All Normal Extensions of **S5**^{2} are Finitely Axiomatizable

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Road-Map For the Proof

- Recap
- Necessary mathematical machinery
- Proof in several steps
- Complexity results

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Axioms of **S5**^{2}

- All tautologies of propositional calculus
- $\Box_i(\varphi \to \psi) \to (\Box_i\varphi \to \Box_i\psi)$
- $\Box_i\varphi \to \varphi$
- $\Box_i\varphi \to \Box_i\Box_i\varphi$
- $\Diamond_i\Box_i\varphi \to \varphi$
- $\Box_1\Box_2\varphi \leftrightarrow \Box_2\Box_1\varphi$

Two modal operators: $\Box_1$ and $\Box_2$
Closed under MP and Necessitation (from φ infer $\Box_i\varphi$).

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Facts on **S5**^{2} (1)

Complete with respect to $\{n \times n : n \geq 1\}$, for natural number n [Segerberg].

where we have:

$(x_1, x_2)R_1(y_1, y_2)$ iff $x_2 = y_2$

$(x_1, x_2)R_2(y_1, y_2)$ iff $x_1 = y_1$

Facts on $\mathbf{S5}^2$ (2)

Every proper extension L of $\mathbf{S5}^2$ has poly-size model property; that is, there is a polynomial $P(n)$ such that any L-consistent formula φ has a model over a frame validating L with at most $P(|\varphi|)$ points, where $|\varphi|$ is the length of the formula φ.

Facts on **S5**^{2} (3)

$\mathcal{F} = (W, R_1, R_2)$ is a **S5**^{2} frame where:

- W is non-empty
- R_i's are equivalence relations on W such that

$$\mathcal{F} \models (\forall w, v, u)(wR_1v \wedge vR_2u) \rightarrow (\exists z)(wR_2z \wedge zR_1u)$$

p-morphism

For two **S5**^{2} frames $\mathcal{F} = (W, R_1, R_2)$ and $\mathcal{G} = (U, S_1, S_2)$, *p*-morphism $f : U \to W$ from $\mathcal{G}$ to $\mathcal{F}$, for each $i = 1, 2$ is defined as follows:

$$(\forall t \in U)(\forall w \in W)(f(t) R_i w \leftrightarrow (\exists u \in U)(t S_i u \wedge f(u) = w))$$

Definitions on *p*-morphism

S5$^{\mathbf{2}}$ frames $\mathcal{F}$ is rooted if and only if

$$(\forall w, v)(\exists u)(wR_1 u \wedge uR_2 v))$$

Then define $\mathbf{F}_{\mathbf{S5^2}}$ as the set of representatives of the isomorphism types of the finite rooted **S5**$^{\mathbf{2}}$ frames. We will, from now on, consider the frames in $\mathbf{F}_{\mathbf{S5^2}}$. Why?

Definitions on *p*-morphism: an earlier result

Let L be a normal extension of $\mathbf{S5^2}$. $\mathcal{F} \in \mathbf{S5^2}$ is called L-frame if $\mathcal{F}$ validates each formula in L. Then, define $\mathbf{F_L}$ the set of all L-frames in $\mathbf{F_{S5^2}}$.

Bezhanisvili proved somewhere else that: L is complete wrt $\mathbf{F_L}$.

This is the reason why we will only consider the frames in $\mathbf{F_{S5^2}}$. This is the first step towards our aim.

Define $\mathbf{M_L} = min(\mathbf{F_{S5^2}} \backslash \mathbf{F_L})$.

Definitions on *p*-morphism: a relation

We will introduce our first partial order in $\mathbf{F}_{\mathbf{S5}^2}$: $\leq$.
For $\mathcal{F}$ and $\mathcal{G}$ in $\mathbf{F}_{\mathbf{S5}^2}$,

$$\mathcal{F} \leq \mathcal{G} \text{ iff } \mathcal{F} \text{ is a } p\text{-morphic image of } \mathcal{G}.$$

For each $\mathcal{G}$ in a subset A of $\mathbf{F}_{\mathbf{S5}^2}$, there is a frame $\mathcal{F} \in min(A)$ such that $\mathcal{F} \leq \mathcal{G}$.

