Some Logical Approaches to Lakatos's *Proofs and Refutations*

Can Başkent

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Outline

- Why Lakatos, Why *Proofs and Refutations*?
- Interrogative Approaches
- Dialetheic Approaches
- Computational Approaches

This is a work-in-progress, comments are welcomed. For a full list of references, see the preceding papers of mine on the subject.

Why Lakatos, Why *Proofs and Refutations*?

- *Almost* historically accurate
- Strong game theoretical flavor
- Didactic way to approach theorem development
- Dialogical method of presenting the arguments
- Underlying dialectical framework

3Ds of PR: **didactic, dialogical, dialectical**

Why Lakatos, Why *Proofs and Refutations?*

***Almost* historically accurate**

- It is quite clear that Lakatos distorted the history of mathematics to allow his "rational reconstruction" in PR

- A very very detail oriented research on the history of mathematics

Why Lakatos, Why *Proofs and Refutations?*

Strong game theoretical flavor

- I suggested earlier that Lakatos's method presents a very clear game theoretical framework

- Game is between the prover and the refuter. They present examples and counterexamples. They modify, refute or improve the proof

- It can be seen as a dialog game

Why Lakatos, Why *Proofs and Refutations?*

A didactic way to approach theorem development

- Theorems and proofs are developed, improved, and refuted.
- Similar to natural sciences, theories develop or regress in mathematics
- Axiomatic method has never been the method for the increase of mathematical knowledge. Instead he suggests heuristics - experience / experiment based problem solving, similar to Pólya
- Is mathematics deductive or inductive?

Why Lakatos, Why *Proofs and Refutations?*

Dialogical method of presenting the arguments

- The method is also Socratic.
- Lakatos himself admires elenchus
- Focuses on the mathematical practice, not on axiomatization

Why Lakatos, Why *Proofs and Refutations?*

Underlying dialectical framework

- Strong dialectic (*à la* Hegel and Marx) flavor: proof and its refutation are both present
- Some objects tend to behave as examples and counterexamples
- I also claim, dialectic nature of PR leads to paraconsistency

Lakatos on Formalism

“I call a deductive system a "Euclidean theory" if the propositions at the top (axioms) consists of perfectly well-known terms (primitive terms) and if there are infallible truth-value injections … which flows downward through the deductive channels of truth-transmission (proofs) and inundates the whole system. Since a Euclidean theory contains only indubitably true propositions, it operates neither with conjectures nor with refutations. In a fully-fledged Euclidean theory meaning, like truth, is injected at the top and it flows down safely through meaning-preserving channels of nominal definitions from the primitive terms to the (abbreviatory and therefore theoretically superfluous) defined terms. A Euclidean theory is *eo ipso* [by form] consistent, for all the propositions occurring in it are true, and a set of true propositions is certainly consistent.”

Lakatos on Proofs - 1

“Proofs in axiomatized theories can be submitted to a peremptory verification procedure, and this can be done in a fool proof, mechanical way. Does this mean that for instance if we prove Euler's theorem in Steenrod and Eilenberg's fully formalized postulate system it is impossible to have any counterexample? Well, it is certain that we won't have any counterexample formalizable in the system assuming the system is consistent; but we have no guarantee at all that our formal system contains the full empirical or quasi-empirical stuff in which we are really interested and with which we dealt in the informal theory. There is no formal criterion as to the correctness of formalization.

Lakatos on Proofs - 2

Well-known examples of 'falsified' formalizations are (1) the formalization of the theory of manifolds by Riemann, where thereis no account of Mobius-strips; (2) the Kolmogorov axiomatization of probability theory, in which you cannot formalize such intuitive statements as 'every number turns up in the set of natural numbers with the same probability'(*). As a final but most interesting example I should mention (3) Gödel's opinion that the Zermelo-Fraenkel and kindred systems of formalized set theory are not correct formalizations of pre-formal set theory as one cannot disprove in them Cantor's continuum hypothesis."

