

# Towards Paraconsistency Game Theory

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## Slogan: Paraconsistency for Game Theory!

**Paraconsistency helps us understand  
game theory better,  
and  
game theory helps us understand  
paraconsistency better.**

# Plan

- ▶ Paraconsistent Game Semantics
- ▶ A Self-Referential Paradox in Games
- ▶ A Non-Self-Referential Paradox in Games

# Part One

## Paraconsistent Game Semantics

# Outlook of Part One

- ▶ Logic of Paradox
- ▶ First-Degree Entailment
- ▶ Connexive Logic

# Classical Game Semantics

During the semantic verification game, the given formula is broken into subformulas by **two** players (Abelard and Heloise) **step by step**, and the game **terminates** when it reaches the propositional atoms.

If we end up with a propositional atom which is true, then Eloise the verifier wins the game. **Otherwise**, Abelard the falsifier wins. We associate **conjunction with Abelard, disjunction with Heloise**.

A win for the verifier is when the game terminates with a true statement. The verifier is said to have a winning strategy if she can force the game to her win, regardless of how her opponent plays.

# Classical Game Semantics

Just because the game may end with a true/false atom does not necessarily suggest the truth/falsity of the given formula in general.

In classical logic, however, the major result of game theoretical semantics states that the verifier has a winning strategy **if and only if** the given formula is true in the model.

# Classical Games

Classical semantic games are

- ▶ Two-player,
- ▶ Determined,
- ▶ Sequential,
- ▶ Zero-sum,
- ▶ Complete: winning strategies necessarily and sufficiently guarantee the truth value.

**Question** How do these attributes of semantical games depend on the underlying logical structure? How can we give game semantics for *deviant* logics?



# Logic of Paradox

# Logic of Paradox and GTS

Consider Priest's Logic of Paradox (LP) (Priest, 1979).

LP introduces an additional truth value  $P$ , called *paradoxical*, that stands for both true and false.

|     | $\neg$ |
|-----|--------|
| $T$ | $F$    |
| $P$ | $P$    |
| $F$ | $T$    |

| $\wedge$ | $T$ | $P$ | $F$ |
|----------|-----|-----|-----|
| $T$      | $T$ | $P$ | $F$ |
| $P$      | $P$ | $P$ | $F$ |
| $F$      | $F$ | $F$ | $F$ |

| $\vee$ | $T$ | $P$ | $F$ |
|--------|-----|-----|-----|
| $T$    | $T$ | $T$ | $T$ |
| $P$    | $T$ | $P$ | $P$ |
| $F$    | $T$ | $P$ | $F$ |

# Game Models

We define the verification game as a tuple

$\Gamma = (\pi, \rho, \delta, \sigma)$  where

- $\pi$  is the set of players,
- $\rho$  is the set of well-defined game rules,
- $\delta$  is the set of designated truth values: the truth values preserved under validities: they determine the theorems of the logic.
- $\sigma$  is the set of positions: subformula and player pairs.

It is possible to extend it to concurrent games as well.

## Game Rules for LP

The introduction of the additional truth value  $P$  requires an additional player in the game, let us call him *Astrolabe* (after Abelard and Heloise's son).

Since we have three truth values in LP, we need three players forcing the game to their win. If the game ends up in their truth set, then that player wins.

Then, how to associate moves with the connectives?

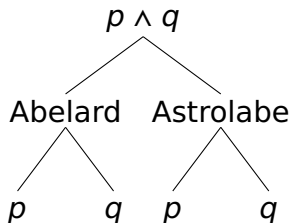
# Game Rules for LP

Denote this system with  $GTS^{LP}$ .

|              |   |
|--------------|---|
| $p$          | whoever has $p$ in their extension, wins                        |
| $\neg F$     | Abelard and Heloise switch roles                                |
| $F \wedge G$ | Abelard and Astrolabe choose between $F$ and $G$ simultaneously |
| $F \vee G$   | Eloise and Astrolabe choose between $F$ and $G$ simultaneously  |

## An Example

Consider the conjunction. Take the formula  $p \wedge q$  where  $p, q$  are  $P, F$  respectively. Then,  $p \wedge q$  is  $F$ .



Abelard makes a move and chooses  $q$  which is false. This gives him a win. Interesting enough, Astrolabe chooses  $p$  giving him a win.

In this case both seem to have a winning strategy. Moreover, the win for Abelard does not entail a loss for Astrolabe.

