A Geometric - Epistemic Approach to Lakatosian Heuristics

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Aim

We will present a formalization for Lakatosian heuristics and present a simple example implementing it.

Why Lakatos?

His view of mathematics as a quasi-emprical science sheds light to our understanding of mathematics and its heuristics.

Moreover, epistemic theories of learning has a lot to do with Lakatosian heuristics.

Proofs and Refutations

A Lakatosian Heuristics

Lakatos's *Proofs and Refutations* exhibits a careful analysis of a very significant mathematical development, namely the evolution of Euler's Theorem $V - E + F = 2$ for three dimensional polyhedra where V, E and F are the number of vertices, edges, and faces of the polyhedron respectively.

Lakatosian Heuristics

Development of Proof

For Lakatos, the *development* of a mathematical theorem together with its proof was a very significant aspect of the growth of knowledge in mathematics.

As Kiss put it, "in Lakatos's heuristics, the theorem is not ready when we start to prove it. It is stated in a possibly false generality, and it can be formulated several times in the process [of its development]." [4].

In other words, in Lakatosian heuristics, one starts off with rather loose and overly generalized statements and makes them more and more precise *along the course of their development*.

Lakatosian Heuristics

Basics

1. Primitive conjecture.
2. Proof (a rough thought experiment or argument, decomposing the primitive conjecture into subconjectures and lemmas).
3. Global counterexamples.
4. Proof re-examined. The guilty lemma is spotted. The guilty lemma may have previously remained hidden or may have been misidentified.
5. Proofs of the other theorems are examined to see if the newly found lemma occurs in them.
6. Hitherto accepted consequences of the original and now refuted conjecture are checked.
7. Counterexamples are turned into new examples, and new fields of inquiry open up. [3]

Lakatosian Heuristics Methods

Strategies

Lakatos employed three main strategies to implement his method of proofs and refutations: *monster-barring*, *exception-barring* and *lemma incorporation*.

Monster-barring deals with the objects which are *not in mind* when the conjecture is put forward. They are, in this sense, monsters and should be excluded from our domain of discourse.

Exception-barring accepts that the theorem in its stated form is not valid due to the emergence of some genuine counterexamples targeting the correctness of the theorem itself.

Lemma incorporation depicts the way we turn the counterexamples into new examples of the modified and reformulated form of the theorem.

Basics

Idea

In order to give a formal account of the Lakatosian heuristics, we will use subset space logic which was first put forward in early 90s by Moss and Parikh. Their goal was to formalize reasoning about sets and points [8].
SSL had two modal operators K for knowledge and $\Box$ for effort.
The key idea of Moss and Parikh's approach to the concept of closeness can be formulated as follows.

In order to get close, one needs to make some effort.

Therefore, to gain *knowledge*, we need to make some *effort*. By spending some effort, we eliminate some of the existing possibilities, and obtain a smaller set of possibilities.

Basics

Semantics

The triple $\mathcal{S} = \langle S, \sigma, v \rangle$ is called a subset space model where S is a set, σ is a collection of subsets of S (not necessarily a topology) and, $v : P \rightarrow \wp(S)$ is a valuation function.

We will interpret the formulae at the neighborhood situations (s, U) where $s \in U \in \sigma$.

Basics

Semantics

$$
\begin{array}{llll}
s, U \models p & \text{if and only if} & s \in v(p) & \\
s, U \models \varphi \wedge \psi & \text{if and only if} & s, U \models \varphi & \text{and } s, U \models \psi \\
s, U \models \neg\varphi & \text{if and only if} & s, U \not\models \varphi & \\
s, U \models \mathsf{K}\varphi & \text{if and only if} & t, U \models \varphi & \text{for all } t \in U \\
s, U \models \Box\varphi & \text{if and only if} & s, V \models \varphi & \text{for all } V \in \sigma \\
 & & & \text{such that } s \in V \subseteq U \\
s, U \models \mathsf{L}\varphi & \text{if and only if} & t, U \models \varphi & \text{for some } t \in U \\
s, U \models \Diamond\varphi & \text{if and only if} & s, V \models \varphi & \text{for some } V \in \sigma, \\
 & & & \text{such that } s \in V \subseteq U.
\end{array}
$$

Basics

Axiomatizatoin

The epistemic modality K is S5 whereas the dynamic modality $\Box$ is S4. Moreover, we will need an additional axiom to state the interaction between the two modalities: $K\Box\varphi \rightarrow \Box K\varphi$. As it is observable from the semantics, notice that the atomic sentences are independent from the neighborhood; thus: $(p \rightarrow \Box p) \wedge (\neg p \rightarrow \Box\neg p)$.

SSL is sound and complete with respect to the given axiomatization, and it is decidable.

Controlled Subset Spaces

∃-sickness!

The problem we focus on can be thought of as an instantiation of the "∃-sickness" problem. Recall that the $\Diamond$ operator only states that "there exists" a subset of the given observation set, but does not precisely indicate which subset is the intended one [1].

Controlled Subset Spaces

Models

Recall that f is a *contraction mapping*, if for every subset U in its domain, we have $fU \subseteq U$.

Let $\mathcal{F}$ be an arbitrary collection of contraction mappings, and further let $F \subseteq \mathcal{F}$ be some selection of such contraction mappings.

