Deviant Games for Deviant Logics

Towards Non-Classical Game Semantics

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Semantic games help us understand the non-classical and inconsistency-friendly elements in the theory of games.

Overview

Semantic Games

Elements of Semantic Games

Game theoretical semantics suggests a very intuitive and natural approach to formal semantics and proofs.

The semantic verification game (for classical logic) is played by two players: the verifier (Heloise) and the falsifier (Abelard). The verifier's goal is to verify the truth of a given formula in a given model. Dually, the falsifier's goal is to falsify it.

In the game, the given formula is broken into subformulas step by step by the players. The play of the game terminates when it reaches the propositional literals and when there is no move to make. If the play ends with a propositional literal which is true in the model in question, then the verifier wins the game. Otherwise, the falsifier wins.

Elements of Semantic Games

The rules of the semantic verification game are specified syntactically based on the form of the formula. We associate conjunction with the falsifier, disjunction with the verifier. The negation operator switches the roles of the players: the verifier becomes the falsifier and the falsifier becomes the verifier.

Informally, a player is said to have a *winning strategy* if he has a set of rules that guides him throughout the play and tells him which move to make, and consequently gives him a win regardless of how the opponent plays.

The major result of this approach states that Heloise the verifier has a winning strategy in the verification game if and only if the given formula is true in the given model.

The rules of the semantic verification game are specified syntactically based on the form of the formula. We associate conjunction with the falsifier, disjunction with the verifier. The negation operator switches the roles of the players: the verifier becomes the falsifier and the falsifier becomes the verifier.

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Non-Classical Logics and Game Semantics

First Steps Towards Non-Classicity

In the case of classical logics, the verification games are constructed as

- zero-sum (a win for a player is a loss for the other),
- two-player,
- determined (one player always has a winning strategy),
- sequential (players do not make moves at the same time),
- non-cooperative and competitive

games.

First Steps Towards Non-Classicity

What about the semantic games where

- Abelard and Heloise both may win,
- Abelard and Heloise both may lose,
- Heloise may win, Abelard may not lose,
- Abelard may win, Heloise may not lose,
- There is a tie,
- There is an additional player,
- Players play simultaneously,
- Players may cooperate.

Such possibilities can occur, for instance, when both p and $\neg p$ are true, so that both players can be expected to have winning strategies. We can also imagine verification games with additional truth values and additional players beyond verifiers and falsifiers, and also construct games where players may play concurrently in a parallel fashion.

First Steps Towards Non-Classicity

And, in the classical case, the existence of winning strategies and the truth values of formulas are closely connected.

In particular, can players have winning strategies that cannot determine the truth value of the formula?

Can the truth value of a formula be established if more than one player has a winning strategy?

Reference

CB, *Game Theoretical Semantics for Some Non-Classical Logics*, Journal of Applied Non-Classical Logics, vol. 26, no. 3, pp. 208-39, 2016.

Deviant Logics and their Deviant Games

A Folk Tale

Two men want to marry a princess. The king says they have to race on a horseback. The slowest one wins, and can marry the princess. How can one win this game and marry the princess?

A Folk Tale

> *Two men want to marry a princess. The king says they have to race on a horseback. The slowest one wins, and can marry the princess. How can one win this game and marry the princess?*

The solution simply suggests that the men need to swap their horses. Since the fastest one loses, and players race with each other's horse, what they need to do is to become the fastest in the dual game. The fastest one on the switched horse in the dual game wins the original game.

A Folk Tale

There is a lot to be learned from this tale. It clarifies how negation works game theoretically.

One of the most obvious difficulties arises when the same game is considered with three or more players. For $n > 2$ players, the solution requires a different understanding of negation – a permutation.

The similar complexity also carries over to binary connectives perceived as choice functions for certain players.

Reference

L. Olde Loohuis & Y. Venema, *Logics and algebras for multiple players*, The Review of Symbolic Logic, vol. 3, 485–519, 2010.

A Folk Tale

Conceived as choice functions, conjunction and disjunction (and their numerous non-classical cousins) give raise to various non-classical interpretations game semantically.

An interesting example is bunched implication logic (of Pym and O'Hearn) with a multiplicative conjunction $*$ and separating implication $-\!\!*$, or more fundamentally, relevant logic.

Deviant Logics: A Selection

Let us now consider a selection of well-studied non-classical logics and their game semantics.

- Logic of Paradox
- First-Degree Entailment
- Connexive Logic
- Four-valued Logic
- Logic of Formal Inconsistency
- Logic of Nonsense

Logic of paradox (LP, for short) introduces an additional truth value P, called paradoxical, which intuitively stands for both true and false.