# Correctness

## Theorem

In  $\text{GTS}^{\text{LP}}$  verification game for  $\varphi$ ,

- ▶ Eloise has a winning strategy if  $\varphi$  is true,
- ▶ Abelard has a winning strategy if  $\varphi$  is false,
- ▶ Astrolabe has a winning strategy if  $\varphi$  is paradoxical.

# Correctness

## Theorem

In a  $GTS^{LP}$  game for a formula  $\varphi$  in a LP model  $M$ ,

- ▶ If Eloise has a winning strategy, but Astrolabe does not, then  $\varphi$  is true (and only true) in  $M$ ,
- ▶ If Abelard has a winning strategy, but Astrolabe does not, then  $\varphi$  is false (and only false) in  $M$ ,
- ▶ If Astrolabe has a winning strategy, then  $\varphi$  is paradoxical in  $M$ .



# First-Degree Entailment

# First-Degree Entailment

Semantic valuations are *functions* from formulas to truth values.

If we replace the valuation function with a valuation *relation*, we obtain *First-degree entailment* (FDE) which is due to Dunn (Dunn, 1976).

We use  $\varphi \mathbf{r} 1$  to denote the truth value of  $\varphi$  (which is 1 in this case).

Since,  $\mathbf{r}$  is a relation, we allow  $\varphi \mathbf{r} \emptyset$  or  $\varphi \mathbf{r} \{0, 1\}$ .

Thus, FDE is a paraconsistent (inconsistency-tolerant) and paracomplete (incompleteness-tolerant) logic.

# First-Degree Entailment

For formulas  $\varphi, \psi$ , we define  $\mathbf{r}$  as follows.

|                                    |            |  |
|------------------------------------|------------|--|
| $\neg\varphi\mathbf{r}1$           | <i>iff</i> | $\varphi\mathbf{r}0$                       |
| $\neg\varphi\mathbf{r}0$           | <i>iff</i> | $\varphi\mathbf{r}1$                       |
| $(\varphi \wedge \psi)\mathbf{r}1$ | <i>iff</i> | $\varphi\mathbf{r}1$ and $\psi\mathbf{r}1$ |
| $(\varphi \wedge \psi)\mathbf{r}0$ | <i>iff</i> | $\varphi\mathbf{r}0$ or $\psi\mathbf{r}0$  |
| $(\varphi \vee \psi)\mathbf{r}1$   | <i>iff</i> | $\varphi\mathbf{r}1$ or $\psi\mathbf{r}1$  |
| $(\varphi \vee \psi)\mathbf{r}0$   | <i>iff</i> | $\varphi\mathbf{r}0$ and $\psi\mathbf{r}0$ |

## Game Semantics for FDE

The truth values  $\{0\}$ ,  $\{1\}$  and  $\{0, 1\}$  work exactly as the truth values  $F, T, P$  respectively in LP. In fact, LP can be obtained from FDE by introducing a restriction that no formula gets the truth value  $\emptyset$ .

Recall that for  $GTS^{LP}$ , we allowed parallel plays for selected players depending on the syntax of the formula: we associated conjunction with Abelard and Astrolabe, disjunction with Heloise and Astrolabe.

## Game Semantics for FDE

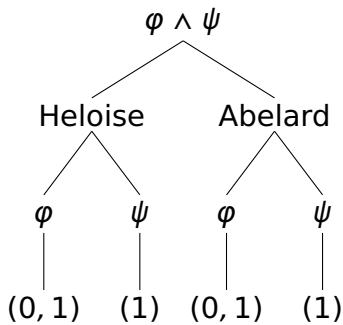
For FDE, the idea is to allow each player play at each node.

Therefore, it is possible that both players (or none) may have a winning strategy.

## An Example

Consider two formulas with the following relational semantics:  $\varphi \mathbf{r}0$ ,  $\varphi \mathbf{r}1$  and  $\psi \mathbf{r}1$ . In this case, we have  $(\varphi \wedge \psi) \mathbf{r}1$  and  $(\varphi \wedge \psi) \mathbf{r}0$ .

We expect both Abelard and Heloise have winning strategies, and allow each player make a move at each node.



