Paraconsistency, Topological Semantics, Homotopies and Games

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Outlook of the Talk

- What is Paraconsistency?
- Topological Semantics
- Paraconsistency and Topology
- Games
- Conclusion

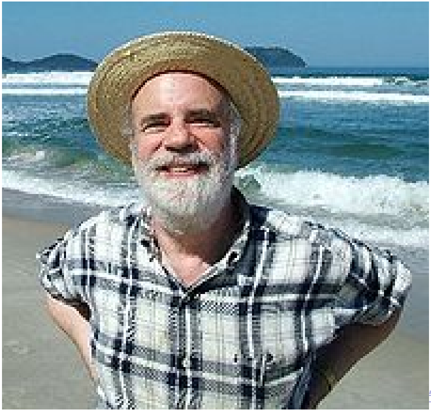
Motto

"I predict a time when there will be mathematical investigations of calculi containing contradictions, and people will actually be proud of having emancipated themselves from contradictions."

Wittgenstein, *Philosophical Remarks*

DISCLAIMER!

No Kripke Structures in this talk!



Paraconsistency

The well-studied notion of deductive explosion describes the situation where any formula can be deduced from an inconsistent set of formulae, i.e. for all formulae φ and ψ, we have $\{\varphi, \neg\varphi\} \vdash \psi$, where $\vdash$ denotes the classical logical consequence relation.

In this respect, both "classical" and intuitionistic logics are known to be explosive. Paraconsistent logic, on the other hand, is the umbrella term for logical systems where the logical consequence relation $\vdash$ is not explosive (Priest, 2002).

Motivation for Paraconsistency

Motivation for paraconsistency is usually this: we may be in a situation where our theory/information is inconsistent, but we still would like to make inference **sensibly**.

There are several class of situations where paraconsistency could be thought of a natural approach.

- Computer databases
- Scientific theories
- Law
- Counterfactuals
- Various human behavior

(Priest, 2002)

The First Semantics for Modal Logics

The history of the topological semantics of (modal) logics can be traced back to early 1920s making it the first semantics for variety of modal logics (Goldblatt, 2006). The major revival of the topological semantics of modal logics and its connections with algebras, however, is due to McKinsey and Tarski (McKinsey & Tarski, 1946; McKinsey & Tarski, 1944).

What is a Topology?

Definition

The structure $\langle S, \tau \rangle$ is called a topological space if it satisfies the following conditions.

1. $S \in \tau$ *and* $\emptyset \in \tau$
2. τ *is closed under arbitrary unions and finite intersections*

Definition

The structure $\langle S, \sigma \rangle$ is called a topological space if it satisfies the following conditions.

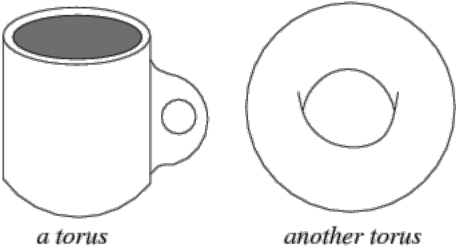
1. $S \in \sigma$ *and* $\emptyset \in \sigma$
2. σ *is closed under finite unions and arbitrary intersections*

What is a Topology

Collections σ and τ are called topologies.
The elements of τ are called *open* sets whereas the elements of σ are called *closed* sets. Therefore, a set is open if its complement in the same topology is a closed set and vice versa.

Homeomorphism

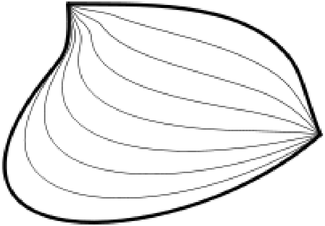
Two topological spaces are called *homeomorphic* if there is function from one to the other which is a continuous bijection with a continuous inverse.



Homeomorphic spaces share the same topological properties (compactness, connectedness etc).

Homotopy

Two continuous functions are called *homotopic* if there is a continuous deformation between the two.



Many algebraic-topological concepts are homotopy invariant (I don't know much about them).

Topological Semantics

In topological semantics, the modal operator for necessitation corresponds to the topological *interior* operator Int where $\text{Int}(O)$ is the largest open set contained in O. Furthermore, one can dually associate the topological closure operator Clo with the possibility modal operator $\Diamond$ where the closure $\text{Clo}(O)$ of a given set O is the smallest closed set that contains O.

