Meditations on Subset Space Logic

Can Başkent

Department of Computer Science
Graduate Center, the City University of New York
cbaskent@gc.cuny.edu // www.canbaskent.net

October 6, 2009 - Graduate Center

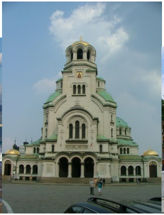
Contents

Some parts of this talk is joint work with Rohit Parikh.

This talk is based on some previous talks given at:

- "International Workshop in Modal Logic" - Istanbul/Turkiye
- "Panhellenic Logic Symposium" - Patras/Greece
- "ASL Logic Colloquium" - Sofia/Bulgaria
- "GAMES 2009 Workshop" - Udine/Italy






How to Start?

What about the games / epistemic situations with uncertainty?

Consider the dart game: you aim at a point, and the dart hits at a point *around* your aim. By construction, there is *some* uncertainty involved. Assuming the players are rational, you can assume *some* level of uncertainty as they will not aim at somewhere other than the dart board.

Thus, notion of *closure* which is conceptually familiar from topology can be used to understand uncertainty in dynamic situations.

Road Map

We will consider two well-defined logics: an epistemic one and a dynamic/game theoretical one. Then merge them in a meaningful way.

En route, I will also discuss dynamic epistemic settings as a first step towards complete dynamic setting.

Epistemic constructions will then emphasize the strategies and will make them the focus of our work[1].

[1] Thanks to R.Ramanujam for pointing this out.

Hidden Agenda

We will utilize a dynamic logic which depends on Propositional Dynamic Logic. Thus, our game theoretical approach is a step towards the geometrical understanding of dynamic logics (one-sorted or many-sorted).

Basics

Subset space logic (SSL) formalizes reasoning about sets and points with an underlying motivation of embedding the geometrical notion of *closeness* into epistemic logic [4].

The key idea of SSL can be formulized as follows: "In order to *get close*, one needs to spend some *effort*." Thus, In SSL, the knowledge is defined with respect to both a *point* and a *neighborhood* of that point.

A subset space model is a triple $\langle S, \sigma, v \rangle$ where S is a set of points and $\sigma \subseteq \wp(S)$ and v is a valuation function.

Syntax and Semantics

We have two modalities: Knowledge (K) and Effort ($\Box$) with the usual syntax.

$$
\begin{array}{lll}
s, U \models p & \text{iff} & s \in v(p) \\
s, U \models \varphi \wedge \psi & \text{iff} & s, U \models \varphi \text{ and } s, U \models \psi \\
s, U \models \neg\varphi & \text{iff} & s, U \not\models \varphi \\
s, U \models K\varphi & \text{iff} & t, U \models \varphi \text{ for all } t \in U \\
s, U \models \Box\varphi & \text{iff} & s, V \models \varphi \text{ for all } V \subseteq U \text{ for } V \in \sigma
\end{array}
$$

Axioms

The axioms of SSL simply reflect the fact that the K modality is S5-like whereas the $\Box$ modality is S4-like. Moreover, we need an additional axiom to state the interaction between the two modalities: $\mathsf{K}\Box\varphi \rightarrow \Box\mathsf{K}\varphi$.
Yet another important fact is that the atomic sentences are independent from their neighborhoods, thus the following axiom for atomic sentence F is valid in SSL: $(F \rightarrow \Box F) \wedge (\neg F \rightarrow \Box\neg F)$.
Moreover, SSL is sound and complete with respect to the aforementioned axiomatization.
Furthermore, it is decidable.

Dynamic Epistemology on Subset Spaces

Public announcement logic deals with knowledge updates with a state elimination based paradigm [6].
Consider $[\varphi]\psi$ with the intended meaning that *after the public announcement of* φ, ψ *holds*. The important restriction is the fact that both φ and ψ should be basic modal formulae. Thus, an announcement cannot be announced.

