Paraconsistent Dynamic Epistemology

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Outlook of the Talk

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Dynamic Epistemology

Considered a kind of dynamic epistemic logic, public announcement logic works as follows.

An external agent makes a truthful and public announcement, then the agents update their epistemic states by eliminating the possible worlds that do not agree with the announcement.

For example, you may think that today is either Tuesday or Wednesday, then on TV you hear that it is actually Tuesday today. Then, you eliminate the possibility that today is Wednesday and come to know that today is Tuesday. Thus, after an announcement, you come to know the announcement.

Non-Kripkean Dynamic Epistemology

Traditionally, public announcement logic (PAL, henceforth) adopts Kripke semantics (Plaza, 1989; Gerbrandy, 1999). In a relatively recent work, a topological semantics for public announcement logic was given (Başkent, 2012).

Topological models exhibit some unexpected properties: the backward induction procedure or announcements may stabilize in more than ω steps.

Paraconsistent Dynamic Epistemology

In this work, we extend such results by focusing on the relation between topologies, public announcements and inconsistency-friendly logics, particularly paraconsistent logic (Başkent, 2015).

One of the main motivation of this work comes from *impossible worlds* - worlds which satisfy contradictions.

Topologies and Closed Sets

Given a non-empty set S, a topology σ is a collection of subsets of S satisfying the following conditions.

- The empty set and S are in σ,
- The collection σ is closed under arbitrary intersections and finite unions.

We call the tuple (S, σ) a *topological space*. The members of the topology is called *closed*.

A function defined on a topological space is *continuous* if the inverse image of a closed/open is a closed/open.

A function is called *homeomorphism* if it is a continuous function between topological spaces with a continuous inverse.

Topological Models

Let $M = (S, \sigma, v)$ be a *topological model* where (S, σ) is a topology and v is a valuation.

For an announcement φ, we define the *updated model* $M'_\varphi = (S', \sigma', v')$ as follows. Set $S' = S \cap |\varphi|$, $\sigma' = \{O \cap S' : O \in \sigma\}$, and $v' = v \cap S'$.

Thus, in PAL, an announcement is made and the states that do not satisfy the announcement are eliminated in a way that preserves the topological structure.

The new topology σ', which we obtained by relativizing σ, is a familiar one, and is called *the induced topology*.

Syntax

The language of topological PAL includes the epistemic modality K and the public announcement modality $[\cdot]$.

Topological Modal Operations

In a topology, we have the *interior* and *closure* operators (Int, Clo) which return the largest open set contained in the given set, and the smallest closed set containing the given set respectively.

We put $|\mathsf{K}\varphi| = \mathsf{Int}(|\varphi|)$. Dually, we have $|\mathsf{L}\varphi| = \mathsf{Clo}(|\varphi|)$. Intuitively, extension of a modal formula is the interior (or the closure) of the extension of the formula.

Epistemic modal operators necessarily produce topological entities. However, it is not necessary that $|p|$ for a proposition p will be open or closed, as it simply does not follow from the definition.

Semantics

The semantics of propositional variables and Booleans are standard.

$$M, s \models K\varphi \quad \text{iff} \quad \exists O \in \sigma.(s \in O \land \forall s' \in O, M, s' \models \varphi)$$
$$M, s \models [\varphi]\psi \quad \text{iff} \quad M, s \models \varphi \text{ implies } M', s \models \psi$$

Topological models can distinguish a variety of epistemic properties that Kripke models cannot since the topological semantics for the epistemic modality K has Σ_2 complexity as it is of the form $\exists\forall-$, while Kripkean semantics offers Π_1 complexity as it is of the form $\forall-$.

Axioms

PAL with topological semantics admits the following standard reduction axioms.

- $[\varphi]p \leftrightarrow (\varphi \rightarrow p)$
- $[\varphi]\neg\psi \leftrightarrow (\varphi \rightarrow \neg[\varphi]\psi)$
- $[\varphi]\psi \wedge \chi \leftrightarrow [\varphi]\psi \wedge [\varphi]\chi$
- $[\varphi]K\psi \leftrightarrow (\varphi \rightarrow K[\varphi]\psi)$

In PAL, the rules of derivation are normalization ($\vdash \varphi \therefore \vdash \Box\varphi$) and modus ponens.

Theorem ((Başkent, 2012))

PAL in topological models is complete and decidable.

Homotopic Announcements

For a public announcement φ, we say φ is "functionally representable in M" if there is an open and continuous function $f_{\varphi}^{M} : (S, \sigma) \mapsto (S', \sigma')$ where $M'_{\varphi} = (S', \sigma', v')$ is the updated model.

Theorem

Every public announcement is functionally representable.

Homotopic Models

Let S and S' be two topological spaces with continuous functions $f, f' : S \mapsto S'$. A homotopy between f and f' is a continuous function $H : S \times [0, 1] \mapsto S'$ such that for $s \in S$, $H(s, 0) = f(s)$ and $H(s, 1) = g(s)$.

The definition of homotopy can easily be extended to topological models. Given a topological model $M = (S, \sigma, v)$ we call the family of models $\{M_t = (S_t, \sigma_t, v_t)\}_{t \in [0,1]}$ generated by M and homotopic functions *homotopic models*. We put $v_t = f_t(v)$.

Theorem

Given M, consider a family of updated homeomorphic models $\{M_i\}_{i<\omega}$ each of which is obtained by an announcement φ_i representable by f_i. Then f_is are homotopic.