Let us set a piece of notation and terminology. The extension, i.e. the points at which the formula is satisfied, of a formula φ in the model M will be denoted as $[\varphi]$.

Thus, we will have $[\Box\varphi] = \text{Int}([\varphi])$.

Similarly, we will put $[\Diamond\varphi] = \text{Clo}([\varphi])$.

History

Use of topological semantics for paraconsistent logic is not new. To our knowledge, the earliest work discussing the connection between inconsistency and topology goes back to Goodman (Goodman, 1981)[1].

In a recent work, Priest discussed the dual of the intuitionistic negation operator and considered that operator in topological framework (Priest, 2009). Similarly, Mortensen discussed topological seperation principles from a paraconsistent and paracomplete point of view and investigated the theories in such spaces (Mortensen, 2000). Similar approaches from modal perspective was discussed by Béziau, too (Béziau, 2005).

[1] Thanks to Chris Mortensen for pointing this work out.

How to Connect?

Recall: $[\Box\varphi] = \text{Int}([\varphi])$ and $[\Diamond\varphi] = \text{Clo}([\varphi])$. Namely, in basic modal logic, only modal formulas produce topological objects.

Stipulate that:
extension of *any* propositional variable will be an open set, or
extension of *any* propositional variable will be a closed set
(Mortensen, 2000).

Problem of Negation -1

Negation can be difficult as the complement of an open set is not generally an open set, thus may not be the extension of a formula in the language. For this reason, we will need to use a new negation symbol $\dot{\sim}$ that returns the open complement (interior of the complement) of a given set.

Problem of Negation - 2

A similar idea can also be applied to closed sets where we assume that the extension of any propositional variable will be a closed set. In order to be able to avoid a similar problem with the negation, we stipulate yet another negation operator which returns the closed complement (closure of the complement) of a given set. In this setting, we will the symbol $\sim$ that returns the closed complement of a given set.

In other words, we *separate* Boolean negation into two dual negations.

Incomplete Topological Theories

Let us consider the boundary $\partial(\cdot)$ of a set X where $\partial(X)$ is defined as $\partial(X) := \mathsf{Clo}(X) - \mathsf{Int}(X)$. Consider now, for a given formuala φ, the boundary of its extension $\partial([\varphi])$ in the topology of opens τ. Let $x \in \partial([\varphi])$. Since $[\varphi]$ is open, $x \notin [\varphi]$. Similarly, $x \notin [\dot{\sim}\varphi]$ as the open complement is also open by definition. Thus, neither φ nor $\dot{\sim}\varphi$ is true at the boundary. Thus, in τ, any theory that includes the theory of the propositions that are true at the boundary is **incomplete**.

Inconsistent Topological Theories

Take $x \in \partial([\varphi])$ where $[\varphi]$ is a closed set in σ. By the above definition, since we have $x \in \partial([\varphi])$, we obtain $x \in [\varphi]$ as $[\varphi]$ is closed. Yet, $\partial[(\varphi)]$ is also included in $[\sim \varphi]$ which we have defined as a closed set. Thus, by the same reasoning, we conclude $x \in [\sim \varphi]$. Thus, $x \in [\varphi \wedge \sim \varphi]$ yielding that $x \models \varphi \wedge \sim \varphi$. Therefore, in σ, any theory that includes the boundary points is **inconsistent**.

Homeomorphism

An immediate observation yields that since extensions of all formulae in σ (respectively in τ) are closed (respectively, open), the topologies which are obtained in both paraconsistent and paracomplete logics are discrete.

Theorem

Let M_1 and M_2 be paraconsistent and paracomplete topological models respectively. If $|M_1| = |M_2|$, then there is a homeomorphism from a paraconsistent topological model to the paracomplete one, and vice versa.

Therefore, paraconsistent and paracomplete models of the same cardinality do have the same topological properties!

Continuity - 1

Theorem

Let $M = \langle S, \sigma, V \rangle$ *and* $M' = \langle S, \sigma', V' \rangle$ *be two paraconsistent topological models with a homeomorphism* f *from* $\langle S, \sigma \rangle$ *to* $\langle S, \sigma' \rangle$*. Define* $V'(p) := f(V(p))$*. Then,* $M \models \varphi$ *iff* $M' \models \varphi$ *for all* φ*.*

Note that this also works for classical logic (Artemov *et al.*, 1997; Kremer & Mints, 2005).

