Some Philosophical Applications of Subset Space Logic

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Organization of the Talk

there is one

- Introduction to Subset Space Logic
- Public Announcement Logic
- Lakatosian Heuristics
- Fitch's Paradox
- Grue Paradox
- Deontic Subset Space Logic
- Conclusion

Aim of this Talk

if there is one

To open up discussion on the formalization of epistemic paradoxes and utilize geometrical interpretations of epistemic logics to analyze them.

Speeding Cars

Consider a policeman measuring the speed of passing cars. His knowledge of the speeds of the cars simply depends on the accuracy (i.e. error range) of his measuring device.

How can he increase his knowledge *without changing his point of view*?

– By using a more accurate/sophisticated measuring device with a smaller error range.

Speeding Cars - Conlusion

"[N]otion of *effort* enters in topology. Thus if we are at some point at s and make a measurement, we will then discover that we are in some neighborhood U of s, but not know where. If we make our measurement finer, then U will shrink, say, to a smaller neighborhood V." [Parikh & Moss] (implied at [Vickers])

Therefore, by spending some effort, we eliminate some of the possibilities, and finally obtain a smaller set of possibilities. The smaller the set of observation is, the larger the information we have.

Therefore, as it was also observed in the above example, to gain *knowledge*, we need to spend some *effort*.

SSL: Model and Language

A subset space (or *topologic*) model is a triple $\mathcal{S} = \langle S, \sigma, \nu \rangle$ where S is a set; $\sigma \subseteq \wp(S)$ a subset of the power set of S; $\nu : P \to \wp(S)$ is a valuation function for the countable set of propositional variables P.

The language $\mathcal{L}$ of SSL is:
$p \mid \top \mid \neg\varphi \mid \varphi \wedge \psi \mid \mathsf{K}\varphi \mid \Box\varphi$

SSL: Semantics

$s, U \models \top$	if and only if	always	
$s, U \models p$	if and only if	$s \in \nu(p)$	
$s, U \models \varphi \wedge \psi$	if and only if	$s, U \models \varphi$	and $s, U \models \psi$
$s, U \models \neg\varphi$	if and only if	$s, U \not\models \varphi$	
$s, U \models \mathsf{K}\varphi$	if and only if	$t, U \models \varphi$	for all $t \in U$
$s, U \models \Box\varphi$	if and only if	$s, V \models \varphi$	for all $V \in \sigma$ such that $s \in V \subseteq U$

(s, U) is called a *neighborhood situation* if U is a neighborhood of s, i.e. if $s \in U$.

Axioms

1. All the substitutional instances of the tautologies of the classical propositional logic
2. $(A \rightarrow \Box A) \wedge (\neg A \rightarrow \Box \neg A)$ for atomic sentence A
3. $\mathsf{K}(\varphi \rightarrow \psi) \rightarrow (\mathsf{K}\varphi \rightarrow \mathsf{K}\psi)$
4. $\mathsf{K}\varphi \rightarrow (\varphi \wedge \mathsf{KK}\varphi)$
5. $\mathsf{L}\varphi \rightarrow \mathsf{KL}\varphi$
6. $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$
7. $\Box\varphi \rightarrow (\varphi \wedge \Box\Box\varphi)$
8. $\mathsf{K}\Box\varphi \rightarrow \Box\mathsf{K}\varphi$ Cross-Axiom

K is S5 and $\Box$ is S4.

Completeness and Decidability

SSL is strongly complete and decidable.

Finite model property fails in SSL.

Exercise

Consider $\Box(\Diamond\varphi \wedge \Diamond\neg\varphi)$ *at* (s, U) *where* U *is the minimal open about* s.