# Game Rules for FDE

|              |   |
|--------------|---|
| $p$          | whoever has $p$ in their extension, wins                      |
| $\neg F$     | players switch roles  |
| $F \wedge G$ | Abelard and Heloise choose between $F$ and $G$ simultaneously |
| $F \vee G$   | Abelard and Heloise choose between $F$ and $G$ simultaneously |

# Correctness

## Theorem

In a  $GTS^{\text{FDE}}$  verification game for a formula  $\varphi$ , we have the following:

- ▶ Heloise has a winning strategy if  $\varphi \mathbf{r} 1$
- ▶ Abelard has a winning strategy if  $\varphi \mathbf{r} 0$
- ▶ No player has a winning strategy if  $\varphi \mathbf{r} \emptyset$



# Connexive Logic

## McCall's Connexive Logic

Connexive logic is a “comparatively little-known and to some extent neglected branch of non-classical logic” (Wansing, 2015). Even if it is under-studied, its philosophical roots can be traced back to Aristotle and Boethius.

Connexive logic is defined as a system which satisfies the following two schemes of conditionals:

- ▶ **Aristotle's Theses:**  $\neg(\neg\varphi \rightarrow \varphi)$
- ▶ **Boethius' Theses:**  $(\varphi \rightarrow \neg\psi) \rightarrow \neg(\varphi \rightarrow \psi)$

We discuss one of the earliest examples of connexive logics CC, which is due to McCall (McCall, 1966).

## McCall's Connexive Logic

CC is axiomatized by adding the scheme  $(\varphi \rightarrow \varphi) \rightarrow \neg(\varphi \rightarrow \neg\varphi)$  to the propositional logic. The rules of inference for CC is modus ponens and adjunction, which is given as  $\vdash \varphi, \vdash \psi \therefore \vdash \varphi \wedge \psi$ .

The semantics for CC is given with 4 truth values:  $T$ ,  $t$ ,  $f$  and  $F$  which can be viewed as "logical necessity", "contingent truth", "contingent falsehood", and "logical impossibility" respectively (Routley & Montgomery, 1968).

In CC, the designated truth values are  $T$  and  $t$ .

# McCall's Connexive Logic

|     |        |
|-----|--------|
|     | $\neg$ |
| $T$ | $F$    |
| $t$ | $f$    |
| $f$ | $t$    |
| $F$ | $T$    |

|     |          |     |     |     |     |
|-----|----------|-----|-----|-----|-----|
|     | $\wedge$ | $T$ | $t$ | $f$ | $F$ |
| $T$ | $T$      | $t$ | $f$ | $F$ |     |
| $t$ | $t$      | $T$ | $F$ | $f$ |     |
| $f$ | $f$      | $F$ | $f$ | $F$ |     |
| $F$ | $F$      | $f$ | $F$ | $f$ |     |

|     |        |     |     |     |     |
|-----|--------|-----|-----|-----|-----|
|     | $\vee$ | $T$ | $t$ | $f$ | $F$ |
| $T$ | $t$    | $T$ | $t$ | $T$ |     |
| $t$ | $T$    | $t$ | $T$ | $t$ |     |
| $f$ | $t$    | $T$ | $F$ | $f$ |     |
| $F$ | $T$    | $t$ | $f$ | $F$ |     |

First, we introduce 4 players for 4 truth values:  $T$  is forced by Heloise,  $F$  by Abelard,  $t$  by Aristotle and  $f$  by Boethius.

## Game Rules for CC

As the *true*s and *false*s are closed under the binary operations respectively, we suggest the following coalitions.

### Truth-maker Coalition:

Heloise ( $T$ ) and Aristotle ( $t$ )

### False-maker Coalition:

Abelard ( $F$ ) and Boethius ( $f$ )

## Game Rules for CC

|              |  |
|--------------|--|
| $p$          | whoever has $p$ in their extension, wins   |
| $\neg F$     | switch the roles: Heloise assumes Abelard's role, Aristotle assumes Boethius' role, Boethius assumes Aristotle's role, Abelard assumes Heloise's role, and the game continues with $F$ |
| $F \wedge G$ | false-makers coalition chooses between $F$ and $G$   |
| $F \vee G$   | truth-makers coalition chooses between $F$ and $G$   |

# Correctness

## Theorem

For the evaluation games for a formula  $\varphi$  in McCall's Connexive logic, we have the following:

- ▶ truth-makers have a winning strategy if and only if  $\varphi$  has the truth value  $t$  or  $T$  in  $M$ ,
- ▶ false-makers have a winning strategy if and only if  $\varphi$  has the truth value  $f$  or  $F$  in  $M$ .