Public Announcement in SSL

Public announcements in SSL simply shrinks the neighborhood. After the announcement φ which is true at the neighborhood situation, we obtain a smaller neighborhood U_φ which can be defined as $U_\varphi = U \cap (\varphi)_2$ where $(\varphi)_2 = \{U : (s, U) \in (\varphi) \text{ for some } s\}$ for the extension (φ). Similarly, for a given subset space model $\mathcal{S} = \langle S, \sigma, V\rangle$, we get the updated model $\mathcal{S}_\varphi = \langle S_\varphi, \sigma_\varphi, V_\varphi\rangle$ after the announcement φ. In this context, $S_\varphi = S \cap (\varphi)_1$ where $(\varphi)_1 = \{s : (s, U) \in (\varphi) \text{ for some } U\}$, and $\sigma_\varphi = \{U \cap S_\varphi : U \in \sigma :\}$, and $V_\varphi = V \cap S_\varphi$, as expected[2].

[2] Thanks to A. Kudinov for bringing some of these points to my attention.

Axioms

The following axiomatizes the PAL in SSL.

Atoms	$[\varphi]p \leftrightarrow (\varphi \rightarrow p)$
Partial Functionality	$[\varphi]\neg\psi \leftrightarrow (\varphi \rightarrow \neg[\varphi]\psi)$
Distribution	$[\varphi](\psi \wedge \chi) \leftrightarrow ([\varphi]\psi \wedge [\varphi]\chi)$
Knowledge Announcement	$[\varphi]K\psi \leftrightarrow (\varphi \rightarrow K[\varphi]\psi)$
Effort Announcement	$[\varphi]\Box\psi \leftrightarrow (\varphi \rightarrow \Box[\varphi]\psi)$

Theorem ([1])

PAL in SSL is sound and complete.

Topological Spaces

It is an easy and nice exercise to see that public announcement logic also works for topological spaces.

Basics

Game logic (GL) uses the constructive ideas which are familiar from PDL in order to give an abstract framework for games [3, 5]. The games in GL have two players which we call ∃loise and ∀belard. In order to be able to construct the set of well-formed formulae of GL, we need a set of atomic propositions Π and a set of atomic games Γ.

Syntax

Syntax of GL is as follows.

$$\gamma := g \mid \varphi? \mid \gamma;\gamma \mid \gamma \cup \gamma \mid \gamma^* \mid \gamma^d$$
$$\varphi := \bot \mid p \mid \neg\varphi \mid \varphi \vee \varphi \mid \langle\gamma\rangle\varphi$$

Define the dual $[\gamma]\varphi := \neg\langle\gamma\rangle\neg\varphi$.
Notice that $\langle\gamma\rangle\varphi$ express that $\exists$ has a φ-strategy in game γ.
Notice further that $[\gamma]\varphi$ express that $\exists$ does not have a $\neg\varphi$-strategy in game γ; so $\forall$ has a φ-strategy in γ.

A Model for Games

A model $\mathcal{M}$ of GL is the triple $\mathcal{M} = \langle S, \{E_g : g \in \Gamma\}, V \rangle$ where S is a set of states, V is a valuation function, and a family of effectivity functions $E_g : S \to \wp(\wp(S))$ which are monotonic [5].

In other words, our models here are neighborhood models.

Focus on Effectivity Function

The effectivity function in the given semantics makes use of the composite games γ which can be obtained from the atomic games by the given operations. Let us now introduce a piece of notation. Let $E_\gamma(X) := \{x \in X : X \in E_\gamma(s)\}$. Then, E_γ can be defined inductively as follows. $E_{\gamma;\lambda}(X) := E_\gamma(E_\lambda(X))$, $E_{\gamma\cup\lambda}(X) := E_\gamma(X) \cup E_\lambda(X)$, $E_{\varphi?}(X) := (\varphi)^{\mathcal{M}} \cap X$, $E_{\gamma^d}(X) := \overline{E_\gamma(\overline{X})}$ and $E_{\gamma^*}(X) := \mu Y.E_\gamma(Y) \cup X$.