(*) Alfréd Rényi, On a new axiomatic theory of probability, Acta Mathematica Hungarica, vol6, no 3-4 (1955) — CB.

Lakatos on Deductivism - 1

"Euclidean methodology has developed a certain obligatory style of presentation. I shall refer to this as 'deductivist style'. This style starts with a painstakingly stated list of axioms, lemmas and/or definitions. The axioms and definitions frequently look artificial and mystifyingly complicated. One is never told how these complications arose. The list of axioms and definitions is followed by the carefully worded theorems. These are loaded with heavy-going conditions; it seems impossible that anyone should ever have guessed them. The theorem is followed by the proof.

Lakatos on Deductivism - 2

In deductivist style, all propositions are true and all inferences valid. Mathematics is presented as an ever-increasing set of eternal, immutable truths. Counterexamples, refutations, criticism cannot possibly enter.

Deductivist style hides the struggle, hides the adventure. The whole story vanishes, the successive tentative formulations of the theorem in the course of the proof-procedure are doomed to oblivion while the end result is exalted into sacred infallibility”

Method of Proofs and Refutations

A logic of mathematical discovery

1. Primitive Conjecture

2. "A proof" - as a thought-experiment

3. Counter-examples: some are global (for the theorem), some are local (for the lemmas)

4. Proof re-examined - "guilty" lemmas are found, proof is fixed and improved

Method of Proofs and Refutations

Counterexamples are suggested and attempts are made to evade them: by denying that they indeed are examples ("monster-barring"), by denying that they are genuine counterexamples ("monster-adjusting"), or by accepting them as "exceptions" to the theorem ("exception-barring").

Progress finally results when the meaning of terms is extended beyond its intuitive limits. "Hidden lemmas" are searched. Refuted lemmas are incorporated into the theorem as restrictions.

For example, we get, instead of "All polyhedra are Eulerian"; "All polyhedra with simply connected faces that are topologically equivalent to a sphere are Eulerian."

Method of Proofs and Refutations

In short

- Informal, quasi-emprical mathematics
- Growth by discussion, argumentation, speculation and guess - similar to empirical sciences
- Completely against Hilbertian or Euclidian school - even at the presentation level

Proofs and Refutations

- *Proofs and Refutations* [PR] focuses on Euler's Theorem for polyhedra: $V - E + F = 2$
- Takes place in a classroom setting where students represent mathematicians who worked on the thoerem: L'Huilier, Gergonne, Cauchy, Hessel, Kepler, Poincaré, Matthiessen etc.
- Focuses on the actual history of the theorem, and its proofs and disproofs

Proofs and Refutations

The classroom discusses teacher's conjecture:

For all polyhedra, $V - E + F = 2$.

Then, the teacher offers Cauchy's well-known proof:

Proofs and Refutations - the proof

Cauchy's proof for *V* - *E* + *F* = 2

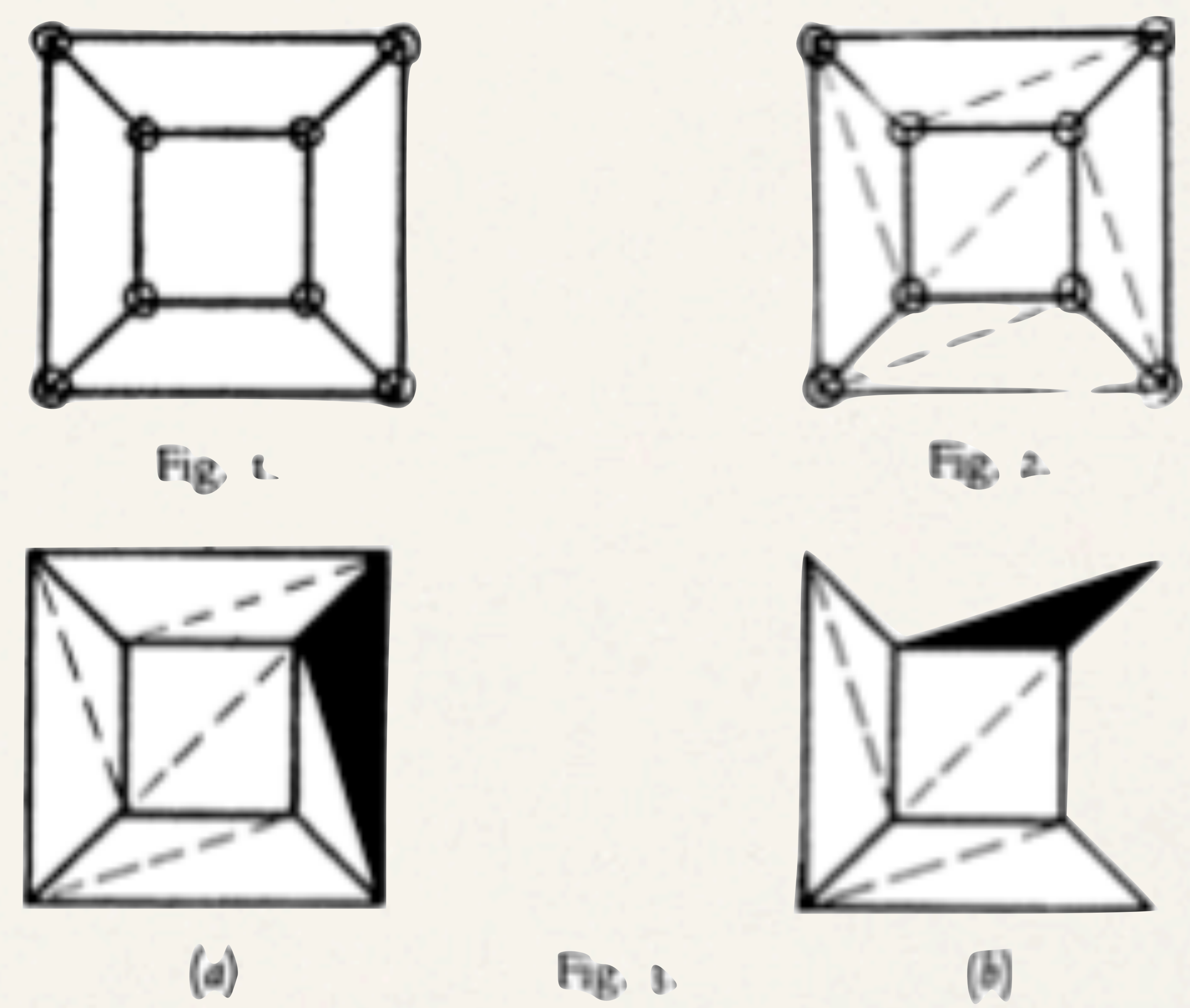
1. Imagine that the polyhedra is hollow and made with thin rubber. If we cut out one of the edges, we can stretch the remaining surface flat on the board. Faces and edges will be deformed, and edges will be curved, but V and E won't alter. But, we will have *V - E - F = 1*

2. Triangulate the stretched surface by drawing diagonals. For each diagonal, *E* and *F* increase by 1, so the total *V - E - F* won't change.

3. Remove the triangles one by one. So, either remove an edge, or remove two edges and a vertex to achieve this. At the end, we obtain one single triangle, for which *V - E - F = 1* holds true.

Proofs and Refutations - the proof

a diagram:



Proofs and Refutations - refutations

Counter-examples to all three steps:

1. How do we know, we can flat the polyhedra after removing a face?

   Consider the nested-cube:

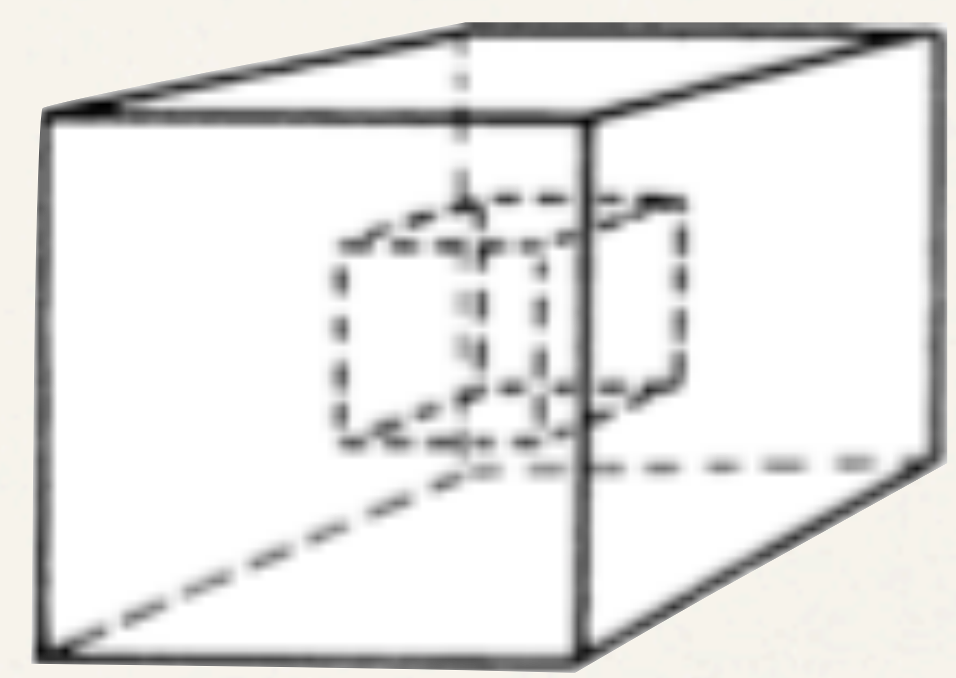


Fig. 5.

Proofs and Refutations - refutations

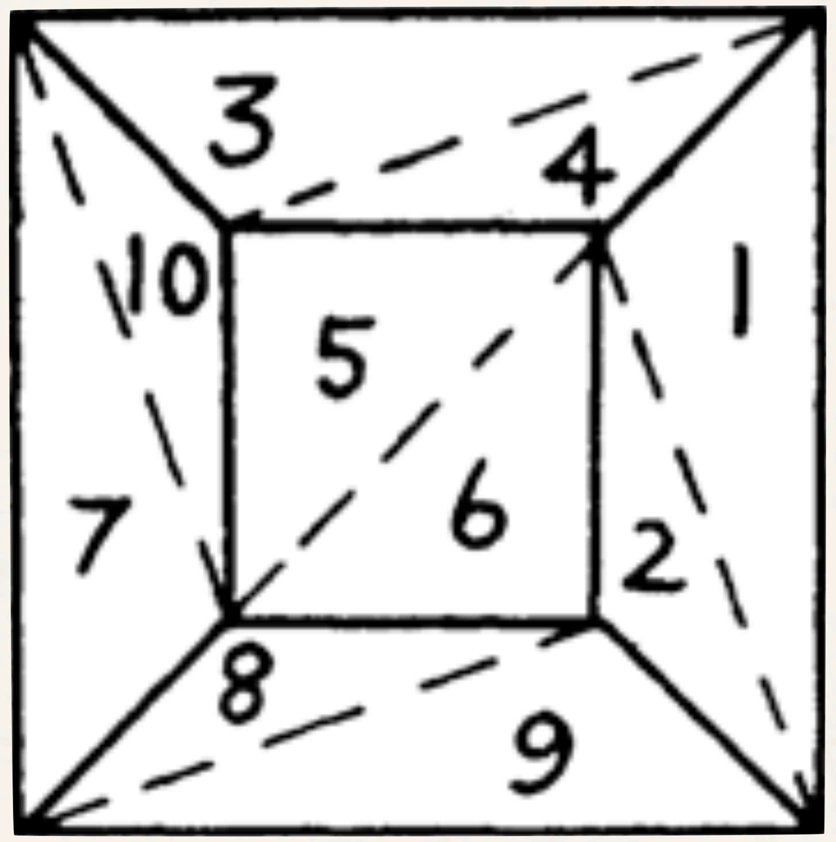
2. How do we know that when we triangulate, $V - E - F$ won't change or that we can triangulate the surface of every polyhedra.