## What Have We Observed?


- ▶ Failure of the biconditional correctness
- ▶ Multiplayer semantic games in a nontrivial way
- ▶ Non-sequential / paralel / concurrent plays
- ▶ Variable sum games
- ▶ Coalitions

If winning strategies are proofs, game semantics for paraconsistent logics present a constructive way to give proofs for inconsistencies.



## Reference

See the paper for more logics:

 CB, *Game Theoretical Semantics for Paraconsistent Logics*, in "Proceedings of the Fifth International Conference on Logic, Rationality and Interaction" (LORI-V), Edited by W. van der Hoek and W. Holliday and W. Wang, pp. 14-26, Springer, 2016.

## **Part Two**

# A Self-Referential Paradox in Games

## Outlook of Lecture Two

- ▶ The Brandenburg - Keisler Paradox
- ▶ Paraconsistent Models
- ▶ A Countermodel

# The Brandenburger-Keisler Paradox

## The Paradox

The Brandenburg-Keisler paradox (BK paradox) is a two-person self-referential paradox in epistemic game theory (Brandenburger & Keisler, 2006).

The following configuration of beliefs is impossible:

### The Paradox

Ann believes that Bob assumes that Ann believes that Bob's assumption is wrong.

The paradox appears if you ask whether "Ann believes that Bob's assumption is wrong".

Notice that this is essentially a 2-person Russell's Paradox.

## The Model

Brandenburger and Keisler use belief sets to represent the players' beliefs.

The model  $(U^a, U^b, R^a, R^b)$  that they consider is called a *belief structure* where  $R^a \subseteq U^a \times U^b$  and  $R^b \subseteq U^b \times U^a$ .

The expression  $R^a(x, y)$  represents that in state  $x$ , Ann believes that the state  $y$  is possible for Bob, and similarly for  $R^b(y, x)$ . We will put  $R^a(x) = \{y : R^a(x, y)\}$ , and similarly for  $R^b(y)$ .

At a state  $x$ , we say Ann believes  $P \subseteq U^b$  if  $R^a(x) \subseteq P$ .

# The Semantics

A modal logical semantics for the interactive belief structures can be given.

We use two modalities  $\Box$  and  $\heartsuit$  for the belief and assumption operators respectively with the following semantics.

$$\begin{aligned}
 x \models \Box^{ab} \varphi & \text{ iff } \forall y \in U^b. R^a(x, y) \text{ **implies** } y \models \varphi \\
 x \models \heartsuit^{ab} \varphi & \text{ iff } \forall y \in U^b. R^a(x, y) \text{ **iff** } y \models \varphi
 \end{aligned}$$

Note the bi-implication in the definition of the assumption modality!

# Completeness

A belief structure  $(U^a, U^b, R^a, R^b)$  is called *assumption complete* with respect to a set of predicates  $\Pi$  on  $U^a$  and  $U^b$  if for every predicate  $P \in \Pi$  on  $U^b$ , there is a state  $x \in U^a$  such that  $x$  assumes  $P$ , and for every predicate  $Q \in \Pi$  on  $U^a$ , there is a state  $y \in U^b$  such that  $y$  assumes  $Q$ .



# Incompleteness

Brandenburger and Keisler showed that no belief model is complete for its (classical) first-order language.

Therefore, "not every description of belief can be represented" with belief structures (Brandenburger & Keisler, 2006).

This is a version of the diagonalization! (Yanofsky, 2003)

## Some Remarks

- ▶ BK paradox is a game theoretical example of a self-referential paradox
- ▶ It is a simple step towards the self-referentiality in games
- ▶ It raises the possibility of discussing discursive/dialogical logics within game theory proper
- ▶ Provides an interesting take on Hintikka's interrogative theory - how to inquire about a paradoxical sentence?

# Paraconsistent Topological Approach

# What is a Topology?

## Definition

The structure  $\langle S, \sigma \rangle$  is called a topological space if it satisfies the following conditions.

1.  $S \in \sigma$  and  $\emptyset \in \sigma$
2.  $\sigma$  is closed under finite unions and arbitrary intersections

Collection  $\sigma$  is called a topology, and its elements are called *closed sets*.

