An Examination of Counterexamples in Proofs and Refutations

Can BAŞKENT

Institute for Logic, Language and Computation
Universiteit van Amsterdam

canbaskent@yahoo.com www.illc.uva.nl/∼cbaskent

November 4, 2006

..the first important notions in topology were acquired in the course of the study of polyhedra.

H. Lebesgue

Lakatosian method of Proofs and Refutations

- Primitive conjecture.
- Proof (a rough thought experiment or argument, decomposing the primitive conjecture into subconjectures and lemmas).
- Global counterexamples.
- Proof re-examined. The guilty lemma is spotted. The guilty lemma may have previously remained hidden or may have been misidentified.
- Proofs of the other theorems are examined to see if the newly found lemma occurs in them.
- Hitherto accepted consequences of the original and now refuted conjecture are checked.
- Counterexamples are turned into new examples, and the new fields of inquiry open up.

Lakatosian method of Proofs and Refutations

- Primitive conjecture.
- Proof (a rough thought experiment or argument, decomposing the primitive conjecture into subconjectures and lemmas).
- Global counterexamples.
- Proof re-examined. The guilty lemma is spotted. The guilty lemma may have previously remained hidden or may have been misidentified.
- Proofs of the other theorems are examined to see if the newly found lemma occurs in them.
- Hitherto accepted consequences of the original and now refuted conjecture are checked.
- Counterexamples are turned into new examples, and the new fields of inquiry open up.

Lakatosian method of Proofs and Refutations

- Primitive conjecture.
- Proof (a rough thought experiment or argument, decomposing the primitive conjecture into subconjectures and lemmas).
- Global counterexamples.
- Proof re-examined. The guilty lemma is spotted. The guilty lemma may have previously remained hidden or may have been misidentified.
- Proofs of the other theorems are examined to see if the newly found lemma occurs in them.
- Hitherto accepted consequences of the original and now refuted conjecture are checked.
- Counterexamples are turned into new examples, and the new fields of inquiry open up.

Lakatosian method of Proofs and Refutations

- Primitive conjecture.
- Proof (a rough thought experiment or argument, decomposing the primitive conjecture into subconjectures and lemmas).
- Global counterexamples.
- Proof re-examined. The guilty lemma is spotted. The guilty lemma may have previously remained hidden or may have been misidentified.
- Proofs of the other theorems are examined to see if the newly found lemma occurs in them.
- Hitherto accepted consequences of the original and now refuted conjecture are checked.
- Counterexamples are turned into new examples, and the new fields of inquiry open up.

Lakatosian method of Proofs and Refutations

- Primitive conjecture.
- Proof (a rough thought experiment or argument, decomposing the primitive conjecture into subconjectures and lemmas).
- Global counterexamples.
- Proof re-examined. The guilty lemma is spotted. The guilty lemma may have previously remained hidden or may have been misidentified.
- Proofs of the other theorems are examined to see if the newly found lemma occurs in them.
- Hitherto accepted consequences of the original and now refuted conjecture are checked.
- Counterexamples are turned into new examples, and the new fields of inquiry open up.

Lakatosian method of Proofs and Refutations

- Primitive conjecture.
- Proof (a rough thought experiment or argument, decomposing the primitive conjecture into subconjectures and lemmas).
- Global counterexamples.
- Proof re-examined. The guilty lemma is spotted. The guilty lemma may have previously remained hidden or may have been misidentified.
- Proofs of the other theorems are examined to see if the newly found lemma occurs in them.
- Hitherto accepted consequences of the original and now refuted conjecture are checked.
- Counterexamples are turned into new examples, and the new fields of inquiry open up.

Lakatosian method of Proofs and Refutations

- Primitive conjecture.
- Proof (a rough thought experiment or argument, decomposing the primitive conjecture into subconjectures and lemmas).
- Global counterexamples.
- Proof re-examined. The guilty lemma is spotted. The guilty lemma may have previously remained hidden or may have been misidentified.
- Proofs of the other theorems are examined to see if the newly found lemma occurs in them.
- Hitherto accepted consequences of the original and now refuted conjecture are checked.
- Counterexamples are turned into new examples, and the new fields of inquiry open up.

Negative and Positive Heuristics

Positive heuristics suggest methods or a plan for articulating and revising the research program forward.

Negative heuristics, on the other hand, suggest methods or plans to avoid for research program to improve.

Aim

We want to analyze the relation between the counterexamples utilized in *Proofs and Refutations* and the heuristic functions of them.