   Consider the cylinder.

Proofs and Refutations - refutations

3. How do we know, when removing the triangles $V - E + F$ won't change.

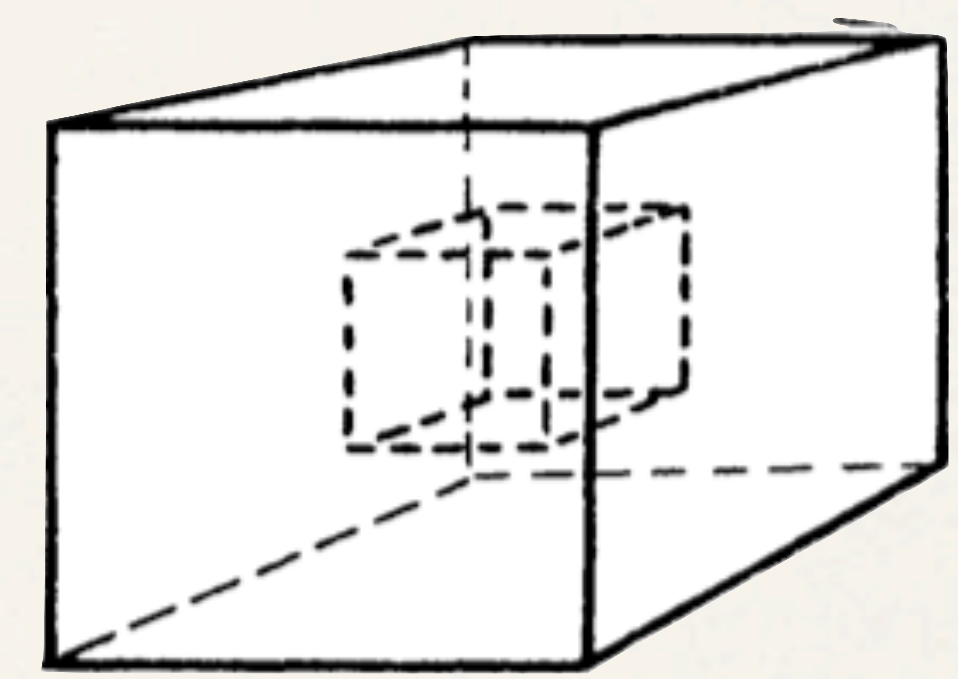
   Consider removing triangles from inside the network - not from outside, thus breaking the connectedness.



It seems that there is a *hidden* method for removing the triangles.

Proofs and Refutations - some counter-examples

Hollow Cube whose Euler Characteristics is 4



Torus whose Euler Characteristics is 0

Twin polyhedra whose Euler Characteristics is 3 (for both)

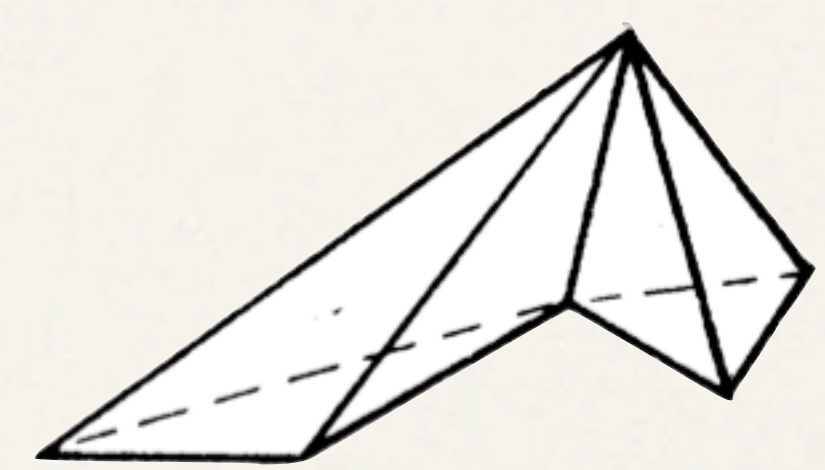


FIG. 6*a*

FIG. 6*b*

Proofs and Refutations - refutations

- Refutations, counter-refutations; examples, counter-examples are presented one after another in PR.
- Some target objects (Is picture-frame a polygon?), some defining terms (what is an edge or a vertex in a cylinder?), some target the theorem (for torus, $V - E + F \neq 2$)
- Then, the thought-experiment (the initial proof) is revised and revised...