# Paraconsistent Topological Semantics

Use of topological semantics for paraconsistent logic is not new. To our knowledge, the earliest work discussing the connection between inconsistency and topology goes back to Goodman (Goodman, 1981).

In classical modal logic, only modal formulas produce topological objects.

However, if we stipulate that:

extension of *any* propositional variable to be a closed set (Mortensen, 2000; Mortensen, 2010), we get a paraconsistent system.

## Problem of Negation

Negation can be difficult as the complement of a closed set is not generally a closed set, thus may not be the extension of a formula in the language.

For this reason, we will need to use a new negation symbol  $\sim$  that returns the closed complement (closure of the complement) of a given set.

# Topological Belief Models

The language for the logic of topological belief models is given as follows.

$$\varphi := p \mid \sim\varphi \mid \varphi \wedge \varphi \mid \Box_a \mid \Box_b \mid \boxplus_a \mid \boxplus_b$$

where  $p$  is a propositional variable,  $\sim$  is the paraconsistent topological negation symbol which we have defined earlier, and  $\Box_i$  and  $\boxplus_i$  are the belief and assumption operators for player  $i$ , respectively.

## Topological Belief Models

For the agents  $a$  and  $b$ , we have a corresponding non-empty type space  $A$  and  $B$ , and define closed set topologies  $\tau_A$  and  $\tau_B$  on  $A$  and  $B$  respectively. Furthermore, in order to establish connection between  $\tau_A$  and  $\tau_B$  to represent belief interaction among the players, we introduce additional constructions  $t_A \subseteq A \times B$ , and  $t_B \subseteq B \times A$ . We then call the structure  $F = (A, B, \tau_A, \tau_B, t_A, t_B)$  a paraconsistent topological belief model.

A state  $x \in A$  *believes*  $\varphi \subseteq B$  if  $\{y : t_A(x, y)\} \subseteq \varphi$ . Furthermore, a state  $x \in A$  *assumes*  $\varphi$  if  $\{y : t_A(x, y)\} = \varphi$ . Notice that in this definition, we identify logical formulas with their extensions.



# Semantics

For  $x \in A$ ,  $y \in B$ , the semantics of the modalities are given as follows with a modal valuation attached to  $F$ .

$$\begin{array}{ll}
 x \models \Box_a \varphi & \text{iff } \exists Y \in \tau_B \text{ with } t_A(x, Y) \rightarrow \forall y \in Y. y \models \varphi \\
 x \models \Box_a \varphi & \text{iff } \exists Y \in \tau_B \text{ with } t_A(x, Y) \leftrightarrow \forall y \in Y. y \models \varphi \\
 y \models \Box_b \varphi & \text{iff } \exists X \in \tau_A \text{ with } t_B(y, X) \rightarrow \forall x \in X. x \models \varphi \\
 y \models \Box_b \varphi & \text{iff } \exists X \in \tau_A \text{ with } t_B(y, X) \leftrightarrow \forall x \in X. x \models \varphi
 \end{array}$$

# The Result


## Theorem

*The BK sentence is satisfiable in some paraconsistent topological belief models.*

Thus, we can construct a state which satisfies the BK sentence - push the inconsistencies to the boundaries.

## Reference

For further technical details:

 CB, *Some Non-Classical Approaches to Branderburger-Keisler Paradox*, *Logic Journal of the IGPL*, vol. 23, no. 4, pp. 533-552, 2015.

## Conclusion

This was a *self-referential* paradox in games.

What about *non-self-referential* paradoxes in games?

## **Part Three**

# A *Non-Self-Referential* Paradox in Games

## Outlook of Part Three

- ▶ Yablo's Paradox
- ▶ What is the big deal?
- ▶ A Countermodel

# Yablo's Paradox

## Yablo's Paradox

Yablo's Paradox, according to its author, is a non-self referential paradox (Yablo, 1985; Yablo, 1993).

Yablo considers the following sequence of sentences.

$S_1 : \forall k > 1, S_k$  is untrue,

$S_2 : \forall k > 2, S_k$  is untrue,

$S_3 : \forall k > 3, S_k$  is untrue,

⋮



## Why is it a Paradox?

By using *reductio*, Yablo argues that the above set of sentences is contradictory. Here, the infinitary nature of the paradox is essential as the each finite set of  $S_n$  is satisfiable.

The scheme of this paradox is not new. To the best of our knowledge, the first analysis of this paradox was suggested in 1953 (Yuting, 1953).