Semantics

Since Boolean cases are as usual, we skip them and give the semantics of the modal operator here.

$$\mathcal{M}, s \models \langle \gamma \rangle \varphi \text{ iff } (\varphi)^{\mathcal{M}} \in E_{\gamma}(s)$$

Krikpe Frames for Game Logic

Construct the relation R_γ: $E_\gamma(X) = \{s \in S : (\exists t \in X) sR_\gamma t\}$
Thus, we can construct R_γ inductively now.

$$
\begin{array}{lll}
sR_{\alpha;\beta}t & \textit{iff} & \exists u : sR_\alpha u \textit{ and } uR_\beta t \\
sR_{\alpha\cup\beta}t & \textit{iff} & sR_\alpha t \textit{ or } sR_\beta t \\
sR_{\alpha?}t & \textit{iff} & s = t \textit{ and } s \models \varphi \\
sR_{\alpha^*}t & \textit{iff} & \exists n \geq 0 : \exists s_0 \ldots s_n, \forall i < n : s_i R_\alpha s_{i+1} \textit{ and,} \\
 & & s = s_0 \textit{ and } t = s_n
\end{array}
$$

Thus: $s \models \langle\gamma\rangle\varphi$ iff $\exists t \in S : sR_\alpha t$ and $t \models \varphi$

Axiomatization

- $\langle \alpha \cup \beta \rangle \varphi \leftrightarrow \langle \alpha \rangle \varphi \vee \langle \alpha \rangle \varphi$
- $\langle \alpha ; \beta \rangle \varphi \leftrightarrow \langle \alpha \rangle \langle \beta \rangle \varphi$
- $\langle \psi ? \rangle \varphi \leftrightarrow (\psi \wedge \varphi)$
- $(\varphi \vee \langle \alpha \rangle \langle \alpha^* \rangle \varphi) \rightarrow \langle \alpha^* \rangle \varphi$ Fixed-Point axiom
- $\langle \alpha^d \rangle \varphi \leftrightarrow \neg \langle \alpha \rangle \neg \varphi$

Inference Rules

- $\varphi, \varphi \to \psi \therefore \psi$
- $\varphi \to \psi \therefore \langle\gamma\rangle\varphi \to \langle\gamma\rangle\psi$
- $(\varphi \vee \langle\gamma\rangle\psi) \to \psi \therefore \langle\gamma^*\rangle\varphi \to \psi$

Completeness?

Still open!

Why?

An important deficiency of GL is the fact that it does not address the epistemic aspects of the games.
Our goal in this work is to offer an extension of GL in order to be able supplement GL with the aforementioned missing component and equip it with a geometrical semantics as the geometrical semantics is the natural candidate for reasoning about closeness and approximation.
Recall that the game logic models use neighborhood semantics, and we now have a natural candidate for it!

Extended Syntax

$$\gamma := g \mid \varphi? \mid \gamma;\gamma \mid \gamma \cup \gamma \mid \gamma^* \mid \gamma^d$$
$$\varphi := \mid p \mid \neg\varphi \mid \varphi \vee \varphi \mid K_\gamma\varphi \mid \Box_\gamma\varphi \mid \langle\gamma\rangle\varphi$$

Semantics

$\mathcal{M} = \langle S, \{\tau_\gamma^{s,i} : \gamma \in \Gamma, s \in S, i \in \mathrm{A}\}, V\rangle$ where S is a set, V is a valuation, the family $\{\tau_\gamma^{s,i}\}$ is a set of subsets of S (i.e. strategies) associated with the agent i at the state s for the game γ.

