Paraconsistent Dynamic Epistemic Logic

topological models for inconsistent knowledge

Can Başkent

Department of Computer Science, University of Bath
can@canbaskent.net www.canbaskent.net/logic @topologically

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Today's Plan

1. Motivation
2. Public Announcement Logic
3. Paraconsistent Public Announcement Logic
4. Conclusion

Slogan: Paraconsistency for Dynamic Knowledge!

Paraconsistency helps us understand

inconsistent knowledge and its dynamics

and

it has a natural topological semantics

incorporating **homotopies** into the theory.

Motivation

Examples

- *Knowing an inconsistent theory* - developing a dynamic logic that can work with real life examples, scientific theories, large databases
- *Impossible worlds* - a not-so-well-studied dual of possible worlds can be used to describe various states
- *Knowing true paradoxes* - rational agents dealing with game theoretical paradoxes
- *Revising a knowledge base with inconsistencies* - removing the inconsistencies is not necessarily the only way

Logical Motivations

- Developing a model with impossible or inconsistent worlds
- Constructing a semantical structure that can work both in classical, non-classical logics and dynamic logics
- Following the dynamic agenda in modal logic and focusing on the **transformations** as the dynamic operators
- Understanding **paraconsistent** systems better - where inconsistencies do not entail everything or where some propositions can be both true and false

Public Announcement Logic

Topological Semantics for Modal Logic

Given a non-empty set S, a **topology** σ is a collection of subsets of S satisfying the following conditions.

- The empty set and S are in σ,
- The collection σ is closed under finite intersections and arbitrary unions.

We call the tuple (S, σ) a **topological space**, and the members of σ as **open sets**.

A **topological model** is a tuple $M = (S, \sigma, v)$ where (S, σ) is a topology and v is a valuation.

The **extension** of φ in M is the set $|\varphi|^M = \{s \in S : s, M \models \varphi\}$.

Topological Semantics for Modal Logic

We define the **interior** operator Int and the **closure** operator Clo as the operators which return the largest open set contained in the given set, and the smallest closed set containing the given set respectively.

The extensions of modal/epistemic formulas depend on such operators. We put $|\mathrm{K}\varphi| = \mathrm{Int}(|\varphi|)$.

$$w, M \models \mathrm{K}\varphi \quad \textit{iff} \quad \exists O \in \sigma.(w \in O \wedge \forall w' \in O, w', M \models \varphi)$$

Topological semantics is the oldest semantics for modal logic (1938).

Public Announcement Logic

In Public Announcement Logic, an external and truthful announcement is made.

Then, the agents update their models by eliminating the states which do *not* agree with the announcement.

Public Announcements - a toy example

Ann, Bob and Cathy go to a cafe and order tea, coffee and lemonade, respectively. After a while *another* waiter brings the beverages. He asks who ordered the tea, Ann waives her hand. Then, he asks who ordered the coffee, and Bob says he did.

Finally, the waiter, *without asking*, gives the lemonade to Cathy.

Public Announcements - a toy example

Ann, Bob and Cathy go to a cafe and order tea, coffee and lemonade, respectively.

Ann	Bob	Cathie
tea	**coffee**	**lemonade**
tea	lemonade	coffee
coffee	lemonade	tea
coffee	tea	lemonade
lemonade	coffee	tea
lemonade	tea	coffee

Public Announcements - a toy example

Ann, Bob and Cathy go to a cafe and order tea, coffee and lemonade, respectively.

Ann	Bob	Cathie
tea	**coffee**	**lemonade**
tea	lemonade	coffee
coffee	lemonade	tea
coffee	tea	lemonade
lemonade	coffee	tea
lemonade	tea	coffee

$\Longrightarrow$ Ann: “I ordered tea!”

Public Announcements - a toy example

Ann, Bob and Cathy go to a cafe and order tea, coffee and lemonade, respectively.

Ann	Bob	Cathie
tea	**coffee**	**lemonade**
tea	lemonade	coffee

Public Announcements - a toy example

Ann, Bob and Cathy go to a cafe and order tea, coffee and lemonade, respectively.