## Impact of Yablo's Paradox

Ketland showed that the paradox is  $\omega$ -inconsistent (Ketland, 2005).

Barrio showed that Yablo's Paradox in first-order arithmetic has a model and not inconsistent, but it is  $\omega$ -inconsistent (Barrio, 2010).

It is easy to see how. Since every finite set of  $S_n$  sentences is satisfiable, then, by compactness there exists a model for the Yablo sentences. By  $\omega$ -inconsistency, it can be argued that the model we are looking for is a non-standard model of arithmetic.

## Impact of Yablo's Paradox

As Hardy puts it "Is Yablo's paradox Liar-like? In some ways yes, and in other ways no" (Hardy, 1995).

Priest offers another analysis regarding the infinitary language that it requires, and suggests a reading of the paradox that does indeed involve circularity (Priest, 1997).

Sorensen disagrees and point out the hierarchical view of Tarskian truth theory arguing that Yablo's paradox in effect "exploit[s] an alternative pattern of semantic dependency" (Sorensen, 1998).

## Applications of Yablo's Paradox

Goldstein presents a set theoretical yabloesque paradox for class membership (Goldstein, 1994).

Leitgeb suggests a yabloesque paradox for non-well-founded definitions underlining the set theoretical limitations of the logical toolbox (Leitgeb, 2005).

Picollo discusses the paradox in second-order logic generalizing the  $\omega$ -inconsistency results (Picollo, 2013).

Non-well-founded Yablo chains form a topological space encouraging Bernardi's topological approach to the paradox (Bernardi, 2009).

Cook and Beall consider Curry-like versions of the paradox (Cook, 2009; Beall, 1999).

# A Yabloesque Paradox in Epistemic Games

## A Yabloesque Paradox in Epistemic Games

Consider the following sequence of assumptions where numerals represent game theoretical players.

$A_1$  : 1 believes that  $\forall k > 1$ ,  $k$ 's assumption  $A_l$  about  $l > k$  is untrue,

$A_2$  : 2 believes that  $\forall k > 2$ ,  $k$ 's assumption  $A_l$  about  $l > k$  is untrue,

$A_3$  : 3 believes that  $\forall k > 3$ ,  $k$ 's assumption  $A_l$  about  $l > k$  is untrue,

⋮

## An Interpretation

Imagine a queue of players, where players are conveniently named after numerals, holding beliefs about each player behind them, but not about themselves. In this case, each player  $i$  believes that each player  $k > i$  behind them has an assumption about each other player  $l > k$  behind them and  $i$  believes that each  $k$ 's assumption is false.

This statement is perfectly perceivable for games, and involves a specific configuration of players' beliefs and assumptions, which can be expressible in the language. However, as we shall show, similar to Yablo's paradox and the BK paradox, this configuration of beliefs is impossible.

# The Syntax

The Yabloesque Brandenburger - Keisler paradox ('YBK Paradox', henceforth) requires  $\omega$ -many players  $i \in I$  and the following syntax.

$$\varphi := p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Box^{ij}\varphi \mid \heartsuit^{ij}\varphi$$

where  $p \in \mathbf{P}$  and  $i \neq j$  for  $i, j \in I$  with  $|I| = \omega$ .

The disjunction and implication are taken as abbreviations in the standard way.



## The Model

The extended belief model is a tuple  $M = (\{U^i\}_{i \in I}, \{R^{ij}\}_{i \neq j \in I}, V)$  where  $R^{ij} \subseteq U^i \times U^j$  and  $V$  is a valuation function.

As before, the expression  $R^{ij}(x, y)$  represents that in state  $x$ , the player  $i$  believes that the state  $y$  is possible for player  $j$ .

We prevent (a trivial form of) self-reference by disallowing players having beliefs about themselves.

# The Semantics

The semantics for the modal operators is given as follows in a similar way.

$$\begin{aligned}
 x \models \Box^{ij}\varphi & \text{ iff } \forall y \in U^j. R^{ij}(x, y) \text{ implies } y \models \varphi \\
 x \models \heartsuit^{ij}\varphi & \text{ iff } \forall y \in U^j. R^{ij}(x, y) \text{ iff } y \models \varphi
 \end{aligned}$$

# The Statement

$$A_1 := \bigwedge_{k>1} \square^{1k} \{ \bigwedge_{l>k} \heartsuit^{kl} \neg A_l \} \quad (1)$$

$$A_2 := \bigwedge_{k>2} \square^{2k} \{ \bigwedge_{l>k} \heartsuit^{kl} \neg A_l \}$$

$$A_3 := \bigwedge_{k>3} \square^{3k} \{ \bigwedge_{l>k} \heartsuit^{kl} \neg A_l \}$$

$$\vdots$$

where numerals identify the players.