	$\neg$
T	F
F	T
P	P

$\wedge$	T	P	F
T	T	P	F
P	P	P	F
F	F	F	F

$\vee$	T	P	F
T	T	T	T
P	T	P	P
F	T	P	F

We stipulate that the introduction of the third truth value requires an additional player that we call Astrolabe. Astrolabe is the paradoxifier in the game forcing the game to an end with the truth value P.

Reference

G. Priest, *The logic of paradox*, Journal of Philosophical Logic, vol. 8, pp. 219–241, 1979.

Here are the new and expanded rules for the semantic games for Logic of Paradox.

Atomic Formulas Heloise wins if it is true, Abelard wins if false and Astrolabe wins if paradoxical,

Negation Abelard and Heloise switch roles, Astrolabe keeps his role,

Conjunction Abelard and Astrolabe choose between the conjuncts,

Disjunction Heloise and Astrolabe choose between the disjuncts.

Game Semantics for Logic of Paradox

Theorem

In a semantics game for φ in logic of paradox,

- *Heloise the verifier has a winning strategy if φ is true,*
- *Abelard the falsifier has a winning strategy if φ is false,*
- *Astrolabe the pardoxifier has a winning strategy if φ is paradoxical.*

Theorem

In a semantics game for φ in logic of paradox,

- *If Heloise the verifier has a winning strategy, then φ is true,*
- *If Abelard the falsifier has a winning strategy, then φ is false,*
- *If Astrolabe the paradoxifier has a winning strategy, but not the other players, then φ is paradoxical.*

The above theorem indicates that Astrolabe the paradoxifier's strategy is strictly dominated in a sense that if some other player also admits a winning strategy, then Astrolabe's strategy will not give him a win. This observation is another reading of the truth table for LP.

Game Semantics for First-Degree Entailment

Semantic evaluations are thought of as functions from logical formulas to truth values. This ensures that each and every formula is assigned a unique truth value.

If we replace the valuation function with a valuation *relation* which can produce multiple or no truth values for logical formulas. The system obtained in this manner is called First-degree entailment.

Reference

A. R. Anderson & N. D. Belnap, *First degree entailments*, Mathematische Annalen, vol. 149, pp. 302–319, 1963.

Game Semantics for First-Degree Entailment

$\neg\varphi \mathbf{r} 1$	*iff*	$\varphi \mathbf{r} 0$
$\neg\varphi \mathbf{r} 0$	*iff*	$\varphi \mathbf{r} 1$
$(\varphi \wedge \psi)\mathbf{r} 1$	*iff*	$\varphi \mathbf{r} 1$ *and* $\psi \mathbf{r} 1$
$(\varphi \wedge \psi)\mathbf{r} 0$	*iff*	$\varphi \mathbf{r} 0$ *or* $\psi \mathbf{r} 0$
$(\varphi \vee \psi)\mathbf{r} 1$	*iff*	$\varphi \mathbf{r} 1$ *or* $\psi \mathbf{r} 1$
$(\varphi \vee \psi)\mathbf{r} 0$	*iff*	$\varphi \mathbf{r} 0$ *and* $\psi \mathbf{r} 0$

Logic of Paradox can be obtained from First-Degree Entailment by imposing a restriction that no formula gets the truth value $\emptyset$.

Game Semantics for First-Degree Entailment

Here are the new and expanded rules for the semantic games for First-Degree Entailment.

Atomic Formulas Heloise wins if φr1, Abelard wins if φr0, neither wins if φr$\emptyset$,

Negation Players switch roles,

Conjunction Abelard and Heloise choose between the conjuncts,

Disjunction Abelard and Heloise choose between the disjuncts.

Game Semantics for First-Degree Entailment

Theorem

In a semantics game for φ in first-degree entailment,

- *Heloise the verifier has a winning strategy if* $\varphi \mathsf{r} 1$,
- *Abelard the falsifier has a winning strategy if* $\varphi \mathsf{r} 0$,
- *No player has a winning strategy if* $\varphi \mathsf{r} \emptyset$.

As Wansing puts it, connexive logic is a "comparatively little-known and to some extent neglected branch of non-classical logic". Even if it is understudied, its roots can be traced back to Aristotle and Boethius.

Connexive logic is defined as a system which satisfies the following two schemes of conditionals:

- Aristotle's Theses: $\neg(\neg\varphi \to \varphi)$
- Boethius' Theses: $(\varphi \to \neg\psi) \to \neg(\varphi \to \psi)$

Reference

H. Wansing, *Connexive Logic*, The Stanford Encyclopedia of Philosophy (Fall 2015 ed.). Retrieved from `http://plato.stanford.edu/entries/logic-connexive`.

Game Semantics for a Connexive Logic

	¬
T	F
t	f
f	t
F	T

∧	T	t	f	F
T	T	t	f	F
t	t	T	F	f
f	f	F	f	F
F	F	f	F	f

∨	T	t	f	F
T	t	T	t	T
t	T	t	T	t
f	t	T	F	f
F	T	t	f	F

The semantics for CC is given with four truth values: T, t, f and F which can be viewed as "logical necessity", "contingent truth", "contingent falsehood", and "logical impossibility", respectively.