Road-Map for the Proof

We will proceed as follows:

- Find a set of formulas that axiomatize any proper normal extension of **S5**^{2}.
- Show that this set is finite by stating equivalent statement about the finiteness of the set of axioms.

Jankov-Fine Formulas

$$\alpha(\mathcal{F}) = \Box_1\Box_2\Bigg(\bigvee_{p\in W} (p \wedge \neg \bigvee_{p'\in W\setminus\{p\}} p') \wedge \bigwedge_{\substack{i=1,2\\ p,p'\in W\\ pR_ip'}} (p \to \Diamond_i p') \wedge \bigwedge_{\substack{i=1,2p\\ p'\in W\\ \neg(pR_ip')}} (p \to \neg\Diamond_i p') \Bigg)$$

$$\chi(\mathcal{F}) = \neg\alpha(\mathcal{F})$$

Why on earth do we need that formula?

- $\mathcal{F} \leq \mathcal{G}$ if and only if $\mathcal{G} \not\models \chi(\mathcal{F})$.
- $\mathcal{G} \in \mathbf{F_L}$ if and only if for no $\mathcal{F} \in \mathbf{M_L}, \mathcal{F} \leq \mathcal{G}$, where $\mathbf{M_L} = min(\mathbf{F_{S5^2}} \setminus \mathbf{F_L})$.
- **Theorem** Every proper normal extension L of **S5^2** is axiomatizable by the axioms of **S5^2** and $\{\chi(\mathcal{F}) : \mathcal{F} \in \mathbf{M_L}\}$.
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A qo-set (1)

$\mathbf{R_i}$ – Depth of $\mathcal{F}$ is the number of R_i-equivalence classes of $\mathcal{F}$. Denote $d_i(\mathcal{F})$.
$n(L)$ is the least n such that, $n \times n \notin \mathbf{F_L}$.

- If $\mathcal{F} \in \mathbf{F_L}$, then $d_1(\mathcal{F}) < n(L)$ or $d_2(\mathcal{F}) < n(L)$.
- In contrast, if $\mathcal{F}$ is not in $\mathbf{F_L}$, i.e. $\mathcal{F} \in \mathbf{M_L}$; then $d_1(\mathcal{F}) \leq n(L)$ or $d_2(\mathcal{F}) \leq n(L)$.
- The previous two results give rise to the following fact: $\mathbf{M_L}$ is finite iff $\{\mathcal{F} \in \mathbf{M_L} : d_i(\mathcal{F}) = k\}$ is finite for each $k \leq n(L)$ where $i = 1, 2$.

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A qo-set (2)

So to prove the finiteness of $\mathbf{M_L}$, prove the finiteness of $\{\mathcal{F} \in \mathbf{M_L} : d_i(\mathcal{F}) = k\}$ for each k while $i = 1, 2$.

But, since $\mathbf{M_L}$ is a $\leq$-antichain in $\mathbf{F_{S5^2}}$, *instead* show $\mathbf{M_L}$ does *not* contain an infinite $\leq$-antichain.

A qo-set (3): A Newer Relation

Fix k. WLOG, let $i = 2$. Let $\mathcal{M}_n$ be the set of $n \times k$ matrices (m_{ij}) and $\mathcal{M}$ is the collection $\bigcup_{n \in \omega} \mathcal{M}_n$.

$(m_{ij}) \preceq (m'_{ij})$ holds if $(m_{ij}) \in \mathcal{M}_n$ and $(m'_{ij}) \in \mathcal{M}_{n'}$ and $n \leq n'$ and there is a surjection $f : n' \to n$ such that $m_{f(i)j} \leq m'_{ij}$.

Observe that $(\mathcal{M}, \preceq)$ is a qo-set.

A qo-set (4)

Define $H : \mathbf{F^k_{S5^2}} \to \mathcal{M}$ by $H(\mathcal{F}) = (m_{ij})$, if $|F_i \cap F^j| = m_{ij}$.
H is an order-reflecting injection, where $\mathbf{F^k_{S5^2}}$ is the set of frames in $\mathbf{F_{S5^2}}$ with R_2-depth k, F_i is the i^{th} equivalence class of R_1 and F^j is the j^{th} equivalence class of R_2.

