Epistemic Investigations on Nabla Modality

Can BAŞKENT

The Graduate Center of the City University of New York
cbaskent@gc.cuny.edu www.canbaskent.net

GROLOG Seminar - Rijksuniversiteit Groningen
June 17th, 2010

Outlook of the Talk

- Nabla Modality
- Dynamic Epistemic Nabla Logic
- Epistemic Nabla Entrenchment
- Conclusion

Classical Modalities vs Nabla

The traditional necessity and possibility operators of modal logic provide a very direct insight and intuition about the semantics of many different modalities.
Especially, supported with simple-to-use Kripkean semantics and intuitive proof theory, such modalities have provided us with variety of mathematical and philosophical developments in the field.
Nevertheless, from a mathematical point of view, one can put these two modalities together in a certain way to obtain an equi-expressible language as the standard propositional modal logic.

Classical Modalities vs Nabla

The motivation is quite similar once the expressive equivalence between Public Announcement Logic (PAL) and Epistemic Logic (EL) is recalled.

Even if PAL and EL are equi-expressible, from *some* point of view, PAL reflects the intuition of dynamic epistemology better and provides a *nice* framework.

In a similar fashion, we will investigate the nabla modality from a syntactic point of view and carry the intuition to epistemic logical contexts.

Nabla Modality

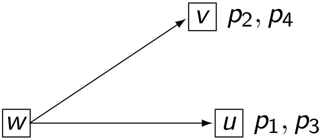
Nabla modality ∇ was initially introduced by Larry Moss for coalgebraic purposes (Moss, 1999). Therefore, it has many applications in fixed point logics, automata theory, category theory - but, we will not discuss them here.

$$\nabla\Phi := (\bigwedge \blacklozenge\Phi) \wedge (\Box \bigvee \Phi)$$

where $\blacklozenge\Phi$ for a set of formulae Φ is an abbreviation for the set $\{\Diamond\varphi : \varphi \in \Phi\}$.

We will call ∇ as *nabla* or *cover* modality interchangeably. The set Φ will be called the *cover set*. The language with the ∇ as the modal primitive will be called $\mathcal{L}_\nabla$.

A Picture



In this case $w \models \nabla\{p_1, p_2\}$, or $w \models \nabla\{p_1, p_2, p_3\}$ but $w \not\models \nabla\{p_1, p_2, q\}$.

Nabla Modality

Classical modal connectives are definable in terms of ∇ modality as follows.

$$\Diamond\varphi \equiv \nabla\{\varphi, \top\}$$
$$\Box\varphi \equiv \nabla\emptyset \vee \nabla\{\varphi\}$$

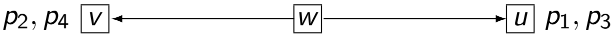
Semantics

If $M = \langle W, R, V\rangle$ is our model where W is a non-empty set, R is a binary relation defined on W, and V is a valuation function mapping propositional variables to subsets of W; then we define the semantics of nabla as follows.

$$M, w \models \nabla\Phi \quad \text{iff} \quad \forall\varphi \in \Phi, \exists v \text{ with } wRv \text{ such that } M, v \models \varphi, \text{ and } \forall v \text{ with } wRv, \exists\varphi \in \Phi \text{ such that } M, v \models \varphi.$$

An Example

Consider the following picture. In this example, observe that the states v and u are accessible from w, and u and v satisfy the propositional letters p_1, p_3 and p_2, p_4 respectively.



Let us say that $\Phi_1 = \{p_1, p_2\}$, $\Phi_2 = \{p_3, p_4\}$, $\Phi_3 = \{p_1, p_4\}$ and finally $\Phi_4 = \{p_2, p_3\}$. Thus, $w \models \bigwedge_{1 \leq i \leq 4} \nabla \Phi_i$. Notice also that in this example, ∇ cannot distinguish Φ_i from Φ_j for $i \neq j$. Thus, nabla cannot always provide the full epistemic picture of the agent.

Tableaux Rules

If the prefix $\sigma.n$ is new to the branch,

$$\frac{\sigma \quad \nabla\Phi}{\begin{array}{ll} \sigma.1 & \varphi_1 \\ \vdots & \\ \sigma.n & \varphi_n \\ \sigma.1 & \varphi_1 \vee \cdots \vee \varphi_n \\ \vdots & \\ \sigma.n & \varphi_1 \vee \cdots \vee \varphi_n \end{array}}$$

See (Bílková *et al.*, 2008)

Tableaux Rules

Tableaux rules for $\neg\nabla\Phi$ can be constructed very similarly.
We leave the proof of the correctness of tableaux rules to the reader as an exercise!