Bisimulation

For $\mathcal{S} = \langle S, \sigma, \upsilon \rangle$ and $\mathcal{T} = \langle T, \tau, \nu \rangle$, a topologic bisimulation is a non-empty relation $\rightleftarrows$ for neighborhood situations in $(S \times \sigma) \times (T \times \tau)$ such that if $(s, U) \rightleftarrows (t, V)$, then we have:

1. **Base Condition**
   1.1 $s \in \upsilon(p)$ if and only if $t \in \nu(p)$ for each p
2. **Back Conditions**
   2.1 $\forall t' \in V$ there exists $s' \in U$ with $(s', U) \rightleftarrows (t', V)$.
   2.2 $\forall V' \subseteq V$ such that $t \in V'$, there is $U' \subseteq U$ with $s \in U'$ such that $(s, U') \rightleftarrows (t, V')$
3. **Forth Conditions**
   3.1 $\forall s' \in U$ there exists $t' \in V$ with $(s', U) \rightleftarrows (t', V)$.
   3.2 $\forall U' \subseteq U$ such that $s \in U'$, there is $V' \subseteq V$ with $t \in V'$ such that $(s, U') \rightleftarrows (t, V')$.

Bisimulation Invariance

Theorem (Bisimulation Invariance for Subset Spaces)

If $(s, U) \rightleftarrows (t, V)$ then they satisfy the same formulae, i.e. $(s, U) \leftrightsquigarrow (t, V)$.

Converse is true only under the special conditions.

Theorem

Let $\mathcal{S} = \langle S, \sigma, u \rangle$ and $\mathcal{T} = \langle T, \tau, v \rangle$ be two finite subset space. Then for each neighborhood situations (s, U) in $S \times \sigma$ and (t, V) in $T \times \tau$; we have $(s, U) \rightleftarrows (t, V)$ if and only if $(s, U) \leftrightsquigarrow (t, V)$.

Evaluation and Bisimulation Games

Position	Player	Admissible Moves
$(\bot, (s, U))$	$\exists$	$\emptyset$
$(\top, (s, U))$	$\forall$	$\emptyset$
$(p, (s, U))$ with $s \in v(p)$	$\forall$	$\emptyset$
$(p, (s, U))$ with $s \notin v(p)$	$\exists$	$\emptyset$
$(\psi_1 \wedge \psi_2, (s, U))$	$\forall$	$\{(\psi_1, (s, U)), (\psi_2, (s, U))\}$
$(\psi_1 \vee \psi_2, (s, U))$	$\exists$	$\{(\psi_1, (s, U)), (\psi_2, (s, U))\}$
$(\mathrm{L}\psi, (s, U))$	$\exists$	$\{(\psi, (t, U)) : t \in U\}$
$(\mathrm{K}\psi, (s, U))$	$\forall$	$\{(\psi, (t, U)) : t \in U\}$
$(\Diamond\psi, (s, U))$	$\exists$	$\{(\psi, (s, V)) : s \in V \subseteq U\}$
$(\Box\psi, (s, U))$	$\forall$	$\{(\psi, (s, V)) : s \in V \subseteq U\}$

Theorems

Theorem (Adequacy Theorem for Topologic Evaluation Games)

$$(\varphi, (s, U)) \in Win_{\exists}(\mathcal{E}(\varphi, (s, U))) \text{ if and only if } s, U \models \varphi.$$

Theorem (Adequacy Theorem for Topologic Bisimulation Games)

$(s, U) \rightleftarrows_n (t, V)$ if and only if $\exists$ has a winning strategy in the topologic bisimulation game of length n.

Semantics

Definition (Plaza)

Let $\mathcal{M} = \langle W, \{R_i\}_{i \in I}, V \rangle$ be a model and i be an agent. For atomic propositions, negations and conjunction the definition is as usual. For modal operators, we have the following semantics:

$$\begin{array}{lll} \mathcal{M}, w \models \mathsf{K}_i \varphi & \textit{iff} & \mathcal{M}, v \models \varphi \textit{ for each } v \textit{ with } wR_iv \\ \mathcal{M}, w \models [\varphi]\psi & \textit{iff} & \mathcal{M}, w \models \varphi \textit{ implies } \mathcal{M}|\varphi, w \models \psi \end{array}$$

Here, the updated model $\mathcal{M}|\varphi = \langle W', \{R'_i\}_{i \in I}, V' \rangle$ is defined by restricting $\mathcal{M}$ (and thus R_is and V) to those states where φ holds.

Reduction Axioms

The proof system of public announcement logic is the proof system of multi-modal S5 epistemic logic with the following additional axioms.

