A Logic of Isolation

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June 10-13, 2024 // Workshop on Logic, Language, Information and Computation (WoLLIC)

In a Nutshell

Topological isolation operator

enjoys the richness of topologies

with a well-defined modality,

and can be used to express “factive ignorance”.

Overview

1. Introduction

2. Isolated Points

3. Logic of Isolated Points

4. Topological Semantics for Logic of Isolation

5. Conclusion

Introduction

Basics of Topological Modal Models i

Topological semantics remains the oldest semantics for modal logics.

Modal operators have traditionally been identified with interior and closure operators, as well as, less often so, with border and boundary operators.

Basics of Topological Modal Models ii

The tuple (S, σ) is topological space for set S and the collection σ of subsets of S satisfying the following axioms:

1. The empty set and S are in σ
2. σ is closed under arbitrary unions,
3. σ is closed under finite intersections.

The collection σ is called a **topology**. We call the elements of σ **opens**, and their complements **closeds**.

The largest open set contained in a set is called the **interior**. The smallest closed set containing a given set is called the **closure**.

The interior operator Int returns the interior of a given set. Dually, the closure operator Clo returns the closure of a given set.

In modal logic, Int is identified with the classical modal operator $\Box$. Dually and similarly, Clo is identified with $\Diamond$.

In a topological model $M = (S, \sigma, V)$, where V is a valuation function in the ordinary sense, the semantics of the topological modal operators is given as follows, for $w \in S$.

$$
\begin{array}{lll}
M, w \models \Box\varphi & \textit{iff} & \exists U \in \sigma\,(w \in U \land \forall v \in U\,.\,M, v \models \varphi) \\
M, w \models \Diamond\varphi & \textit{iff} & \forall U \in \sigma\,(w \in U \to \exists v \in U\,.\,M, v \models \varphi)
\end{array}
$$

Notice the difference in complexity of the topological semantics for modalities!

Under these assumptions, McKinsey and Tarski showed in 1944 that the modal logic of topological spaces is S4 by identifying the axioms of S4 modal logic with the axioms of the closure/interior operator.

S4 is the logic of reflexive and transitive frames, characterised by the following axioms.

- *All axioms of propositional logic*
- $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$
- $\Box\varphi \to \varphi$
- $\Box\varphi \to \Box\Box\varphi$

Reference

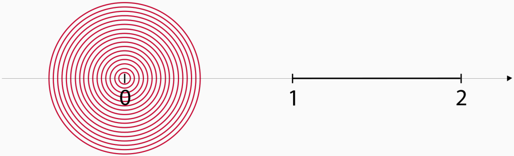
J. C. C. McKinsey and A. Tarski, "The algebra of topology", *Annals of Mathematics*, vol. 45(1), pp. 141–191, 1944.

Isolated Points

Isolated Points

Given a topological space (S, σ) and $X \subseteq S$, the point x is an **isolated** point of X if there is an open neighbourhood U of x such that $U \cap S = \{x\}$.

Here is an illustration from Wikipedia, showing that 0 is an isolated point in the set $[1, 2] \cup \{0\}$, with natural Euclidean topology:



Isolated Points: Examples

A point x is isolated in space (S, σ) if and only if $\{x\}$ is open.

In $\mathbb{N}$, each natural number $n \in \mathbb{N}$ is isolated.

Logic of Isolated Points

Logic of Factive Ignorance

Factive ignorance postulates that

If an agent is ignorant of φ, then φ is true.

In a classical Kripke semantics, its semantics is given as follows for a model $M = \langle W, R, V \rangle$

$$M, w \models \mathrm{I}\varphi \quad \textit{if and only if} \quad \forall w' \neq w \textit{ and such that } Rww', \; M, w' \not\models \varphi \textit{ and } M, w \models \varphi$$

where $\mathrm{I}\varphi$ represents that "the agent is ignorant that φ".

Reference

Kubyshkina, E. and Petrolo, M, "A logic for factive ignorance", *Synthese*, vol. 198, pp. 5917–5928, 2021.

Logic of Isolation: Syntax

The syntax of Logic of Isolation is as follows:

$$\alpha ::= p \mid \neg\alpha \mid \alpha \wedge \alpha \mid [\mathrm{i}]\alpha$$

Logic of Isolation: Proof System

The axioms and the proof rules of Logic of Isolation are given as follows:

- All instances of propositional tautologies,
- $[\mathrm{i}]\varphi \to \varphi$,
- $([\mathrm{i}]\varphi \wedge [\mathrm{i}]\psi) \to [\mathrm{i}](\varphi \wedge \psi)$,
- From $\vdash \varphi$ and $\vdash \varphi \to \psi$ infer $\vdash \psi$,
- From $\vdash \varphi \to \psi$ infer $\vdash \varphi \to ([\mathrm{i}]\psi \to [\mathrm{i}]\varphi)$

We call this system S^i.