Ann	Bob	Cathie
tea	**coffee**	**lemonade**
tea	lemonade	coffee

$\Longrightarrow$ Bob: "I ordered coffee!"

Public Announcements - a toy example

Ann, Bob and Cathy go to a cafe and order tea, coffee and lemonade, respectively.

Ann	Bob	Cathie
tea	**coffee**	**lemonade**
tea	lemonade	coffee

$\Longrightarrow$

Ann	Bob	Cathie
tea	**coffee**	**lemonade**

For an announcement φ, define the **updated model** $M'_\varphi = (S', \sigma', v')$: Set $S' = S \cap |\varphi|$, $\sigma' = \{O \cap S' : O \in \sigma\}$, and $v' = v \cap S'$.

The new topology σ', which is obtained by relativizing σ with respect to φ is called the *induced topology*.

The language of topological PAL includes the epistemic modality K and the public announcement modality [·].

$$w, M \models [\varphi]\psi \quad \textit{iff} \quad w, M \models \varphi \quad \textit{implies} \quad w, M' \models \psi$$

Topological Semantics for Public Announcements

The axiomatization of the topological PAL does not differ from the traditional PAL with Kripke semantics.

1. All the substitutional instances of the tautologies of the *classical* propositional logic
2. $\mathsf{K}(\varphi \to \psi) \to (\mathsf{K}\varphi \to \mathsf{K}\psi)$
3. $\mathsf{K}\varphi \to \varphi$
4. $\mathsf{K}\varphi \to \mathsf{KK}\varphi$
5. $\neg\mathsf{K}\varphi \to \mathsf{K}\neg\mathsf{K}\varphi$
6. $[\varphi]p \leftrightarrow (\varphi \to p)$
7. $[\varphi]\neg\psi \leftrightarrow (\varphi \to \neg[\varphi]\psi)$
8. $[\varphi](\psi \wedge \chi) \leftrightarrow ([\varphi]\psi \wedge [\varphi]\chi)$
9. $[\varphi]\mathsf{K}\psi \leftrightarrow (\varphi \to \mathsf{K}[\varphi]\psi)$

Topological Semantics for Public Announcements

The rules of deduction in topological PAL are as expected: normalization for both modalities and modus ponens.

Theorem

PAL in topological models is complete and decidable with respect to the given axiomatization.

Paraconsistent Public Announcement Logic

Topological Semantics for ParaPAL

Notice that the topological semantics for the classical (modal) logic does not impose any condition on the topological qualities on (the extensions of) propositional variables.

Stipulate that the extension of propositional variables are also closed sets (or dually, open sets for intuitionistic logic).

Then, what about negation? - as the compliment of a closed set is not necessarily a closed set.

Negation in Paraconsistency

Define **a new negation** as the "closure of the complement".

In this case, boundary points $\partial(\cdot)$, the points that are shared by the closure of a given set and the closure of its complement, are the points that satisfy the contradictions: $\partial U := \text{Clo}(U) - \text{Int}(U)$.

Let us denote the paraconsistent negation by $-$.

Negation in Paraconsistency

Take $p \wedge -p$, where $|p| = U \in \sigma$ for a closed set topology σ.

Then $|p \wedge -p|$ is $U \cap \text{Clo}(\overline{U})$ which is $\partial(U)$.

Therefore, the contradictions are satisfied on the boundary points.

We now have a **paraconsistent logic** in which contradictions do not trivialize the system.

We call this system ParaPAL.

Public Announcements

In PAL, an external and truthful announcement is made. Then, the agents update their models by eliminating the states which do *not* agree with the announcement.

In paraconsistent spaces, public announcements obtain a broader meaning.

When φ is announced in a paraconsistent space, it simply means "Keep φ". It can be the case that some of the possible worlds that satisfy φ may also satisfy $-\varphi$.

The main problem caused by the inconsistencies is that they trivialize the theory and collapse the model. Therefore, if there exists some contradictions that do *not* trivialize the theory, there seems to be no need to eliminate them.

We obtain ParaPAL models in the exact same way.

Let $M = (S, \sigma, v)$ be a topological model where (S, σ) is a closed set topology where every $K \in \sigma$ is a closed set.

