Epistemic Investigations on Nabla Modality

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Outlook of the Talk

- Nabla Modality
- Dynamic Epistemic Nabla Logic
- Epistemic Nabla Entrenchment
- Conclusion

Classical Modalities vs Nabla

The traditional necessity and possibility operators of modal logic provide a very direct insight and intuition about the semantics of many different modalities.
Especially, supported with simple-to-use Kripkean semantics and intuitive proof theory, such modalities have provided us with variety of mathematical and philosophical developments in the field.
Nevertheless, from a mathematical point of view, one can put these two modalities together in a certain way to obtain an equi-expressible language as the standard propositional modal logic.

Nabla Modality

Nabla modality ∇ was initially introduced by Larry Moss for coalgebraic purposes (Moss, 1999).

$$\nabla\Phi := (\bigwedge \blacklozenge\Phi) \wedge (\Box \bigvee \Phi)$$

where $\blacklozenge\Phi$ for a set of formulae Φ is an abbreviation for the set $\{\Diamond\varphi : \varphi \in \Phi\}$.
We will call ∇ as *nabla* or *cover* modality interchangeably. The set Φ will be called the *cover set*. The language with the ∇ as primitive will be called $\mathcal{L}_\nabla$.

Nabla Modality

Classical modal connectives are definable in terms of ∇ modality as follows.

$$\Diamond\varphi \equiv \nabla\{\varphi, \top\}$$
$$\Box\varphi \equiv \nabla\emptyset \vee \nabla\{\varphi\}$$

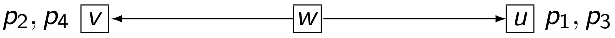
Semantics

If $M = \langle W, R, V \rangle$ is our model where W is a non-empty set, R is a binary relation defined on W, and V is a valuation function mapping propositional variables to subsets of W; then we define the semantics of nabla as follows.

$$M, w \models \nabla\Phi \quad \textit{iff} \quad \forall\varphi \in \Phi, \exists v \textit{ with } wRv \textit{ such that } M, v \models \varphi, \textit{ and } \forall v \textit{ with } wRv, \exists\varphi \in \Phi \textit{ such that } M, v \models \varphi.$$

An Example

Consider the following picture. In this example, observe that the states v and u are accessible from w, and u and v satisfy the propositional letters p_1, p_3 and p_2, p_4 respectively.



Let us say that $\Phi_1 = \{p_1, p_2\}$, $\Phi_2 = \{p_3, p_4\}$, $\Phi_3 = \{p_1, p_4\}$ and finally $\Phi_4 = \{p_2, p_3\}$. Thus, $w \models \bigwedge_{1 \leq i \leq 4} \nabla \Phi_i$. Notice also that in this example, ∇ cannot distinguish Φ_i from Φ_j for $i \neq j$. Thus, nabla cannot always provide the full epistemic picture of the agent.

Tableaux Rules

If the prefix $\sigma.n$ is new to the branch,

$$\frac{\sigma \quad \nabla\Phi}{\begin{array}{ll} \sigma.1 & \varphi_1 \\ \vdots & \\ \sigma.n & \varphi_n \\ \sigma.1 & \varphi_1 \vee \cdots \vee \varphi_n \\ \vdots & \\ \sigma.n & \varphi_1 \vee \cdots \vee \varphi_n \end{array}}$$

Tableaux Rules

Tableaux rules for $\neg\nabla\Phi$ can be constructed very similarly.
We leave the proof of the correctness of tableaux rules to the reader as an exercise!

Closure of Cover Sets

First note that the formulae in $\mathcal{L}_\nabla$ are invariant under bisimulation. Furthermore, nabla operator is closed under union, that is if $w \models \nabla\Phi$ and $w \models \nabla\Psi$, then $w \models \nabla(\Phi \cup \Psi)$.
However, it is not closed under intersection [why?].
Nevertheless, by imposing an intuitive additional constraint, and a slight abuse of the formal language in which nabla is defined, we can make nabla closed under superset relation. If $w \models \nabla\Phi$ and $w \models \Diamond\varphi$, then $w \models \nabla(\Phi \cup \{\varphi\})$.

Minimum Nabla

Given $w \models \nabla\Phi$, we say Φ' is minimum if $\Phi' \subseteq \Phi$ with $w \models \nabla\Phi'$, and there is no $\Phi'' \subseteq \Phi'$ with $w \models \nabla\Phi''$.

Theorem

The problem of finding the MinimumNabla set is NP-complete.

An Epistemic Reading

What does it mean epistemically that $M, w \models \nabla\Phi$? Let us proceed step by step. The first conjunct of the semantics of nabla modality says that every formula in the set Φ is epistemically possible. The second conjunct, on the other hand, manifests that every accessible state realizes or witnesses some formula that is in Φ. In short, $M, w \models \nabla\Phi$ says the agent at the current state w considers each φ in Φ possible and knows the disjunction of all formulae in Φ.

Syntax

The formal syntax we will use is a conglomerate of nabla logic and arbitrary public announcement logic, and is given as follows.

$$p \mid \top \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \nabla\Phi \mid [\varphi]\varphi \mid \boxdot\, \varphi$$

We have two additional operators $[\varphi]$ and $\boxdot$. The formula $[\varphi]\psi$ reads "if φ is true, then after the announcement of φ, ψ shall be true as well". The crucial point is that it is common knowledge among the knowers that announcements are truthful. Furthermore, the formula $\boxdot\varphi$ reads "after every possible announcement, φ is true".