Continuity - 2

Corollary

Let $M = \langle S, \sigma, V \rangle$ *and* $M' = \langle S, \sigma', V' \rangle$ *be two paraconsistent topological models with a continuous* f *from* $\langle S, \sigma \rangle$ *to* $\langle S, \sigma' \rangle$*. Define* $V'(p) = f(V(p))$*. Then* $M \models \varphi$ *implies* $M' \models \varphi$ *for all* φ*.*

Corollary

Let $M = \langle S, \sigma, V \rangle$ *and* $M' = \langle S, \sigma', V' \rangle$ *be two paraconsistent topological models with an open* f *from* $\langle S, \sigma \rangle$ *to* $\langle S, \sigma' \rangle$*. Define* $V'(p) = f(V(p))$*. Then* $M' \models \varphi$ *implies* $M \models \varphi$ *for all* φ*.*

Continuity - 3

Recall that a *homotopy* is a description of how two continuous function from a topological space to another can be deformed to each other. We can now state the formal definition.

Definition

Let S and S' be two topological spaces with continuous functions $f, f' : S \to S'$. A homotopy between f and f' is a continuous function $H : S \times [0,1] \to S'$ such that if $s \in S$, then $H(s,0) = f(s)$ and $H(s,1) = g(s)$

Continuity - 4

In other words, a homotopy between f and f' is a family of continuous functions $H_t : S \to S'$ such that for $t \in [0, 1]$ we have $H_0 = f$ and $H_1 = g$ and the map $t \to H_t$ is continuous from $[0, 1]$ to the space of all continuous functions from S to S'.
Notice that homotopy relation is an equivalence relation.

Continuity - 5

Define $H : S \times [0,1] \to S'$ such that if $s \in S$, then $H(s,0) = f_0(s)$ and $H(s,1) = f_1(s)$. Then, H is a homotopy. Therefore, given a (paraconsistent) topological modal model M, we generate a family of models $\{M_t\}_{t \in [0,1]}$ whose valuations are generated by homotopic functions.

Definition

Given a model $M = \langle S, \sigma, V \rangle$, we call the family of models $\{M_t = \langle S, \sigma, V_t \rangle\}_{t \in [0,1]}$ generated by homotopic functions and M homotopic models. In the generation, we put $V_t = f_t(V)$.

Theorem

Homotopic paraconsistent (paracomplete) topological models satisfy the same modal formulae.

Classical Case

We saw that homotopic paraconsistent (paracomplete) topological models satisfy the same modal formulae.
We can extend it to classical case.

Theorem

Homotopic (classical) topological models satisfy the same modal formule.

Proof is by induction.

Open Questions

- Classification of homotopic models
- Modal logic of algebraic topological structures: modal logical equivalent of *nullstellensatz* of Hilbert?

Impossibility

The following configuration of beliefs is impossible (in consistent frameworks):

Ann believes that Bob assumes that Ann believes that Bob's assumption is wrong.
(Brandenburger & Keisler, 2006)

Notice that this is essentially a 2-person Russell's Paradox.

Approaches

It is possible to analyze the situation from neighborhood semantical perspective (Pacuit, 2007).
However, notice that the arguments and therefore the paradox solely depends on the consistency assumption.
What happens if we switch to the paraconsistent frameworks: stay away from trivial theories, accept some contradictions!

Self-Reference

Recently, a category theoretical approach has been presented (Abramsky & Zvesper, n.d.).
They focus on the fixed points and extend their analysis to category theory.
Lawvere's Theorem says that if $g : X \to V^X$ is surjective, then every function $f : V \to V$ has a fixed point (Lawvere, 1969).
BK paradox occurs if f plays the role of a Boolean negation.

Conditions

Lawvere's Theorem says that if $g : X \to V^X$ is surjective, then every function $f : V \to V$ has a fixed point (Lawvere, 1969).
There is an important restriction:

- X should be cartesian closed (actually, should only admit exponents)

Usually people consider the category of sets **Set**.

Co-Heyting

However, there is also another, a little unfamiliar, category which is cartesian closed: **co-Heyting** algebras.