$\mathcal{S} = \langle S, \sigma, v, \mathcal{F} \rangle$ is called a controlled subset space where S is a set, σ is any collection of the subsets of S, $v : P \rightarrow \wp(S)$ is a valuation function and $\mathcal{F}$ is a collection of contraction mappings $\mathcal{F} = \{f : f$ is a contracting mapping and $f : S \rightarrow S\}$ defined on S.

Controlled Subset Spaces

Semantics

$$
\begin{array}{lll}
s, U \models_{\mathcal{S}} [F]\varphi & \textit{iff} & s, fU \models_{\mathcal{S}_F} \varphi \textit{ for each } f \in F \\
s, U \models_{\mathcal{S}} \langle F \rangle\varphi & \textit{iff} & s, fU \models_{\mathcal{S}_F} \varphi \textit{ for some } f \in F.
\end{array}
$$

where $\mathcal{S} = \langle S, \sigma, v, \mathcal{F} \rangle$ is the subset space, and the *image space* $\mathcal{S}_F = \langle S, \sigma_F, v, \mathcal{F} \rangle$ where $\sigma_F := \{fU : f \in F, U \in \sigma\}$.

Controlled Subset Spaces

Why?

We will use this structure to formalize Lakatosian heuristics. The mathematical methods will be represented by the functions, so that we will be able to *compare* them in terms of theory improvement.

Observations

Controlled State Elimination

Consider the observation set $U = \{s, c, k, t\}$ where s, c, k, t represent the sphere, the cylinder, the cube and the torus respectively.
The set U is the set of possible objects which can be seen as possible worlds at which some formulae about them will be valid or not. Let us assume, for simplicity, that the current state that we are occupying is s; thus U is the observation set for the agent at the state s.

Observations

Controlled State Elimination

Let f be the function which returns the input object x as the output only if the given object x satisfies the Euler conjecture $V(x) - E(x) + F(x) = 2$. More precisely, f is given as follows.

$$f(x) = \begin{cases} x & : \quad \text{if } V(x) - E(x) + F(x) = 2 \\ \text{undefined} & : \quad \text{otherwise} \end{cases}$$

Observe that f is a well-defined contraction mapping. The underlying motivation to define f as such is to mimic the characteristic function of the set of objects whose Euler characteristics are 2.

Observations

Controlled State Elimination

If we happen to consider some other Euler characteristics, we only need to include them as functions. In a similar fashion, let f' be the contraction mapping for the Euler characteristics 0. Similarly, the precise definition is as follows.

$$f'(x) = \begin{cases} x & : \quad \text{if } V(x) - E(x) + F(x) = 0 \\ \text{undefined} & : \quad \text{otherwise} \end{cases}$$

Observations

Controlled State Elimination - Lemma Incorporation

The method of lemma incorporation suggests us to extend our set of functions in consideration. One of the ways to achieve this to introduce the conditions that stems from the modified lemma into the formulation of the function. For instance, in PR, the notion of genus was introduced to discuss non-simple polyhedra. Then the general form of the Euler conjecture, as we discussed previously, becomes $V(x) - E(x) + F(x) = 2 - 2.g(x)$ in the oriented objects such as torus where $g(x)$ is the genus (i.e. the number of holes) of the object x.

$$h(x) = \begin{cases} x & : \text{ if } V(x) - E(x) + F(x) = 2 \wedge g(x) = 0 \\ \text{undefined} & : \text{ otherwise} \end{cases}$$

Examples

Simple Cases

Thus, $s, U \models \mathsf{L}(\chi = 0)$
where χ is the Euler's Formula and s, t are sphere and torus respectively where we have $\chi(s) = 2$ and $\chi(t) = 0$.

Similarly, we have $t, U \models \mathsf{L}(\chi = 0)$ or $t, U \models \Diamond(\chi = 0)$.
More precisely, $t, hU \models \mathsf{K}(\chi = 0)$

Generalization

Epistemic Concepts

Let $\Theta(\vec{x})$ be the theorem in question with free variables $\vec{x}$. We can formalize the Lakatosian heuristics of the development of the theorem $\Theta(\vec{x})$ with the input vector $\vec{x}$ as follows. For simplicity let us assume that the current epistemic neighborhood situation we are in is given (s, U). Furthermore, let A be the set of conditions incorporated.

$$f(\vec{x}) = \begin{cases} \vec{x} & : \text{if } \Theta(\vec{x}) \wedge A \text{ hold} \\ \text{undefined} & : \text{otherwise} \end{cases}$$

Thus: $s, fU \models \mathsf{K}\Theta(s)$

Criticism

Lakatosian Methodology

It has been claimed that Lakatos's *rationally reconstructed account* of the history of the development of the Euler theorem often diverged from the actual history of the subject. Kvasz, for instance, asserted that this was due to "his confusion of dialectic with logic" [6]. Furthermore, Koetsier stated that "there is no doubt that *Proofs and Refutations* contains a highly counterfactual rational reconstruction." [5].

Possibility

Formalism of Lakatosian Heuristics

The underlying idea for the possibility of employing such methods in Lakatosian philosophy of mathematics is Lakatos's understanding that mathematics is a quasi-empirical activity. For Lakatos, thought-experiments reflect the empiric side of the mathematical practice. Our formalization reflects this point, too. We start from a single formula f, then extend it to a set of formula F, and finally experiment with the different formulae in F to see how they interact with the geometrical objects in question, and finally modify our set of possible worlds if necessary.

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Thanks!

Questions or Comments?

Talk slides and the paper are available at:

www.canbaskent.net