore \vdash \bigcirc\varphi$,
- $\vdash \varphi \to (\neg\psi \wedge \bigcirc\varphi) \therefore \vdash \varphi \to \neg(\varphi' U\psi)$.

Equilibria

Best Response

We define the *best response* BR_i of an agent i in a two-player game as $BR_i = \sim_{-i} \cap \prec_i$ where $-i$ denotes the opponent of i.

Associate a diamond-like modality $\triangle_i$ with the relation BR_i.

Best Response	$\neg\triangle_i\top$
Nash Equilibrium	$\bigwedge_{i\in\mathsf{A}} \neg\triangle_i\top$
Pareto Optimality	$\bigvee_{I\subset\mathsf{A}} \triangle_i\top \wedge \bigwedge_{\mathsf{A}\setminus I} \neg\triangle_{-i}\top$

Updating Preferences

The preference update will be carried out by a *distinguishing formula* φ.

Given two histories H, H'; if $H, t \models \varphi$ but $H', t \models \neg\varphi$, then we call φ "distinguishing formula" for (H, t) and (H', t).

In this case, if $H \preceq_i H'$, after a preference update by φ, we will then have $H \not\preceq_i H'$ at t.

The updated preference orders $\preceq_i^*$ are defined as follows

$$\preceq_i^* := \preceq_i \setminus \{(H, H') : H, t \models_M \varphi \text{ and } H', t \models_M \neg\varphi \text{ for any } t\}.$$

Syntax and Semantics

The syntax of history based preference update logic is as expected:

$$\varphi := \ p \mid \neg\varphi \mid \varphi \wedge \varphi \mid K_i\varphi \mid \bigcirc\varphi \mid \varphi U \varphi \mid \lozenge_i \mid [!\varphi]\varphi$$

Given a model M and a distinguishing formula φ, the semantics of the preference update modality is given as follows.

$$H, t \models_M [\varphi!]\psi \quad \textit{iff} \quad H, t \models_{M!\varphi} \psi$$

The additional axioms for the dynamic preference update modality are given as follows.

- $[\varphi!]p \leftrightarrow p$
- $[\varphi!]\neg\psi \leftrightarrow \neg[\varphi!]\psi$
- $[\varphi!]\psi \wedge \chi \leftrightarrow [\varphi!]\psi \wedge [\varphi!]\chi$
- $[\varphi!]K_i\psi \leftrightarrow K_i[\varphi!]\psi$
- $[\varphi!]\Diamond_i\psi \leftrightarrow$ $(\neg\varphi \wedge \Diamond_i[\varphi!]\psi) \vee \Diamond_i(\varphi \wedge [\varphi!]\psi)$

The proof rule we need is necessitation for the dynamic modality: $\vdash [\varphi]\psi \therefore \vdash \psi$.

We denote the history based preference update logic by HBPL*.

The Theorem

The Theorem

HBPL* is sound and complete.

Thank you!

Come see the poster

Talk slides and the papers are available at

`CanBaskent.net/Logic`

References

Fagin, Ronald, Halpern, Joseph Y., Moses, Yoram, & Vardi, Moshe Y. 1995.
Reasoning About Knowledge.
MIT Press.

Pacuit, Eric. 2007.
Some Comments on History Based Structures.
Journal of Applied Logic, **5**(4), 613–24.

Pacuit, Eric, Parikh, Rohit, & Cogan, Eva. 2006.
The Logic of Knowledge Based Obligation.
Synthese, **149**(2), 311–341.

A Brief Bibliography II

Parikh, Rohit, & Ramanujam, R. 2003.
A Knowledge Based Semantics of Messages.
Journal of Logic, Language and Information, **12**(4), 453 – 467.