Logic of Isolation: Some Theorems

The following are all theorems of S^i.

- $([\mathrm{i}]\varphi \wedge [\mathrm{i}]\psi) \rightarrow [\mathrm{i}](\varphi \wedge \psi)$
- $[\mathrm{i}](\varphi \vee \psi) \rightarrow ([\mathrm{i}]\varphi \vee [\mathrm{i}]\psi)$
- $[\mathrm{i}]\varphi \rightarrow [\mathrm{i}][\mathrm{i}]\varphi$

One can also derive $\vdash [\mathrm{i}]\varphi \leftrightarrow [\mathrm{i}]\psi$ from $\vdash \varphi \leftrightarrow \psi$.

Logic of Isolation: Semantics

Relational modal semantics for S^i is given as follows:

$M, w \models [\mathrm{i}]\varphi$ iff ($M, w \models \varphi$ and $\forall w' \neq w\,(wRw'$ implies $M, w' \not\models \varphi$))

Theorem

S^{i} *is sound and strongly complete with respect to the class of all relational frames.*

Reference

D. Gilbert, E. Kubyshkina, M. Petrolo, and G. Venturi, "Logics of Ignorance and Being Wrong", *Logic Journal of the IGPL*, vol.30(5), pp. 870–885, 2021.

Topological Semantics for Logic of Isolation

Logic of Isolation: Neighbourhood Semantics

A *neighborhood frame* is a pair $\langle X, N \rangle$ such that $X \neq \emptyset$ and $N : X \mapsto \wp(\wp(X))$.

A *neighborhood model* is a pair $\langle F, V \rangle$, where F is a neighborhood frame and $V : Prop \mapsto \wp(X)$ is a valuation function.

Standard topological definition of isolation can be extended to neighborhood systems, where less mathematical structure is insisted upon:

x is an isolated point of S if there is a neighborhood U of x, i.e., $U \in N(x)$, such that $U \cap S = \{x\}$.

Logic of Isolation: Neighbourhood Semantics

Given a model M and a formula α, the *truth set* of α in M is denoted by $[\![\alpha]\!]^M$. We will omit the superscript for easy reading.

$$
\begin{array}{lcl}
[\![p]\!] & := & V(p) \\
[\![\neg\varphi]\!] & := & X \setminus [\![\varphi]\!] = ([\![\varphi]\!])^c \\
[\![\varphi \wedge \psi]\!] & := & [\![\varphi]\!] \cap [\![\psi]\!] \\
[\![[\mathrm{i}]\varphi]\!] & := & \{x : \exists U \in N(x) \text{ s.t. } U \cap [\![\varphi]\!] = \{x\}\}
\end{array}
$$

Logic of Isolation: Reflexive-Insensitivity

The language of S^{i} is reflexive-insensitive.

The satisfaction of formulas in the language of S^{i} in a model $M = \langle W, R, V \rangle$ is not affected when arbitrary elements of the form (w, w), where $w \in W$, are either added to or removed from R.

Particularly, because the definition of $[\![\cdot]\!]$ utilizes only sets of each $N(x)$ that contain x, the addition or removal of sets that do not contain x will be immaterial.

Logic of Isolation: Reflexive-Insensitivity

For a given neighborhood frame $\langle X, N \rangle$, consider the set

$$S_x := \{Y \in \wp(X) : x \notin Y\}$$

Given a neighborhood frame $F = \langle X, N \rangle$, construct the frames $F = \langle X, N^+ \rangle$ and $F = \langle X, N^- \rangle$, where N^+ and N^- are defined as follows for all $x \in X$:

$$N^+(x) := N(x) \cup S_x$$

$$N^-(x) := N(x) \setminus S_x$$

Logic of Isolation: Reflexive-Insensitivity

If $M = \langle X, N, V\rangle$, then $M^+ = \langle X, N^+, V\rangle$ and $M^- = \langle X, N^-, V\rangle$.

Theorem

Let M be a neighborhood model. Then M, M^+, and M^- (as well as the intermediate models) are all pointwise equivalent. That is,

$$[\![\alpha]\!]^{M^-} = [\![\alpha]\!]^{M} = [\![\alpha]\!]^{M^+}$$

for all formulas α in the language of S^i.

Logic of Isolation: Neighbourhood Models

Supplemented Neighbourhood Models A neighborhood frame is *supplemented* when its neighborhood function is closed under supersets: for every x, if $Y \subseteq N(x)$ and $Y \subseteq Z$, then $Z \subseteq N(x)$.

