Public Announcements, Topology and Paraconsistency

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Outlook of the Talk

- Public Announcement Logic with Topological Semantics
- Paraconsistent Public Announcement Logic with Topological Semantics
- Some results

Examples

- Knowing an inconsistent theory
- Impossible worlds
- Knowing true paradoxes

Logical Motivations

- Describe a model with impossible worlds - worlds that satisfy contradictions
- Choose a semantical structure that can work both in classical, non-classical logics and dynamic logics

Topological Semantics for Modal Logic

Given a set S, a topology σ is a collection of subsets of S satisfying the following conditions.

- The empty set and S are in σ,
- The collection σ is closed under finite intersection and arbitrary unions.

We call the tuple (S, σ) a topological space. Let $M = (S, \sigma, v)$ be a topological model where (S, σ) is a topology and v is a valuation.

Define the extension of φ in M: $|\varphi|^M = \{s \in S : s, M \models \varphi\}$.

Topological Semantics for Modal Logic

In a topology, for a given set, we define the *interior* operator Int and the *closure* operator Clo as the operators which return the largest open set contained in the given set, and the smallest closed set containing the given set respectively.

The extensions of modal/epistemic formulas depend on such operators. We put $|\mathsf{K}\varphi| = \mathsf{Int}(|\varphi|)$.

$$w, M \models \mathsf{K}\varphi \quad \textit{iff} \quad \exists O \in \sigma.(w \in O \wedge \forall w' \in O, w', M \models \varphi)$$

Topological Semantics for Public Announcements

For an announcement φ, define the *updated* model $M'_\varphi = (S', \sigma', v')$: Set $S' = S \cap |\varphi|$, $\sigma' = \{O \cap S' : O \in \sigma\}$, and $v' = v \cap S'$.

The new topology σ', which we obtained by relativizing σ is called the *induced topology*.

The language of topological PAL includes the epistemic modality K and the public announcement modality $[\cdot]$

$$w, M \models [\varphi]\psi \quad \textit{iff} \quad w, M \models \varphi \quad \textit{implies} \quad w, M' \models \psi$$

Topological Semantics for Public Announcements

The axiomatization of the topological PAL does not differ from the traditional PAL with Kripke semantics.

1. All the substitutional instances of the tautologies of the *classical* propositional logic
2. $\mathsf{K}(\varphi \to \psi) \to (\mathsf{K}\varphi \to \mathsf{K}\psi)$
3. $\mathsf{K}\varphi \to \varphi$
4. $\mathsf{K}\varphi \to \mathsf{K}\mathsf{K}\varphi$
5. $\neg\mathsf{K}\varphi \to \mathsf{K}\neg\mathsf{K}\varphi$
6. $[\varphi]p \leftrightarrow (\varphi \to p)$
7. $[\varphi]\neg\psi \leftrightarrow (\varphi \to \neg[\varphi]\psi)$
8. $[\varphi](\psi \wedge \chi) \leftrightarrow ([\varphi]\psi \wedge [\varphi]\chi)$
9. $[\varphi]\mathsf{K}\psi \leftrightarrow (\varphi \to \mathsf{K}[\varphi]\psi)$

Topological Semantics for Public Announcements

The rules of deduction in topological PAL are as expected: normalization and modus ponens. Based on this axiomatization and the semantics, we observe the following.

Theorem

PAL in topological models is complete and decidable.

Topological Semantics for ParaPAL

- Topological semantics is versatile - it can carry over to *other* logics.
- Stipulate that the extension of propositional variables are also closed sets.
- Notice that the topological semantics for the classical (modal) logic does not impose any condition on the topological qualities on (the extensions of) propositional variables.
- Then, what aboution negation as the compliment of a closed set is not necessarily a closed set.

Negation in Paraconsistency

Define negation as the "closure of the complement".

In this case, boundary points, the points that are shared by the closure of a given set and the closure of its complement, are the points that satisfy the contradictions. Let us denote this paraconsistent negation by $-$.

Take $p \wedge -p$, where $|p| = U \in \sigma$ for a closed set topology $U \in \sigma$.

Then $|p \wedge -p|$ is $K \cap \text{Clo}(\overline{K})$ which is $\partial(K)$ where $\partial(\cdot)$ is the boundary operator which is defined as $\partial K := \text{Clo}(K) - \text{Int}(K)$.

Therefore, the contradictions hold on the boundary points.

Thus, we now have a paraconsistent logic in which contradictions do not trivialize the system.

Public Announcements

In PAL, an external and truthful announcement is made. Then, the agents update their models by eliminating the states which do *not* agree with the announcement.

In paraconsistent spaces, public announcements obtain a broader meaning. Namely, when φ is announced in a paraconsistent space, it simply means "Keep φ". It can very well be the case that some of the possible worlds that satisfy φ may also satisfy $-\varphi$, namely, those states may be impossible worlds.

The main problem caused by the inconsistencies is that they trivialize the theory and collapse the model. Therefore, if there exists some contradictions that do *not* trivialize the theory, there seems to be no need to eliminate them.

ParaPAL Models

We obtain ParaPAL models in the exact same way.

Let $M = (S, \sigma, v)$ be a topological model where (S, σ) is a closed set topology where every $K \in \sigma$ is a closed set.

For an announcement φ, we obtain an *updated* model $M'_\varphi = (S', \sigma', v')$ where $S' = S \cap |\varphi|^M$, $\sigma' = \{K \cap S' : K \in \sigma\}$, and $v' = v \cap S'$.

We stipulate that the extension of each propositional variable is closed. The intention here is to impose that the extension of each *formula* must be a closed set as closedness is preserved with the logical connectives in this framework.

ParaPAL Semantics

The semantics is as we described

$$
\begin{array}{lll}
|{-}\varphi|^M & = & \mathrm{Clo}(S \setminus |\varphi|^M) \\
w, M \models \mathrm{K}\varphi & \textit{iff} & \exists K \in \sigma.(w \in K \wedge \forall w' \in K : w', M \models \varphi) \\
w, M \models [\varphi]\psi & \textit{iff} & w, M \models \varphi \quad \textit{implies} \quad w, M' \models \psi
\end{array}
$$

Quick Results

Call the non-dynamic segment of ParaPAL as *Paraconsistent Topological Logic* (PTL).

Theorem

ParaPAL reduces to PTL by the following reduction axioms:

- $[\varphi]p \leftrightarrow (\varphi \rightarrow p)$
- $[\varphi]-\psi \leftrightarrow (\varphi \rightarrow -[\varphi]\psi)$
- $[\varphi]\psi \wedge [\varphi]\chi \leftrightarrow [\varphi]\psi \wedge [\varphi]\chi$
- $[\varphi]\mathsf{K}\psi \leftrightarrow (\varphi \rightarrow \mathsf{K}[\varphi]\psi)$

Quick Results

Remark

ParaPAL and PTL are equi-expressible.

Yet, when compared to the classical PAL, ParaPAL provides a more expressive framework as some contradictions can be true in some models.

Remark

ParaPAL is more expressive than PAL.

In ParaPAL, we can have true statements such as $[p](q \wedge -q)$ or $[\top]\bot$ as we discussed earlier.

Thanks for your attention!

Talk slides and the paper are available at

`www.CanBaskent.net/Logic`