Therefore, for each $\leq$-antichain Δ in $\mathbf{F^k_{S5^2}}$, then $H(\Delta)$ is a $\preceq$-antichain.

So, *instead*, we will show there is no infinite $\preceq$-antichains in $\mathcal{M}$. But, *instead* of dealing with $\preceq$, we will define new a quasi-order: $\sqsubseteq$.

A qo-set (5): The Newest Relation

For $(m_{ij}) \in \mathcal{M}_n$ and $(m'_{ij}) \in \mathcal{M}_{n'}$:

$(m_{ij}) \sqsubseteq_1 (m'_{ij})$ if there is an injective order-preserving map $\varphi : n \to n'$ such that $m_{ij} \leq m'_{\varphi(i)j}$ for each $i < n$ and $j < k$.

$(m_{ij}) \sqsubseteq_2 (m'_{ij})$ if there is a map $\psi : n' \to n$ such that $m_{\psi(i)j} \leq m'_{ij}$ for each $i < n$ and $j < k$.

$\sqsubseteq$ is the intersection of $\sqsubseteq_1$ and $\sqsubseteq_2$.

Thus, if $(m_{ij}) \sqsubseteq (m'_{ij})$, then $(m_{ij}) \preceq (m'_{ij})$.

BQOs: finally

Therefore, *instead*, we will show there is no infinite $\sqsubseteq$-antichains in $\mathcal{M}$.

FACT: There is no infinite antichains in a BQO.

BQOs: recap

- $(\omega, \leq)$ is a BQO.
- Any suborder of a BQO, and the intersection of two BQOs are BQOs.
- If $(Q, \leq)$ is a BQO, then, $(\wp(Q), \leq)$ is a BQO.
- If $(Q, \leq)$ is a BQO, then $(\bigcup_{\alpha \in On} Q^\alpha, \leq^*)$ is a BQO. Hence, the suborders $(Q^k, \leq^*)$ and $\bigcup_{n<\omega} Q^n, \leq^*$ are BQOs.

Define $\leq^*$ on the class $\bigcup_{\alpha \in On} Q^\alpha$ by $(x_i)_{i<\alpha} \leq^* (y_i)_{i<\beta}$ if there is an order-preserving map $\varphi : \alpha \to \beta$ such that $x_i \leq y_{\varphi(i)}$ for each $i < \alpha$.

Result-1

- $(\mathcal{M}, \sqsubseteq_1)$ is a BQO.
- $(\mathcal{M}, \sqsubseteq_2)$ is a BQO.
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Result (theorem)

THEOREM: *All normal extensions of* **S5**^{2} *are finitely axiomatizable*.

Result (proof)

- $\mathbf{S5^2}$ is finitely axiomatizable.
- If L is a *proper* extension of $\mathbf{S5^2}$, then it is axiomatizable by the axioms of $\mathbf{S5^2}$ and $\{\chi(\mathcal{F}) : \mathcal{F} \in \mathbf{M_L}\}$.
- Since $\sqsubseteq$ is a BQO, it has no $\sqsubseteq$-infinite antichains, and there is no $\preceq$-antichains in $\mathcal{M}$.
- Therefore for each $k \in \omega$, $\mathbf{F^k_{S5^2}}$ has no infinite antichains. Thus, for each $k \leq n(L)$, the set $\{\mathcal{F} \in \mathbf{M_L} : d_i(\mathcal{F}) = k\}$ has finite number of elements.
- Hence, $\mathbf{M_L}$ is finite, and there are only finitely many $\chi(\mathcal{F})$ formulas that axiomatize L.

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SAT

- **S5**^{2} has a exponential size model property, and its satisfiability problem is NEXP-TIME.
- Every proper normal extension of **S5**^{2} is decidable in polynomial time. Therefore, together with the poly-size model property, it implies that the satisfiability for the normal proper extension is NP-complete.

POLY-SIZE MODEL PROPERTY For the each proper normal extension L of **S5**^{2}, there is a polynomial $P(n)$ s.t. for any L-consistent formula ϕ has a model over a frame validating L, and model has at most $P(|\phi|)$ points where $P(|\phi|)$denotes the length of ϕ.

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Thanks for your attention