Semantics

Definition

Let $M = \langle W, R, V \rangle$ be the given model where W is a nonempty set of states, R is a binary relation on W, and V is a valuation mapping each propositional variable to a subset of W. The semantics of Booleans and Nabla are given already. Then, for model M and $w \in W$, we define the semantics of dynamic modalities as follows.

$$
\begin{array}{lll}
M, w \models [\varphi]\psi & \text{iff} & M, w \models \varphi \text{ implies } M|\varphi, w \models \psi \\
M, w \models \boxdot\varphi & \text{iff} & \text{for all } \psi \in \mathcal{L}_\nabla,\ M, w \models [\psi]\varphi
\end{array}
$$

The updated model $M|\varphi$ is the model $M|\varphi = \langle W', R', V' \rangle$ where $W' = \{w : M, w \models \varphi\}$, $R' = R \cap (W' \times W')$ and $V' = V \cap W'$.

Axioms

The axioms of dynamic nabla logic is as follows.

1. All instances of propositional tautologies
2. S5 axioms for ∇ modality
3. $[\varphi]p \leftrightarrow (\varphi \to p)$
4. $[\varphi]\neg\psi \leftrightarrow (\varphi \to \neg[\varphi]\psi)$
5. $[\varphi](\psi \wedge \chi) \leftrightarrow ([\varphi]\psi \wedge [\varphi]\chi)$
6. $[\varphi][\psi]\chi \leftrightarrow [(\varphi \wedge [\varphi]\psi)]\chi$
7. $\boxdot\varphi \to [\psi]\varphi$ for $\psi \in \mathcal{L}_\nabla$
8. $[\varphi]\nabla\Psi \leftrightarrow (\varphi \to \nabla[\varphi]\Psi)$ where $[\varphi]\Psi$ is an abbreviation for $\{[\varphi]\psi : \psi \in \Psi\}$
9. $\boxdot\nabla\Psi \to [\varphi]\nabla\Psi$ for $\varphi \in \mathcal{L}_\nabla$

Completeness

Theorem

Arbitrary nabla public announcement logic is complete with respect to the given axiomatization.

Proof.

Every formula in the dynamic nabla public announcement logic can be reducible to a formula in the language of arbitrary announcement logic by the above axioms. Since arbitrary announcement logic is complete, so is dynamic epistemic nabla logic.

What is Epistemic Entrenchment?

Cover modality, as the name implies, gives a set of formulae that *covers* the epistemically possible set of accessible states. However, as we have emphasized, there can be many different ways to cover the set of accessible states.

In the previous sections, we discussed how to obtain a minimal set. However, the procedure of obtaining a minimal set does not respect the order of importance that can be imposed on the knowable formulae.

What is Epistemic Entrenchment?

As Gärdenfors and Makinson stated it

> *"Even if all sentences in a knowledge set are accepted or considered as facts (so that they are assigned maximal probability), this does not mean that all sentences are of equal value for planning or problem-solving purposes. Certain pieces of our knowledge and beliefs about the world are more important than others when planning future actions, conducting scientific investigations, or reasoning in general." (Gärdenfors & Makinson, 1988).*

What is Epistemic Entrenchment?

Therefore, following the same approach, we will now assume an order on the knowable formulae under the cover modality. Based on this, we will first define an algorithm to find the minimal set of most important formulae. Let us know recall the basics of this approach which is widely called *epistemic entrenchment* in the literature.

Entrenchment Relation

The relation $\varphi \leq \psi$ denotes that "ψ is at least as epistemically entrenched as φ".

Properties

1. If $\varphi \leq \psi$ and $\psi \leq \chi$, then $\varphi \leq \chi$ *transitivity*
2. If $\varphi \vdash \psi$, then $\varphi \leq \psi$ *dominance*
3. For any φ, ψ; we have $\varphi \leq \varphi \wedge \psi$ or $\psi \leq \varphi \wedge \psi$ *conjunctiveness*
4. When $\Phi \neq \mathcal{L}_\nabla$, $\varphi \notin \Phi$ if and only if $\varphi \leq \psi$ for all ψ *minimality*
5. If $\varphi \leq \psi$ for all φ, then $\vdash \psi$ *maximality*

A Selection for Cover Set

We will apply epistemic entrenchment to the set of formulae Φ to obtain a smaller set $\Phi' \subseteq \Phi$ such that for every formula $\varphi' \in \Phi'$ there is a formula $\varphi \in \Phi$ such that $\varphi' \leq \varphi$.
We will call Φ' a minimal entrenched subset of Φ.

An NP-complete Selection for Cover Set

Theorem

The problem of selecting the minimal and the epistemically most entrenched subset $\Phi' \subseteq \Phi$ of a given cover Φ that can cover the all accessible states from any given state is NP-complete.

Proof.

Knapsack problem or weighted subset cover problem which are NP-complete can easily be reduced to this problem. We leave the details to the reader.

Points We Have not Covered Here

- Language Splitting
- Distribution Property of Nabla
- Game Semantics for Nabla
- Topological Semantics

Future Work

- More coalgebraic and algebraic analysis
- Application to deontic, doxastic etc logics.

References

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Thanks for your attention!

Talk slides and the paper are available at:

www.canbaskent.net