$$
\begin{array}{lll}
s, U \models p & \textit{iff} & s \in V(p) \\
s, U \models \varphi \wedge \psi & \textit{iff} & s, U \models \varphi \textit{ and } s, U \models \psi \\
s, U \models \neg\varphi & \textit{iff} & s, U \not\models \varphi \\
s, U \models \mathrm{K}_\gamma \varphi & \textit{iff} & t, U \models \varphi \textit{ for all } t \in U \in \tau_\gamma^{s,i} \\
s, U \models \Box_\gamma \varphi & \textit{iff} & s, V \models \varphi \textit{ for all } V \subseteq U \textit{ for } V \in \tau_\gamma^{s,i} \\
s, U \models \langle\gamma\rangle\varphi & \textit{iff} & (s, U) \in (\varphi)^{\mathcal{M}} \textit{ for } s \in U \in \tau_\gamma^{t,i}
\end{array}
$$

Axioms

We will adopt the S5 axiomatization for the epistemic modality and S4 axiomatization for the effort modality. The axiomatization of EGL follows the intuition behind the basic game logic.

- $\langle \gamma \cup \delta \rangle \varphi \leftrightarrow \langle \gamma \rangle \varphi \vee \langle \delta \rangle \varphi$
- $\langle \gamma ; \delta \rangle \varphi \leftrightarrow \langle \gamma \rangle \langle \delta \rangle \varphi$
- $\langle \psi ? \rangle \varphi \leftrightarrow (\psi \wedge \varphi)$
- $(\varphi \vee \langle \gamma \rangle \langle \gamma^* \rangle \varphi) \leftrightarrow \langle \gamma^* \rangle \varphi$
- $\langle \gamma^d \rangle \leftrightarrow \neg \langle \gamma \rangle \neg \varphi$

and

- $\mathsf{K}_\gamma \Box_\gamma \varphi \rightarrow \Box_\gamma \mathsf{K}_\gamma \varphi$ and $F \rightarrow \Box F$ for literal F
- $\mathsf{L}_\gamma \langle \gamma \rangle \varphi \leftrightarrow \langle \gamma \rangle \mathsf{L}_\gamma \varphi$
- $\Diamond_\gamma \langle \gamma \rangle \varphi \leftrightarrow \langle \gamma \rangle \Diamond_\gamma \varphi$

Strategy Based Interpretation

Strategies specifies *how/where* we know the information.

Epistemically, it addresses where we can know the information in question (`go to point` x in the neighborhood U).

Dynamically, it addresses how we can reach this knowledge situation (`Shrink/Improve your information to the subset` V `at` x).

Explicit Strategies

It is possible to reduce dynamic game logic to a case where the players ∃ and ∀ are explicitly stated and worked out. Completeness then follows [7].

Research Directions

Further Work

- **Completeness** of Game Logic is still unproven.
- **Geometrical Semantics** for Dynamic Logics
- **Uncertainty** in games discussed with the idea of closeness/neighborhoods

Some References I

- BAŞKENT, C.; *Topics in Subset Space Logic*, ILLC - UvA.
- BALTAG, A., MOSS, L.; *Logics for Epistemic Programs*; Synthese, vol. 139. No. 2, pp. 165224 (2004).
- HAREL, D., KOZEN, D., TIURYN, J.; *Dynamic Logic*, MIT Press (2009)
- MOSS, L., PARIKH, R.; *Topological Reasoning and The Logic of Knowledge* in Moses, Y. (ed.) Proceedings of TARK, pp. 95-105. Morgan Kaufmann (1992)
- PARIKH, R., PAULY, M.; *Game Logic - An Overview*; Studia Logica, vol. 75, No. 2, pp. 165-182 (2003).

Some References II

- PLAZA, J.; *Logic of Public Communication*, in 4th International Symposium on Methodologies for Intelligent Systems, pp. 201-216, 1989.
- VAN BENTHEM, J., GHOSH, S., LIU, F.; *Modelling Simultaneous games with Concurrent Dynamic logic*,Proceedings of the Workshop on Logic, Rationality and Interaction, LORI 2007, Beijing, pp. 243 - 258, 2007.
- VAN BENTHEM, J.; *Logic Games are Complete for Game Logics*; Studia Logica, vol. 75, No. 2, pp. 183203 (2003).

Thanks!

Talk slides and the preliminary report are available at:

www.canbaskent.net