The above configuration of beliefs is impossible.

# Quick Observations

## Lemma

1.  $\heartsuit^{ij}\varphi \rightarrow \square^{ij}\varphi$
2.  $\heartsuit^{ij}(\varphi \wedge \psi) \equiv \heartsuit^{ij}\varphi \wedge \heartsuit^{ij}\psi$
3.  $\square^{ij}(\heartsuit^{jk}\varphi \wedge \heartsuit^{jl}\psi) \equiv \square^{ij}\heartsuit^{jk}\varphi \wedge \square^{ij}\heartsuit^{jl}\psi.$
4.  $\diamond^{ij}(\heartsuit^{jk}\varphi \vee \heartsuit^{jl}\psi) \equiv \diamond^{ij}\heartsuit^{jk}\varphi \vee \diamond^{ij}\heartsuit^{jl}\psi$
5.  $\heartsuit^{ij}(\square^{jk}\varphi \wedge \square^{jl}\psi) \equiv \heartsuit^{ij}\square^{jk}\varphi \wedge \heartsuit^{ij}\square^{jl}\psi$

# Quick Observations

## Lemma

If  $w \models A_n$ , then for all  $p, q$  with  $n < p < q$ ;  
 $w \models \Box^{np} \heartsuit^{pq} \neg A_q$ .

## Theorem

If  $w \models A_n$ , then for all  $p, p', q$  with  $n < p < q$  and  
 $n < p' < q$ ; we have  $R^{pq}(v) = R^{p'q}(v')$  for all  $v \in U^p$  and  
 all  $v' \in U^{p'}$ .

# Quick Observations

## Corrollory

If  $w \models A_n$ , then  $\Box^{np} \heartsuit^{pq} \phi \leftrightarrow \Box^{np'} \heartsuit^{p'q} \phi$  for  $n < p < q$  and  $n < p' < q$ .

# Discussion and Further Remarks

## Assumption Modality and the Diagonal Formula

Assumption modality  $\heartsuit^{ij}$  is an essential part of the construction of the paradox. Without it, it is not possible to generate the YBK paradox.

For example, the following set of sentences about players' beliefs is not inconsistent.

$$A'_1 := \bigwedge_{k>1} \square^{1k} \{ \bigwedge_{l>k} \square^{kl} \neg A'_l \}$$

$$A'_2 := \bigwedge_{k>2} \square^{2k} \{ \bigwedge_{l>k} \square^{kl} \neg A'_l \}$$

$$A'_3 := \bigwedge_{k>3} \square^{3k} \{ \bigwedge_{l>k} \square^{kl} \neg A'_l \}$$

⋮



# Non-Well-Foundedness

As Yablo also argued, what we would have is a “downward facing tree with  $\omega$  branches descending from each node” (Yablo, 2004).

The set of sentences above, in other words, generate trees that are infinitely-branching which satisfy it.

# Categoricity

As argued by Ketland, the set of Yablo sentences is not satisfiable on the standard model of arithmetic, thus they are " $\omega$ -inconsistent" (Ketland, 2005).

This observation suggests that the YBK paradox can be satisfied in a game with  $\omega + 1$  players. As every finite set of  $A_n$ s in the YBK Sentence are satisfiable, by compactness, there must exist a model for the Yablo sentences.


# Conclusion


Yablo's Paradox is

- ▶ An interactive,  $\omega$ -player paradox,
- ▶ A modal paradox,
- ▶ A paradox of well-foundedness in some ways

What to do: Develop a *Curryesque* epistemic game theoretical paradox in which negation and falsity predicates are not used.

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 [canbaskent.net/logic](http://canbaskent.net/logic)

# Conclusion

Game Theory relates non-classical logics to *homo-economicus*.

It helps us understand how we make decisions, reach equilibria, reveal preferences and put utilities in goods.

# Thank you!

Slides and the papers are available at:

[www.CanBaskent.net/Logic](http://www.CanBaskent.net/Logic)

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