For instance, (with a brief abuse of notation), in CC, we have $t \wedge f \equiv F$.

Reference

S. McCall, *Connexive implication*, Journal of Symbolic Logic, vol. 31, pp. 415–433, 1966.

We introduce four players for four truth values. The truth value T is forced by Heloise, F by Abelard, t by Aristotle and f by Boethius.

We let players form coalitions: "Heloise and Aristotle" vs "Abelard and Boethius", truth-maker and false-maker coalitions, respectively.

Here are the new and expanded rules:

Atomic Formulas Heloise wins if φ has the truth value T, Aristotle wins it has the truth value t, Boethius wins if it has the truth value f and Abelard wins if it has the truth value F,

Negation Heloise assumes Abelard's role, Aristotle assumes Boethius' role, Boethius assumes Aristotle's role and Abelard assumes Heloise's role,

Conjunction False-makers simultaneously choose between the conjuncts,

Disjunction Truth-makers simultaneously choose between the disjuncts.

Theorem

In a semantics game for φ in connexive logic,

- *Truth-makers have a winning strategy iff φ has the truth value t or T,*
- *False-makers have a winning strategy iff φ has the truth value f or F.*

Can we have a coalition with three players for each team with the corresponding and similar rules?

Game Semantics for Belnap's Four-Valued Logic

Belnap's four-valued logic introduces two additional truth values besides the classical ones.

The truth value P represents over-valuation and N represents undervaluation. That is P stands for both truth values and N stands for neither of the truth values.

	$\neg$
T	F
P	P
N	N
F	T

$\wedge$	T	P	N	F
T	T	P	N	F
P	P	P	F	F
N	N	F	N	F
F	F	F	F	F

$\vee$	T	P	N	F
T	T	T	T	T
P	T	P	T	P
N	T	T	N	N
F	T	P	N	F

The truth combinations $P \vee N \equiv T$ and $P \wedge N \equiv F$ suggest that our standard approach may not work for Belnap's system.

Game Semantics for Belnap's Four-Valued Logic

We introduce four players for four truth values. The truth value T is forced by Heloise, F by Abelard, P by Astrolabe and N by Bernard (of Clairvaux).

Here are the rules for the semantic games for Belnap's system.

Atomic Formulas Heloise wins if the formula is T, Astrolabe wins if P, Bernard wins if N and Abelard wins if F,

Negation Heloise assumes Abelard's role, Abelard assumes Heloise's role, Astrolabe and Bernard keep their previous roles

Conjunction where only Bernard has a winning strategy for one of the conjuncts and only Astrolabe has a winning strategy for the other conjunct, then Abelard wins,

Conjunction for other cases, Abelard, Astrolabe and Bernard choose simultaneously between the conjuncts,

Game Semantics for Belnap's Four-Valued Logic

Disjunction where only Bernard has a winning strategy for one of the disjuncts and and only Astrolabe has a winning strategy for the other disjunct, then Heloise wins,

Disjunction for other cases, Abelard, Astrolabe and Bernard choose simultaneously between the conjuncts,

The unusual game rules about $P \wedge N$ and $P \vee N$ stem from the truth table for Belnap's system, and the price we have to pay is to incorporate the existence of winning strategies into the semantics.

Theorem

In a semantics game for φ in Belnap's four-valued system,

- *Heloise the verifier has a winning strategy if φ has the truth value T,*
- *Abelard the falsifier has a winning strategy if φ has the truth value F,*
- *Astrolabe the paradoxifier has a winning strategy if φ has the truth value P,*
- *Bernard the nullifier has a winning strategy if φ has the truth value N.*
- *If Heloise the verifier has a winning strategy, then φ is T,*
- *If Abelard the falsifier has a winning strategy, then φ is F,*
- *If only Astrolabe the paradoxifier has a winning strategy, then φ is P,*
- *If only Bernard the nullifier has a winning strategy, then φ is N.*

Logics of Formal Inconsistencies extend da Costa systems and generate a broad class of paraconsistent logics.

p	$\neg p$	$\circ p$	$p \wedge \neg p$
T	T	F	T
	F	T	F
		F	F
F	T	T	F
		F	F

Reference

W. A. Carnielli, M. E. Coniglio and J. Marcos, *Logics of formal inconsistency*, in “Handbook of Philosophical Logic”, vol. 14, D. Gabbay and F. Guenthner (editors), pp. 15–107, Springer, 2007.

Here are the rules for the game semantics for logic of formal inconsistency.