Proofs and Refutations - an excerpt - 1

GAMMA Why not? A polyhedron is a solid whose surface consists of polygonal faces. And my counterexample (hollow cube) is a solid bounded by polygonal faces.

TEACHER Let us call this definition Def.1.

DELTA Your definition is incorrect. A polyhedron must be a surface: it has faces, edges, vertices, it can be deformed, stretched out on a blackboard, and has nothing to do with the concept of 'solid'. A polyhedron is a surface consisting of a system of polygons.

TEACHER Call this Def.2.

Proofs and Refutations - an excerpt - 2

DELTA So really you showed us two polyhedra - two surfaces, completely inside the other. A woman with a child in her womb is not a counterexample to the thesis that human beings have one head.

TEACHER Can you refute our conjecture now if by polyhedron we mean a surface?

ALPHA Certainly. Take two tetrahedra which have an edge in common Or, take two tetrahedra which have a vertex in common. Both these twins are connected, both constitute one single surface. And, you may check that for both V-E+F=3.

Proofs and Refutations - an excerpt - 3

DELTA I admire your perverted imagination, but of course I did not mean that any system of polygons is apolyhedron. By polyhedron I meant a system of polygons arranged in such a way that (1) exactly two polygons meet at every edge and (2) it is possible to get from the inside of any polygon to the inside of any other polygon by a route which never crosses any edge at a vertex. Your first twins will be excluded by the first criterion in my definition, your second twins by the second criterion.

ALPHA I admire your perverted ingenuity in inventing one definition after another as barricades against the falsification of your pet ideas. Why don't you just define a polyhedron as a system of polygons for which the equation V- E + F= 2 holds, and this Perfect Definition would settle the dispute for ever? There would be no need to investigate the subject any further.

Proofs and Refutations - an excerpt - 4

DELTA But there isn't a theorem in the world which couldn't be falsified by monsters.

TEACHER I am sorry to interrupt you. As we have seen, refutation by counterexamples depends on the meaning of the terms in question. If a counterexample is to be an objective criticism, we have to agree on the meaning of our terms. We may achieve such an agreement by defining the term where communication broke down. I, for one, didn't define 'polyhedron'. I assumed familiarity with the concept, i.e. the ability to distinguish a thing which is a polyhedron from a thing which is not a polyhedron - what some logicians call knowing the extension of the concept of polyhedron. It turned out that the extension of the wasn't at all obvious: definitions are frequently proposed and argued about when counterexamples emerge.

Interrogative Approach for Inquiry

- **Hintikka's interrogation game for knowledge increase: either ask a question or perform a deduction.**

  Inquiry is "the action of seeking for truth, knowledge or information" about something. An inquirer is an active observer, and enforces "nature to give answers to questions".

  It is therefore a learning process - a heuristics.

Interrogative Approach

- However, there should be some method for asking questions - a strategy
- It is a game with two types of moves: interrogative and deductive
- For Hintikka, deductive moves are tableaux constructions, interrogative moves are adding premises to the theory

Interrogative Approach

- For Hintikka, deductive moves are tableaux constructions, interrogative moves are adding premises to the theory
- We are familiar with the interrogative moves from Lakatos's PR
- Inquiring about counterexamples are answering questions: is torus a polyhedra? If yes, then what about the initial proof? If not, what is it? What is a polyhedra then?
- You learn from counterexamples: sometimes the answer is in the proof - so you perform deduction; sometimes, you need to expand the theorem to include the counterexamples as examples

Interrogative Approach

- **Hintikka's system allows *bracketing* the irrelevant answers.**

  To me, this is not clear and seems controversial.