Impossibilities and Inconsistencies

By *impossible worlds*, let us denote those states which satisfy some contradictions, define them as $\{x : x \models \varphi \wedge -\varphi \text{ for some } \varphi\}$ for a negation symbol $-$.

Then, the natural question is how to epistemically update an epistemic model with impossible worlds.

Models

If propositional variables are closed sets, then arbitrary intersections and finite unions of them will remain closed - except negation the compliment of a closed set is not necessarily a closed set.

We define negation as the "closure of the complement" as negation and denote it by $-$(Başkent, 2013; Goodman, 1981; Mortensen, 2000).

Consider the formula $p \wedge -p$. Assume $|p| = O$ in a closed set topology. Then, $|p \wedge -p|$ is $O \cap \text{Clo}(\overline{O})$ which is $\partial(O)$, where $\partial(\cdot)$ is the boundary operator which is defined as $\partial O := \text{Clo}(O) - \text{Int}(O)$ and $\overline{O}$ denotes the set theoretical compliment of O.

Therefore, the contradictions hold at the boundary points.

How to Update

In paraconsistent spaces, public announcements obtain a broader meaning.

When φ is announced in a paraconsistent space, it means "Keep the states that satisfy φ". It can very well be the case that some of the states that satisfy φ may also satisfy its negation $-\varphi$.

The methods of "eliminating the states that do not satisfy the announcement" and "keeping the states that satisfy the announcement" are not identical in paraconsistent PAL, unlike in classical logic.

Paraconsistent Updates

Let $M = (S, \sigma, v)$ be a topological model. For a formula $[\varphi]$, we obtain an *updated model* $M'_\varphi = (S', \sigma', v')$ where $S' = S \cap |\varphi|$, $\sigma' = \{K \cap S' : K \in \sigma\}$, and $v' = v \cap S'$.

Alternatively, one may wish to exclude the states that satisfy the negation of the announcement from the space. Now, define $M^-_\varphi := (S \setminus |-\varphi|, \sigma', v')$ as the model obtained after the announcement of $[\varphi]$. We will call M^-_φ as the *reduced model*.

Lemma

In classical PAL, for a model M, updated model M', and reduced model M^- are identical. In paraconsistent PAL, $M^- \subseteq M'$.

Semantics

Let us give the semantics of ParaPAL now. Note that in ParaPAL, we have $|-p| = \text{Clo}(S \setminus |p|)$. Also, $\bot$ is true nowhere (even if $p \wedge -p$ can be true). The semantics for propositional variables and Booleans are as usual.

Let us reinstate the semantics of the modal and dynamic operators.

$$
\begin{array}{lll}
w, M \models K\varphi & \text{iff} & \exists O \in \sigma.(w \in O \wedge \forall w' \in O : w', M \models \varphi) \\
w, M \models [\varphi]\psi & \text{iff} & w, M \models \varphi \text{ implies } w, M'_{\varphi} \models \psi
\end{array}
$$

Reduction Axioms

Theorem

ParaPAL reduces to epistemic paraconsistent logic by the following reduction axioms:

- $[\varphi]p \leftrightarrow (\varphi \rightarrow p)$
- $[\varphi]-\psi \leftrightarrow (\varphi \rightarrow -[\varphi]\psi)$
- $[\varphi]\psi \wedge \chi \leftrightarrow [\varphi]\psi \wedge [\varphi]\chi$
- $[\varphi]K\psi \leftrightarrow (\varphi \rightarrow K[\varphi]\psi)$

Further Results I

The biggest advantage of using a topological background theory to express dynamic epistemic matters in a paraconsistent logic is to have the ability to make use of the topological properties of the model in understanding dynamic epistemic reasoning.

Definition A set X is called connected if $A \cap B \neq \emptyset$ whenever A, B are closed non-empty subsets and $X = A \cup B$. It is called totally disconnected if all of its subsets with more than one element are disconnected.

If the public announcement *disconnects* a space, then we can reduce the inconsistency to consistency by means of a public announcement.

Further Results II

Theorem Let $M = (S, \sigma, v)$ be ParaPAL model where (S, σ) is an arbitrary topological space. Then if there exists a formula φ such that the topological space (S', σ') obtained after the announcement is totally disconnected, then $M'_\varphi = (S', \sigma', v')$ cannot be inconsistent.

The existence of the public announcement φ that can turn arbitrary topological spaces to totally disconnected topological spaces is not guaranteed in each and every model.

Theorem[(Başkent, 2012)] Let X be a connected topological space of closed sets with a paraconsistent topological model on it. Then, the only subtheory that is not inconsistent is the empty theory.

We can improve the above result within the context of ParaPAL.

Further Results III

Theorem Let $M = (S, \sigma, v)$ be a ParaPAL model where (S, σ) is a connected topological space of closed sets. Then, the announcement of $\perp$ produces an updated model of M that has consistent theories.

Given a ParaPAL model M and an arbitrary formula φ, what is the connection between M and M'_{φ} in terms of continuous transformations? For this question, we will use the functional representation of announcements, which we defined earlier.

Theorem Every announcement is functionally representable in ParaPAL.

Conclusion

We have observed that the methods of “eliminating the states that do not satisfy the announcement” and “keeping the states that satisfy the announcement” are not identical in paraconsistent PAL, unlike in classical logic; and paraconsistent framework presents a finer model to update knowledge.

Thank you for your attention!

Talk slides and the papers are available at:

`www.CanBaskent.net/Logic`

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