Closure of Cover Sets

First note that the formulae in $\mathcal{L}_\nabla$ are invariant under bisimulation. Furthermore, nabla operator is closed under union, that is if $w \models \nabla\Phi$ and $w \models \nabla\Psi$, then $w \models \nabla(\Phi \cup \Psi)$.
However, it is not closed under intersection.
Nevertheless, by imposing an intuitive additional constraint, and a slight abuse of the formal language in which nabla is defined, we can make nabla closed under superset relation. If $w \models \nabla\Phi$ and $w \models \Diamond\varphi$, then $w \models \nabla(\Phi \cup \{\varphi\})$. So, we can add possible formulae to the cover set.

Minimum Nabla

Given $w \models \nabla\Phi$, we say Φ' is minimum if $\Phi' \subseteq \Phi$ with $w \models \nabla\Phi'$, and there is no $\Phi'' \subset \Phi'$ with $w \models \nabla\Phi''$.

Theorem

The problem of finding the MinimumNabla set is NP-complete.

Minimum Nabla

Proof.

Given Φ', it is easy to verify in polynomial time that Φ' is indeed a minimum cover set. A nondeterministic algorithm needs to guess the subset Φ' and check in polynomial time if it covers.

We know that the MinimumCover problem is NP-complete: given a collection C of a set S and a natural number n, the problem of finding whether C contains a cover of S of size n or less. We then reduce MinimumNabla to MinimumCover by exhibiting a polynomial transformation from MinimumCover to MinimumNabla.

Minimum Nabla

Proof.
Set $S := [w]$, define $C := \{\hat{\varphi} : \hat{\varphi} = (\varphi) \cap [w]\}_{\varphi \in \Phi}$. Observe that for each $\hat{\varphi} \in C$, $(\hat{\varphi}) \subseteq S$.
Now, the problem is, for some $n = |I|$, to find a minimal cover $C_{min} = \{\hat{\varphi}_i\}_{i \in I}$. By our setting, if C_{min} covers S, then those formulae φ_i for which $\hat{\varphi}_i$ is in C_{min}, we will have $w \models \nabla\{\varphi_i\}_{i \in I}$ for some index set I. □

(φ): extension of φ
$[w]$: set of accessible states from w

A Simple Algorithm

Require: $w \models \nabla\Phi$: Given true, w: a state, Φ: set of formulae

$\Phi' := \Phi$

for $\varphi \in \Phi'$ **do**

  if $(\varphi) \cap [w] \subseteq \bigcup_{\psi \in \Psi}\{(\psi) \cap [w]\}$ where $\Psi \subseteq \Phi - \{\varphi\}$ **then**

    $\Phi' := \Phi' - \{\varphi\}$

  end if

end for

print "Φ' `is minimum!`"

(φ): extension of φ

$[w]$: set of accessible states from w

Distribution Property

It has recently been shown that ∇ has a distribution property (Palmigiano & Venema, 2007).
Let us start with recalling relation lifting.

Definition

Given a relation $R \subseteq S_1 \times S_2$, its power lifting relation $\mathrm{P}(R) \subseteq \wp(S_1) \times \wp(S_2)$ is defined as follows

$$\mathrm{P}(R) := \{(X, X') : \quad \forall x \in X, \exists x' \in X' \text{ such that } (x, x') \in R \text{ and } \forall x' \in X', \exists x \in X \text{ such that } (x, x') \in R\}$$

A relation R is called full on S_1 and S_2 if $(S_1, S_2) \in \mathrm{P}(R)$, and we write $R \in S_1 \bowtie S_2$.

Distribution Property

An important observation made in (Palmigiano & Venema, 2007) is the distribution property of nabla algebras:

$$\nabla\Phi \wedge \nabla\Psi \equiv \bigvee_{R \in \Phi \bowtie \Psi} \nabla\{\varphi \wedge \psi : (\varphi, \psi) \in R\}$$

Notice that this property is very significant to combine the (partial or full) knowledge of the knower. Let us now observe how agent's point of view behaves under some certain assumptions.

Distribution Property

Theorem

Let Φ_i be a set of formula indexed by agent i. Let $w \models \bigwedge_{i \in I} \nabla \Phi_i$. Then, we can construct a set of formulae Φ based on Φ_is such that each formula φ in Φ is a conjunction of the form $\varphi = \bigwedge_i \varphi_i$ such that $\varphi_i \in \Phi_i$ for each i.

This gives us a way to formalize the joint epistemic status of the agents in nabla logic. If we describe the epistemic status of each epistemic agent i with the cover set Φ_i, then the overall epistemic picture is from agent's point of view such that each agent's perspective is independent from the other and therefore, there is no interaction among the agents. However, syntactically, we can also express the same epistemic picture by putting agent's point of view together by using the distribution property.