  If you don't know the answer, how can you decide if a particular answer is irrelevant?

Interrogative Approach

- **Hintikka's system is Socratic**. This is another similarity to the method of proofs and refutations. Hintikka says of elenchus:

  "The story, as I see it, begins with Socrates and his method of elenchus, or in other words, his questioning method. We all think we know what this method is all about. In reality, however, Socratic elenchus is full of logical subtleties even though on the surface it proceeds deceptively smoothly. Socrates is engaged in a question-answer dialogue with an interlocutor. He begins with an initial thesis which is often obtained as a response to Socrates' initial or, as I shall call it, principal question put to his dialogue partner. Socrates then addresses further questions to the other party, and eventually the subsequent answers lead him to a conclusion concerning the initial thesis, typically, to the rejection of this thesis. "

Interrogative Approach

- Interrogation is a dynamic epistemic logic of questions
- Various logical frameworks have been developed to represent Hintikka's approach although we will not discuss them

Dialetheic Approach

- Dialethism is the view that says that there are true contradictions
- Notice that dialethists do not say that all contradictions are true
- Therefore, dialethism can be considered, from perhaps a bit computer scientific perspective, as an enterprise to *decide* which contradictions are true, and when they are true

Dialetheic Approach

- A Lakatosian paradigm: *"Proofs that do not prove"*

- I claim Lakatosian analysis of practice based mathematics requires dialethism in its meta-theory. The practice never ends, counter-examples emerge and disappear, theorems are refuted and verified, counter-examples are turned into examples or discarded etc..

- For a classical philosopher of science, there is a puzzle here: which one is the final theory? How should I know I have reached *the truth*?

- Lakatosian method is intrinsically Hegelian (who thought that contradictions can be realized in various situations) and perhaps Marxist.

Dialetheic Approach

- Lakatosian strategy at first looks like a belief-revision based dynamic epistemology.
- However, the method itself relies on contradictions - it tries to fix them, but on the other hand, due to its strong rejection of formalism, leaves some aside.
- This is largely due to its dialectic nature. Because "dialectic is a notion of contradiction". Hegel himself saw the dialectic logic applicable to a larger domain than the formal logic (where the principle of non-contradiction applies).

Dialetheic Approach

- Hegel writes:

  “ ... Common experience... says that... there is a host of contradictory things, contradictory arrangements, whose contradiction exists not merely in external reflection, but in themselves”.

Dialetheic Approach

- Priest writes:

  "Hegel distinguished between dialectics and formal logic - which was for him the Aristotelian logic of his day. The law of non-contradiction holds in formal logic; but formal logic is correctly applicable only in a limited and well defined area (notably the static and changeless); in dialectical logic, which applies in a much more general domain, the law of non-contradiction fails."

Dialetheic Approach

- Lakatos did not hide that Hegel had a significant influence on him.
- The method of proofs and refutations is dialectic
- Therefore, I claim is dialetheic, too
- It is not unusual to accept **and** reject some objects as examples (or counterexamples), theorems may change, theories develop, yet the object-level dialetheism is maintained

Computational Approaches

- Pease modeled Lakatos's PR aiming
  - to identify the areas where Lakatos was vague, and aspects he omitted
  - to test and compare the hypothesis

Computational Approaches

- Such methods give precise and step-by-step description of Lakatosian methodology
- Moreover, it can be claimed that Lakatosian method employs a primitive notion of computation: a traceable process based on some primitive computations such as *monster barring*, l*emma incorporation*

Conclusion

- I believe PR is a treasure for logicians, philosophers and mathematicians
- Future work possibilities abound
- The **logic** of scientific discovery is a very central notion in philosophy of science, and the subject needs working logicians attention

Thank you!

"BETA: But I had no problems at the beginning! And now I have nothing *but* problems!"

Talks slides and papers are available at

CanBaskent.net/Logic