Furthermore, we have already seen an example of it: topology of closed sets - a natural semantics for paraconsistency.

Figure: Arend Heijting

Co-Heyting: definitions

Let L be a bounded distributive lattice. If there is defined a binary operation $\Rightarrow: L \times L \to L$ such that for all $x, y, z \in L$,

$$x \leq (y \Rightarrow z) \text{ iff } (x \wedge y) \leq z,$$

then we call $(L, \Rightarrow)$ a Heyting algebra.
Dually, if we have a binary operation $\setminus : L \times L \to L$ such that

$$(y \setminus z) \leq x \text{ iff } y \leq (x \vee z),$$

then we call $(L, \setminus)$ a co-Heyting algebra.
We call $\Rightarrow$ implication, $\setminus$ subtraction.

Co-Heyting: definitions

In Boolean algebras, Heyting and co-Heying algebras give two different operations. We interpret $x \Rightarrow y$ as $\neg x \vee y$, and $x \setminus y$ as $x \wedge \neg y$.

In other words, a co-Heyting algebra is a generalization of a Boolean algebra that allows a generalization in which *principium contradictionis* is relaxed.

Paraconsistent BK Paradox

Therefore, even if we have paraconsistent framework. we will have fixed points.
How:

- Take a co-Heyting algebra - which is a natural candidate for paraconsistency.
- Observe that it admits exponents: $x^y \equiv x \wedge \neg y$.
- Thus, Lawvere's Theorem applies.
- It will still have fixed points: instead of the Boolean negation, take co-Heyting negation as the unary operator.

Questions

This framework raises several questions.

- What are the new holes? Do the previous ones still survive after we replace the Boolean negation with co-Heyting negation?
- Can we find a nontrivial paraconsistent framework where BK paradox does not exist?
- What about non-wellfounded sets?

Answers

Non-well-founded set theory does not help as it does not offer any solution to self-referentiality.
Russell's Paradox exists in non-well-founded set theory, too.

Recall that: $w \models \Box\varphi$ iff $\forall v(v \in w \to v \models \varphi)$

Open Questions?

- How can we logically define homotopy and cohomotopy groups in paraconsistent or paracomplete topological modal models?
- How would paraconsistency be affected under topological products?
- What is the (paraconsistent) logic of regular sets?
- What about some topological framework for BK paradox?

Future Work

- Importing more coalgebraic and algebraic tools to dynamic epistemic formalism
- Application to deontic, doxastic etc. logics
- Connection between compactness

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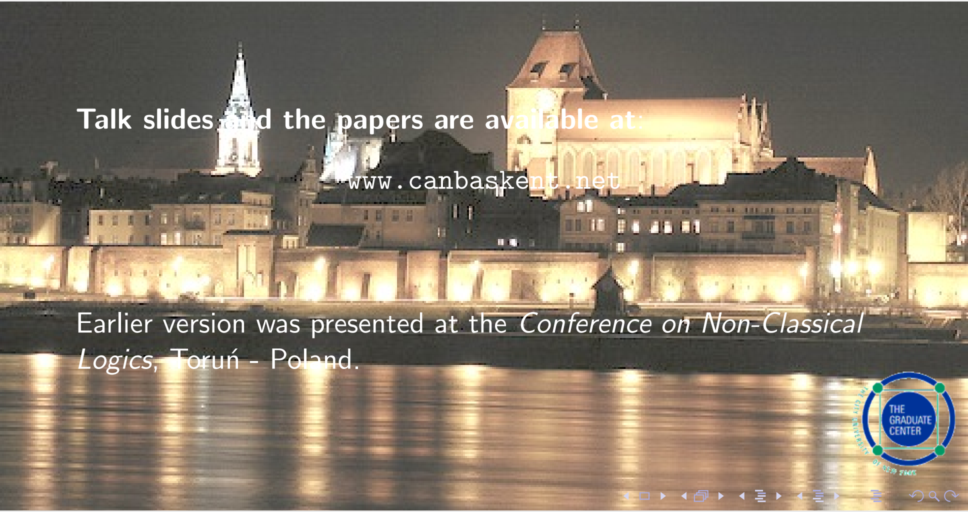
Thanks for your attention!



Talk slides and the papers are available at:

`www.canbaskent.net`

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