An Epistemic Reading

What does it mean epistemically that $M, w \models \nabla\Phi$? Let us proceed step by step. The first conjunct of the semantics of nabla modality says that every formula in the set Φ is epistemically possible. The second conjunct, on the other hand, manifests that every accessible state realizes or witnesses some formula that is in Φ. In short, $M, w \models \nabla\Phi$ says the agent at the current state w considers each φ in Φ possible and knows the disjunction of all formulae in Φ.

Syntax

The formal syntax we will use is a conglomerate of nabla logic and arbitrary public announcement logic, and is given as follows.

$$p \mid \top \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \nabla\Phi \mid [\varphi]\varphi \mid \boxdot\, \varphi$$

We have two additional operators $[\varphi]$ and $\boxdot$. The formula $[\varphi]\psi$ reads "if φ is true, then after the announcement of φ, ψ shall be true as well". The crucial point is that it is common knowledge among the knowers that announcements are truthful. Furthermore, the formula $\boxdot\varphi$ reads "after every possible announcement, φ is true".

Semantics

Definition

Let $M = \langle W, R, V \rangle$ be the given model where W is a nonempty set of states, R is a binary relation on W, and V is a valuation mapping each propositional variable to a subset of W. The semantics of Booleans and Nabla are given already. Then, for model M and $w \in W$, we define the semantics of dynamic modalities as follows.

$$\begin{array}{lll} M, w \models [\varphi]\psi & \text{iff} & M, w \models \varphi \text{ implies } M|\varphi, w \models \psi \\ M, w \models \boxdot\varphi & \text{iff} & \text{for all } \psi \in \mathcal{L}_\nabla,\ M, w \models [\psi]\varphi \end{array}$$

The updated model $M|\varphi$ is the model $M|\varphi = \langle W', R', V' \rangle$ where $W' = \{w : M, w \models \varphi\}$, $R' = R \cap (W' \times W')$ and $V' = V \cap W'$.

Axioms

The axioms of dynamic nabla logic is as follows.

1. All instances of propositional tautologies
2. S5 axioms for ∇ modality
3. $[\varphi]p \leftrightarrow (\varphi \to p)$
4. $[\varphi]\neg\psi \leftrightarrow (\varphi \to \neg[\varphi]\psi)$
5. $[\varphi](\psi \land \chi) \leftrightarrow ([\varphi]\psi \land [\varphi]\chi)$
6. $[\varphi][\psi]\chi \leftrightarrow [(\varphi \land [\varphi]\psi)]\chi$
7. $\boxdot\varphi \to [\psi]\varphi$ for $\psi \in \mathcal{L}_\nabla$
8. $[\varphi]\nabla\Psi \leftrightarrow (\varphi \to \nabla[\varphi]\Psi)$ where $[\varphi]\Psi$ is an abbreviation for $\{[\varphi]\psi : \psi \in \Psi\}$
9. $\boxdot\nabla\Psi \to [\varphi]\nabla\Psi$ for $\varphi \in \mathcal{L}_\nabla$

Playing with the axioms

Let's take the eight axiom and see that it indeed works.

$$
\begin{array}{lll}
[\varphi]\nabla\Psi & \leftrightarrow & [\varphi]((\bigwedge \blacklozenge\Psi) \wedge (\Box \bigvee \Psi)) \\
& & [\varphi](\bigwedge \blacklozenge\Psi) \wedge [\varphi](\Box \bigvee \Psi) \\
& & ([\varphi]\Diamond\psi_1 \wedge \cdots \wedge [\psi]\Diamond\psi_\omega) \wedge (\varphi \rightarrow \Box[\varphi] \bigvee \Psi) \\
& & (\varphi \rightarrow (\Diamond[\varphi]\psi_1 \wedge \cdots \wedge \Diamond[\varphi]\psi_\omega) \wedge (\varphi \rightarrow \Box[\varphi](\psi_1 \vee \cdots \vee \psi \\
& & \varphi \rightarrow ((\Diamond[\varphi]\psi_1 \wedge \cdots \wedge \Diamond[\varphi]\psi_\omega) \wedge \Box[\varphi](\psi_1 \vee \cdots \vee \psi_\omega)) \\
& & \varphi \rightarrow ((\Diamond[\varphi]\psi_1 \wedge \cdots \wedge \Diamond[\varphi]\psi_\omega) \wedge \Box([\varphi]\psi_1 \vee \cdots \vee [\varphi]\psi_\omega)) \\
& & \varphi \rightarrow \nabla[\varphi]\Psi
\end{array}
$$

where $\Psi = \{\psi_1, \ldots, \psi_\omega\}$ and $[\varphi]\Psi$ is an abbreviation for $\{[\varphi]\psi : \psi \in \Psi\}$.