Atoms	$[\varphi]p \leftrightarrow (\varphi \rightarrow p)$
Partial Functionality	$[\varphi]\neg\psi \leftrightarrow (\varphi \rightarrow \neg[\varphi]\psi)$
Distribution	$[\varphi](\psi \wedge \chi) \leftrightarrow ([\varphi]\psi \wedge [\varphi]\chi)$
Knowledge Announcement	$[\varphi]\mathsf{K}_i\psi \leftrightarrow (\varphi \rightarrow \mathsf{K}_i[\varphi]\psi)$

The rule of inference for the $[*]$ operator is called the *announcement generalization* and is described as follows.

From $\vdash \psi$, *derive* $\vdash [\varphi]\psi$.

Semantics

The semantics for topologic PAL differs only on public announcement operator whose semantics is given as follows:

$$s, U \models [\varphi]\psi \quad \textit{if and only if} \quad s, U \models \varphi \textit{ implies } s, U_\varphi \models \psi$$

where $U_\varphi = U \cap (\varphi)$ and
(φ) being the extension of φ.

Compare: $\mathcal{M}, w \models [\varphi]\psi$ iff $\mathcal{M}, w \models \varphi$ implies $\mathcal{M}|\varphi, w \models \psi$

Axioms

Therefore, it is easy to see that the following axiomatize the topologic-PAL:

Atoms	$[\varphi]p \leftrightarrow (\varphi \rightarrow p)$
Partial Functionality	$[\varphi]\neg\psi \leftrightarrow (\varphi \rightarrow \neg[\varphi]\psi)$
Distribution	$[\varphi](\psi \wedge \chi) \leftrightarrow ([\varphi]\psi \wedge [\varphi]\chi)$
Knowledge Announcement	$[\varphi]\mathsf{K}\psi \leftrightarrow (\varphi \rightarrow \mathsf{K}[\varphi]\psi)$
Shrinking Reduction	$[\varphi]\Box\psi \leftrightarrow (\varphi \rightarrow \Box[\varphi]\psi)$

Exercise

Prove the soundness of the axioms associated with modalities.

Completeness

Theorem (Completeness of Topologic PAL)

Topologic PAL is complete with respect to the axiom system given above.

Proof.

By reduction axioms we can reduce each formula in the language of topologic PAL to a formula in the language of (basic) topologic. As topologic is complete, so is topologic PAL.

Philosophy of Science: Lakatos

Proofs and Refutations gives a rationally reconstructed account of the methodological evaluation of Euler's formula for polyhedra: $V - E + F = 2$.

Lakatos in PR follows Socratic heuristics.

Starting from a collection of observations (or assertions) about some peculiar properties of polyhedron, the arguments proceed by revising these observations (or assertions) by some mathematical *thought experiments* as Lakatos himself called.

Philosophy of Science: Lakatos

Let us see an example.
Assume that $(object, U) \models V - E + F = 2$
where U is the collection of observed polyhedral objects. Some may be genuine polyhedra, some not.
Clearly, $V - E + F = 0$ for torus. Thus,
$(torus, U) \not\models V - E + F = 2$.
Thus, we need to get rid of some objects in U that we had previously thought of as genuine polyhedra. For example, we need to get rid of torus, Klein bottle, Mobiüs strip etc. and obtain $U' \subset U$.
The formal way of achieving that is to introduce the Euler characteristic function for both oriented and non-oriented objects. In other words, Euler characteristic function gives a mapping to shrink the initial observation set

Philosophy of Science: Lakatos

The effort in this context corresponds to some mathematical calculations or suggesting a counter example or even refuting a counterexample.
For example, if we establish that the Euler formula holds for simply connected polyhedra, then, we will get rid off the polyhedra which are not simply connected - such as torus. Hence, without changing our point of view, we changed our neighborhood situation by considering some smaller set around the reference point we are occupying.

Semantics

Let $\mathcal{F}$ be a collection of functions from S to S, and further let $F \subseteq \mathcal{F}$. Take two subset spaces $\mathcal{S} = \langle S, \sigma, v \rangle$ and $\mathcal{S}_F = \langle S, \sigma_F, v \rangle$. Here, σ_F is the image of each $U \in \sigma$ under each function $f \in F$. In other words, $\sigma_F := \{fU : f \in F, U \in \sigma\}$. We will call $\mathcal{S}_F$ *the image space of* $\mathcal{S}$ *under* F.