Negation Abelard assumes Heloise's role, Heloise assumes both roles,

Consistency For $\circ\varphi$, the game continues with φ and $\neg\varphi$ with players' roles switched,

Atomic Formulas, Conjunction and Disjunction Same as the classical case

Reference

CB and Pedro Henrique Carrasqueira, *A Game Theoretical Semantics for a Logic of Formal Inconsistency*, Logic Journal of the IGPL, vol. 28, pp. 936-52, 2020.

In a semantics game for φ in Logic of Formal Inconsistency,

Theorem

- *The verifier has a winning strategy if φ is true,*
- *The falsifier has a winning strategy if φ is false.*

The converse of the Theorem is not true. Because in some games players may admit multiple roles.

Moreover, the converse of the correctness theorem cannot be established by imposing various further restrictions, such as uniqueness, on the existence of winning strategies.

Game Semantics for Logics of Nonsense

Logics of nonsense allow a third truth value to express propositions that are nonsense. The initial motivation behind introducing nonsensical propositions was to capture the logical behavior of semantic paradoxes that were thought to be nonsensical.

	$\neg$
T	F
N	N
F	T

$\wedge$	T	N	F
T	T	N	F
N	N	N	N
F	F	N	F

$\vee$	T	N	F
T	T	N	T
N	N	N	N
F	T	N	F

References

- Dimitri Bochvar (1937): On a three-valued logical calculus and its application to the analysis of contradictions. Matematicheskii Sbornik 4(46), pp. 287–308.
- Sören Halldén, *The Logic of Nonsense*, Uppsala Universitets Årsskrift, 1949.

We introduce a third player which we call "Dominator". Dominator forces the game to a nonsense proposition and is allowed to make moves along other players.

We also stipulate that Dominator's strategy is dominant -– his wins determine the truth value.

Reference

CB, *A Game Theoretical Semantics for Logics of Nonsense*, Proceedings of the Eleventh International Symposium on Games, Automata, Logics, and Formal Verification (GandALF 2020), Edited by J.-F. Raskin and D. Bresolin, Electronic Proceedings in Theoretical Computer Science vol. 326, pp. 66–81, 2020.

Here are the rules:

Atomic Heloise the verifier wins if the formula is true, Abelard the falsifier wins if it is false and Dominator wins if it is nonsense;

Negation Heloise the verifier and Abelard the falsifier switch roles, Dominator keeps his role,

Disjunction Heloise the verifier and Dominator choose between the disjuncts simultaneously,

Conjunction Abelard the falsifier and Dominator choose between the conjuncts simultaneously,

Dominant Strategy Dominator's strategy strictly dominates the others's.

Some observations and results:

Theorem

In a semantic game for φ in logic of nonsense,

- *Heloise the verifier has a dominant winning strategy if and only if φ is true,*
- *Abelard the falsifier has a dominant winning strategy if and only if φ is false,*
- *Dominator has a dominant winning strategy if and only if φ is nonsense.*

Some observations and results:

Theorem

In a semantic game in logic of nonsense, Dominator makes a move at each connective.

Theorem

In a semantic game in logic of nonsense for φ, if φ contains a literal with the truth value N, then Dominator has a winning strategy and consequently φ is nonsense.

It is possible to extend the logic of nonsense to 4 truth values, hence to games with 4 players – as a matter of fact to any $< \omega$ values and players.

Non-Classicity: From Semantics to Games

Some Observations

Game semantics for non-classical logics offers an alternative and relaxed understanding of the foundational ideas in game theory. It is an excellent introduction to the theory.

This will help us redefine concepts such as winning strategies, equilibrium and solution concepts.

Some Observations

The next stage of this research programme is to work on the converse direction: given a game, how can we develop a logic whose semantic game matches the given game. For example, what logic's semantic game is three-player, cooperative, non-zero sum?

What logic's semantic game is the prisoners' dilemma? — now we can even talk about *epistemic* semantic games!

Conclusion

Broader Picture

The current work is part of a broader research programme of "Non-classical Semantics for Non-classical Logics".

- Topological Semantics
- Truth Diagrams
- Game Semantics

Reference

Peter C.-H. Cheng, *Truth diagrams versus extant notations for propositional logic*, Journal of logic, language and information, vol. 29, pp. 121–161, 2020.

Reference

CB, *Truth Diagrams for Some Non-Classical and Modal Logics*, under submission.

Future Work

A challenging next step is to use game theoretical semantics to give an explanation for Gödel's Theorems. This would be a nice synthesis of paraconsistency, semantics, computability and proofs.

None of this work has been applied to the game semantics for programming languages, which is a popular and prolific topic.

Another interesting direction is behavioural game theory where *homo economicus* makes "irrational" and "emotional" decisions in a predictable way.

In a Nutshell

Semantic games help us understand the

non-classical, inconsistency-friendly and **behavioural**

elements in the theory of games.

Thank you!

Talk slides are available at my website

CanBaskent.net/Logic