Completeness

Theorem

Arbitrary nabla public announcement logic is complete with respect to the given axiomatization.

Proof.

Well... Every formula in the dynamic nabla public announcement logic can be reducible to a formula in the language of arbitrary announcement logic by the above axioms. Since arbitrary announcement logic is complete, so is dynamic epistemic nabla logic.

What is Epistemic Entrenchment?

Cover modality, as the name implies, gives a set of formulae that *covers* the epistemically possible set of accessible states. However, as we have emphasized, there can be many different ways to cover the set of accessible states.
In the previous sections, we discussed how to obtain a minimal set. However, the procedure of obtaining a minimal set does not respect the order of importance that can be imposed on the knowable formulae.

What is Epistemic Entrenchment?

As Gärdenfors and Makinson stated it

> *"Even if all sentences in a knowledge set are accepted or considered as facts (so that they are assigned maximal probability), this does not mean that all sentences are of equal value for planning or problem-solving purposes. Certain pieces of our knowledge and beliefs about the world are more important than others when planning future actions, conducting scientific investigations, or reasoning in general." (Gärdenfors & Makinson, 1988).*

What is Epistemic Entrenchment?

Therefore, following the same approach, we will now assume an order on the knowable formulae under the cover modality. Based on this, we will discuss how hard it is to find a set of the most important formulae. Let us now recall the basics of this approach which is widely called *epistemic entrenchment* in the literature.

Entrenchment Relation

The relation $\varphi \leq \psi$ denotes that "ψ is at least as epistemically entrenched as φ".

Properties

1. If $\varphi \leq \psi$ and $\psi \leq \chi$, then $\varphi \leq \chi$ *transitivity*
2. If $\varphi \vdash \psi$, then $\varphi \leq \psi$ *dominance*
3. For any φ, ψ; we have $\varphi \leq \varphi \wedge \psi$ or $\psi \leq \varphi \wedge \psi$ *conjunctiveness*
4. When $\Phi \neq \mathcal{L}_\nabla$, $\varphi \notin \Phi$ if and only if $\varphi \leq \psi$ for all ψ *minimality*
5. If $\varphi \leq \psi$ for all φ, then $\vdash \psi$ *maximality*

A Selection for Cover Set

We will apply epistemic entrenchment to the set of formulae Φ to obtain a smaller set $\Phi' \subseteq \Phi$ such that for every formula $\varphi' \in \Phi'$ there is a formula $\varphi \in \Phi$ such that $\varphi' \leq \varphi$.

We will call Φ' a minimal entrenched subset of Φ.

An NP-complete Selection for Cover Set

Theorem

The problem of selecting the minimal and the epistemically most entrenched subset $\Phi' \subseteq \Phi$ of a given cover Φ that can cover the all accessible states from any given state is NP-complete.

Proof.

Knapsack problem or weighted subset cover problem which are NP-complete can easily be reduced to this problem. We leave the details to the reader.

Points We Have not Covered Here

- Language Splitting - an intermission towards first-order splitting
- Game Semantics for Nabla - quite trivial
- Topological Semantics - connection with compactness?

Future Work

- Importing more coalgebraic and algebraic tools to dynamic epistemic formalism
- Application to deontic, doxastic etc. logics
- Connection between compactness

References

Bílková, Marta, Palmigiano, Alessandra, & Venema, Yde. 2008.
Proof systems for the coalgebraic cover modality.
Pages 1–21 of: Areces, C., & Goldblatt, Robert (eds), *Advances in Modal Logic*.

Gärdenfors, Peter, & Makinson, David C. 1988.
Revisions of Knowledge Systems Using Epistemic Entrenchment.
Pages 83–95 of: Vardi, Moshe Y. (ed), *Proceedings of TARK II*.

Moss, Lawrence S. 1999.
Coalgebraic Logic.
Annals of Pure and Applied Logic, **96**(1-3), 277–317.

Palmigiano, Alessandra, & Venema, Yde. 2007.
Nabla Algebras and Chu Spaces.
Pages 394–408 of: Mossakowski, Till, Montanari, Ugo, & Haveraaen, Magne (eds), *Algebra and Coalgebra in Computer Science (CALCO 2007)*, vol. LNCS 4624.
Springer-Verlag.

Thanks for your attention!

Talk slides and the paper are available at:

`www.canbaskent.net`