Each function $f \in F$ is a contracting mapping which was intended to represent the increase in the knowledge. Hence, $fU \subseteq U$ should hold for each function f and for each observation set U.

Semantics

$s, U \models_{\mathcal{S}} [F]\varphi$ iff $s, fU \models_{\mathcal{S}_F} \varphi$ for each $f \in F$

The dual of $[F]$ will be defined as follows:

$s, U \models_{\mathcal{S}} \langle F \rangle\varphi$ iff $s, fU \models_{\mathcal{S}_F} \varphi$ for some $f \in F$

Some Observations

1. $[F](\varphi \rightarrow \psi) \rightarrow ([F]\varphi \rightarrow [F]\psi)$
   It is easy to see that $[F]$ modality realizes the **K** axiom
2. $[F][F]\varphi \rightarrow [F]\varphi$
   This axiom is valid if F is closed under function decomposition.
3. $[F]\varphi \rightarrow [F][F]\varphi$
   This axiom is valid if F is closed under function composition.
4. $[F]\varphi \rightarrow \varphi$
   This axiom is valid if the identity function id_F is in F.
5. $\Box\varphi \rightarrow [F]\varphi$
6. $\mathsf{K}[F]\varphi \rightarrow [F]\mathsf{K}\varphi$
   This is the cross axiom for $[F]$ and K

SSL expressivity

As SSL has a native "effort" modality, so the verification principle is easy to express.

$$\varphi \to \Diamond K\varphi$$

Substitute $\varphi \equiv p \wedge \neg Kp$ to get the Fitch's Paradox.

$$(p \wedge \neg Kp) \to \Diamond K(p \wedge \neg Kp)$$

Then, truth implies knowledge (after some manipulation). Remember, by verticality assumption ($K\varphi \to \varphi$), knowledge already implies truth. Therefore, we obtained that "knowledge $\equiv$ truth".

Brief Interpretation

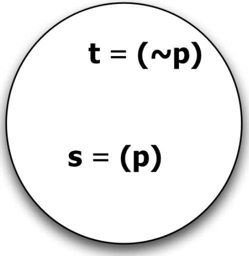
What does the substitution $\varphi \equiv p \wedge \neg \mathrm{K} p$ mean?

If $(s, U) \models p \wedge \neg \mathrm{K} p$ as a presupposition, we then have

- $s \in \nu(p)$, regardless of the neighborhood due to "atomic permenance".
- $s, U \models \neg \mathrm{K} p$ meaning that U has some other point t such that $t \notin \nu(p)$.

Shrinking

If we associate each formula with their extension set, Fitch's paradox appear when we substitute φ with a neighborhood set that includes the following set.



Recall that:
$p \wedge \neg Kp \equiv p \wedge L\neg p$

Observe that the Fitch sentence (as it is) can only be uttered at s.

Knowledge Sets

Consider a neighborhood situation (s, U). Let $KS_U = \{\varphi : (s, U) \models K\varphi\}$, call it the knowledge set with respect to U. Observe that it is independent from s.
Fitch's Paradox takes a knowledge set KS_U with $\varphi \in KS_U$ and relativizes it to a point $t \in U$ by forcing $(t, V) \not\models \varphi$, for some $V \subseteq U$ which is clearly paradoxical.

Exercise

Compare it to stable belief sets.

Exercise

See the paper by [Balbiani, Baltag, van Ditmarsch, Herzig, Toschi, Lima] to see whether Fitch's formula can be publicly announced.

Topology vs Subset Spaces

Recall that the topological interpretations of epistemic logic are **S4**. Therefore, they do not admit the symmetry property as the subset relation $\subset$ is not symmetric. However, the epistemic modality in Subset Space Logic is **S5**.

Moreover, Fitch's paradox requires an **S5** epistemic modality to get a contradiction.
(Use Tableau's to see where **S5** is required)

A Problem of Induction

An object is *grue* if it is green until a future moment t and, blue afterwards.