Given a neighborhood frame, $F = \langle X, N \rangle$, let $F^s = \langle X, N^s \rangle$ be the *supplementation* of F when, for all $x \in X$:

$$N^s(x) = \{Y \subseteq \wp(X) : \exists U \subseteq Y \text{ s.t. } U \in N(x)\}$$

For a model $M = \langle F, V \rangle$, let $M^s = \langle F^s, V \rangle$.

Logic of Isolation: Neighbourhood Models

Anchored Neighborhood System A neighborhood function (and, hence, the resulting system) is *anchored* when, for every point $x \in X$,

$$\forall U \in N(x)(x \in U)$$

Note that we do not force $N(x) \neq \emptyset$ in order to be anchored.

Theorem

Let M be an anchored neighborhood model. Then

$$[\![\alpha]\!]^M = [\![\alpha]\!]^{M^s}$$

Logic of Isolation: Soundness and Completeness

Theorem

S^{i} *is sound with respect to the class of neighborhood frames that are closed under intersections.*

Theorem

S^{i} *is strongly complete with respect to the class of neighborhood frames that are anchored and closed under intersections.*

Logic of Isolation: Soundness and Completeness

Theorem

S^{i} *is strongly complete with respect to the class of neighborhood frames that are anchored, closed under intersections, and supplemented.*

Theorem

S^{i} *is sound and (strongly) complete with respect to the class of all augmented neighborhood frames and all anchored, augmented neighborhood frames.*

Logic of Isolation: Discrete Neighborhoods

A neighborhood system is *discrete* when $\{x\} \in N(x)$, for every $x \in X$.

Consider the following axiom schema:

$$\varphi \leftrightarrow [\mathrm{i}]\varphi \tag{Disc}$$

Call **SDisc** the system obtained by adding (Disc) to $\mathbf{S}^i$.

In the presence of (Disc), no other modal axioms are necessary and neither is the rule "From $\vdash \varphi \to \psi$ infer $\vdash \varphi \to ([\mathrm{i}]\psi \to [\mathrm{i}]\varphi)$".

Logic of Isolation: Discrete Neighborhoods

SDisc can be axiomatized by the following:

- All instances of propositional tautologies,
- $\varphi \leftrightarrow [\mathrm{i}]\varphi$,
- From $\vdash \varphi$ and $\vdash \varphi \to \psi$ infer $\vdash \psi$

Theorem

SDisc *is valid on a relational frame F if and only if, for each w, if wRz, then* $w = z$.

Logic of Isolation: Discrete Neighborhoods

Theorem

SDisc *is sound and strongly complete with respect to the class of relational frames in which, for each* w*, if* wRz*, then* $w = z$*.*

Theorem

SDisc *is sound and strongly complete with respect to the class of discrete neighborhood systems.*

(Hence, due to the semantic insensitivities noted above, also with respect to anchored, discrete, supplemented neighborhood systems.)

Logic of Isolation: Discrete Topologies

Let us now transition to topologies proper.

Satisfaction at points in a topo-model is defined as usual, with the clause for [i] resembling closely the one given for neighborhood systems, but with reference to the topology σ rather than the neighborhood function N:

$$[\![[\mathrm{i}]\varphi]\!] := \{x : \exists U \in \sigma \text{ s.t. } U \cap [\![\varphi]\!] = \{x\}\}$$

Logic of Isolation: Discrete Topologies

With the semantics so defined, S^{i} is sound with respect to the class of all topo-models.

Moreover, SDisc is sound with respect to the class of all discrete topological spaces, since all singletons are open (and closed).

Theorem

SDisc *is complete with respect to the class of all discrete topological spaces.*

Conclusion

Conclusion

We presented a geometric interpretation of $[\mathrm{i}]$ as an isolated points operator in a variety of systems.

The next question is whether there exist intermediate logics (between S^{i} and SDisc) that characterize interesting classes of either neighborhood systems or topologies.

The extensions of S^{i} with only neighborhood systems in mind is an interesting direction to pursue. For example, one can consider adding to S^{i} the axiom $\neg[\mathrm{i}]\top$.

Relationally, this forces all worlds to be non-reflexively serial (for each x, there is a $y \neq x$ such that xRy). This axiom forces a lack of discreteness (hence, the resulting logic is not an intermediate logic, but inconsistent with SDisc), and that it is sound and strongly complete with respect to the class of all neighborhood systems in which $\{x\} \notin N(x)$.

Acknowledgement

CB's work on this paper was supported by a London Mathematical Society Computer Science Scheme 7 grant
[grant number SC7-2022-05].

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In a Nutshell

Topological isolation operator

enjoys the richness of topologies

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Thank you!

Talk slides are available at my website

CanBaskent.net/Logic