For an announcement φ, we obtain an *updated* model $M'_\varphi = (S', \sigma', v')$ where $S' = S \cap |\varphi|^M$, $\sigma' = \{K \cap S' : K \in \sigma\}$, and $v' = v \cap S'$.

We stipulate that the extension of each propositional variable is closed. The intention here is to impose that the extension of each *formula* must be a closed set as closedness is preserved with the logical connectives in this framework.

ParaPAL Semantics

The semantics is as we described:

$$
\begin{array}{lcl}
|{-\varphi}|^M & = & \mathrm{Clo}(S \setminus |\varphi|^M) \\
w, M \models \mathrm{K}\varphi & \textit{iff} & \exists K \in \sigma.(w \in K \wedge \forall w' \in K : w', M \models \varphi) \\
w, M \models [\varphi]\psi & \textit{iff} & w, M \models \varphi \quad \textit{implies} \quad w, M' \models \psi
\end{array}
$$

Results

Call the non-dynamic segment of ParaPAL as *Paraconsistent Topological Logic* (PTL) - the system without the $[\cdot]$ operator.

Theorem

ParaPAL reduces to PTL by the following reduction axioms:

- $[\varphi]p \leftrightarrow (\varphi \to p)$
- $[\varphi]-\psi \leftrightarrow (\varphi \to -[\varphi]\psi)$
- $[\varphi]\psi \wedge [\varphi]\chi \leftrightarrow [\varphi]\psi \wedge [\varphi]\chi$
- $[\varphi]\mathsf{K}\psi \leftrightarrow (\varphi \to \mathsf{K}[\varphi]\psi)$

Results

Remark

ParaPAL and PTL are equi-expressible.

Yet, when compared to the classical PAL, ParaPAL provides a more expressive framework as some contradictions can be true in some models.

Remark

ParaPAL is more expressive than PAL.

In ParaPAL, we can have true statements such as $[p]\mathrm{K}(q \wedge -q)$ or $[p]\mathrm{K}(p \wedge -p)$.

Functional Representation

For an announcement φ, we say

"φ is functionally representable in a topological model $M = (S, \sigma, v)$"

if there is an open and continuous function $f : (S, \sigma) \mapsto (S', \sigma')$ where $M = (S', \sigma', v)$ is the updated model.

Functional Representation

Theorem

Every public announcement is functionally representable.

But functional representation may not be one-to-one.

How can we generalize it?

Homeomorphic Models

Given two models $M = (S, \sigma, v)$ and $M' = (S'\sigma', v')$. We call M and M' *homeomorphic φ-models* if M' is the updated model of M with the public announcement φ, and there is a homeomorphism f from (S, σ) into (S', σ') that functionally represents φ.

Notice that homeomorphic model relation is not symmetric, but it is reflexive and transitive. Homeomorphic φ-models enjoy the same topological qualities after a specific public announcement (here, φ).

Let S and S' be two topological spaces with continuous functions $f, g : S \mapsto S'$.

A homotopy between f and g is a continuous function $H : S \times [0, 1] \mapsto S'$ such that for $s \in S$, $H(s, 0) = f(s)$ and $H(s, 1) = g(s)$.

Theorem

Given M, consider a family of updated homeomorphic models $\{M_i\}_{i<\omega}$ each of which is obtained by an announcement φ_i representable by f_i. Then f_is are homotopic.

Complex Results

Theorem

Let M be a ParaPAL model. If there is a formula φ such that the updated model M' obtained after announcing φ is totally disconnected, then M' cannot be inconsistent.

Theorem

Let M be a ParaPAL model with a compact Hausdorff topological space. Then, the model stabilization for M takes less than ω steps, contrary to the ω step for arbitrary ParaPAL models.

Conclusion

Conclusion

Topological semantics is rich. It provides a semantical framework for a wide variety of logics: intuitionistic, paraconsistent, classical, dynamic.

Paraconsistency has direct applications in dynamic epistemology and multi-agent systems.

This paper combines these two attitudes, and introduces paraconsistent topological semantics and homotopies to dynamic epistemic logic.

A Brief Bibliography

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Thank you!

Any Questions?

Talk slides and the papers are available at

`CanBaskent.net/Logic`