Suppose, you make an observation **now** and see that the emeralds $e_1, e_2, \ldots, e_n$ are green.

This observation confirms the hypothesis that "All emeralds are green".
But, it also confirms the hypothesis that "All emeralds are grue"!

A La Fitch Notion

We can describe the situation as follows

$$\varphi \rightarrow \Diamond K(\varphi \vee \varphi_1 \vee \varphi_2 \vee \ldots)$$

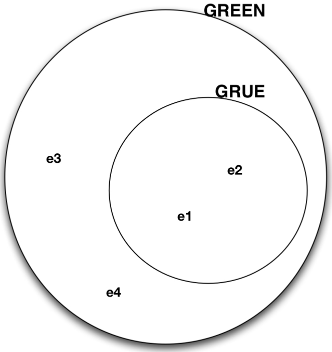
It is thus a classical OR problem. If $\varphi \vee \psi \equiv 1$, you do not know which disjunct is true.

Temporality in SSL

After Heinemann's works, we know that the shrinking modality $\Diamond$ has a temporal flavor.

Why not using it in Goodman's Paradox?

Illustration: Grue and SSL

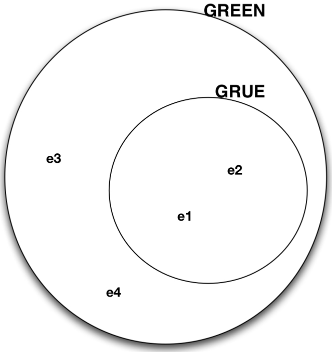


We need to determine the set "GRUE". In this case, subset space will have two sets: "GREEN" and possibly "GRUE" which is a subset of GREEN.

Consider:

$s, GREEN \models \Diamond(s \text{ } is \text{ } GRUE)$ for all $s \in GREEN$

Interpretation: Grue and SSL



$\Diamond$ modality reflects the "attainability" of the GRUE predicate. Can we possibly know that the emerald e_n is GRUE? **If** we can make such an observation -akin to the feasibility of speeding car example-, then we can check whether the below formula is correct.

$s, GREEN \models \Diamond(s\ is\ GRUE)$ for each s

Humeian Induction and SSL

To which extend can the knowledge be improved? Is there an upper bound for knowledge acquisition?

In finite SSL, knowledge acquisition is limited by the nested subsets. You can gain knowledge upto some certain point.

One another modification would be to use strict subset relation $\subset$, instead of $\subseteq$.

But. we get stuck if our knowledge space is the usual topology in $\mathbb{R}$ as it is uncountable.

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Computational Ethics

Is it possible?

"Act as if the maxim of your action were to become by your will a universal law of nature" Kant, *Groundwork of the Metaphysics of Morals*

Does it imply deontic logic of actions?

Computational Kant

Kant beyond Euclidean geometry

Let public announcement $[\varphi]$ be the action (After Aumann's definition of common knowledge), and consequently let φ the corresponding maxim. Let ψ be the consequence of your action.

$$s, U \models [\varphi]\psi$$

already possess the information $s, U \models \varphi$ if we ignore trivialities. Thus, it reduces to the following:

$$s, U_\varphi \models (\varphi \wedge \psi) \tag{1}$$

which is pure Kantian rationalism.

Future Work

- Heuristic games in the sense of Socrates (*Meno*) and Lakatos (*Proofs and Refutations*)
- Non-standart (á la paraconsistent) logics for Fitch's paradox.
- Universal Modalities: it is also possible to extend the language with the universal modalities E and A in order to increase the expressivity. Similar to Segerberg's Fitch analysis.
- Multiagent subset space logic.

References

selected

- Balbiani, Baltag, van Ditmarsch, Herzig, Toschi, Lima; What can we achieve by arbitrary announcements? A dynamic take on Fitchs knowability
- Başkent, An Examination of Counterexamples in 'Proofs and Refutations'
- van Benthem, van Eijck, Kooi; Logics of Communications and Change
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- Plaza, Logics of Public Communications
- Vickers, Topology via Logic

Thanks for your attention!

Talk slides are available at:

www.canbaskent.net

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