Descartes - Euler Conjecture for Polyhedra

Main conjecture is,

$$V - E + F = 2$$

where
V is the number of vertices,
E is the number of edges, and
F is the number of faces.

Cauchy Proof

Step 1 Imagine that the polyhedra is hollow and made of rubber sheet. Cut out one of the faces, strech the remaining faces to a flat surface without tearing. In this process, V and E will not alter. We will have $V - E + F = 1$, since we have removed a face.

Step 2 Triangulate the obtained map. Drawing diagonals for those curvilinear polygons will not alter $V - E + F$ since E and F increases simultaneously.

Step 3 Remove the triangles. It can be done in two ways: either one edge and one face are removed simultaneously; or one face, one vertices and two edges are removed simultaneously. At the end, we will end up with an ordinary triangle for which $V - E + F = 1$ holds trivially.

Cauchy Proof

Step 1 Imagine that the polyhedra is hollow and made of rubber sheet. Cut out one of the faces, strech the remaining faces to a flat surface without tearing. In this process, V and E will not alter. We will have $V - E + F = 1$, since we have removed a face.

Step 2 Triangulate the obtained map. Drawing diagonals for those curvilinear polygons will not alter $V - E + F$ since E and F increases simultaneously.

Step 3 Remove the triangles. It can be done in two ways: either one edge and one face are removed simultaneously; or one face, one vertices and two edges are removed simultaneously. At the end, we will end up with an ordinary triangle for which $V - E + F = 1$ holds trivially.

Cauchy Proof

Step 1 Imagine that the polyhedra is hollow and made of rubber sheet. Cut out one of the faces, strech the remaining faces to a flat surface without tearing. In this process, V and E will not alter. We will have $V - E + F = 1$, since we have removed a face.

Step 2 Triangulate the obtained map. Drawing diagonals for those curvilinear polygons will not alter $V - E + F$ since E and F increases simultaneously.

Step 3 Remove the triangles. It can be done in two ways: either one edge and one face are removed simultaneously; or one face, one vertices and two edges are removed simultaneously. At the end, we will end up with an ordinary triangle for which $V - E + F = 1$ holds trivially.

Lemmas in the Proof

1. Any polyhedron, after a face removed, can be stretched flat onto a flat surface.
2. While triangulating the map, one will always get a new face for every new edge.
3. There are only two alternatives: the disappearance of one edge or else of two edges and a vertex - when one drops the triangles by one. Furthermore, one will end up with a single triangle at the end of this process.

Lemmas in the Proof

1. Any polyhedron, after a face removed, can be stretched flat onto a flat surface.
2. While triangulating the map, one will always get a new face for every new edge.
3. There are only two alternatives: the disappearance of one edge or else of two edges and a vertex - when one drops the triangles by one. Furthermore, one will end up with a single triangle at the end of this process.

Lemmas in the Proof

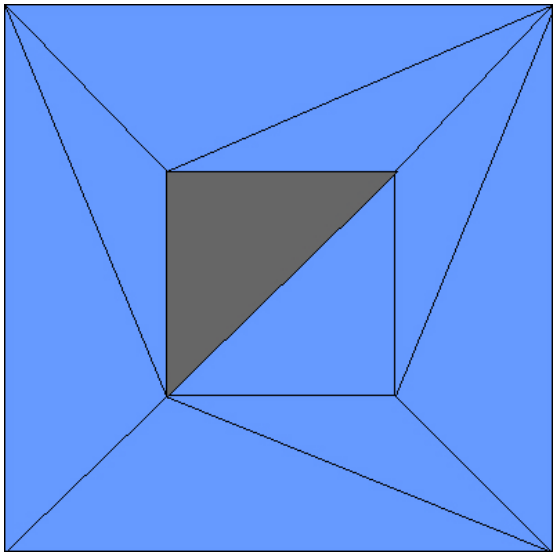
1. Any polyhedron, after a face removed, can be stretched flat onto a flat surface.
2. While triangulating the map, one will always get a new face for every new edge.
3. There are only two alternatives: the disappearance of one edge or else of two edges and a vertex - when one drops the triangles by one. Furthermore, one will end up with a single triangle at the end of this process.

Local but not Global Counterexamples

Directed towards the Lemmas!

Against Lemma 3

Remove a triangle from inside of the triangulated network.
No edge nor vertices were removed.



Lemma 3 There are only two alternatives: the disappearance of one edge or else of two edges and a vertex - when one drops the triangles by one. Furthermore, one will end up with a single triangle at the end of this process.

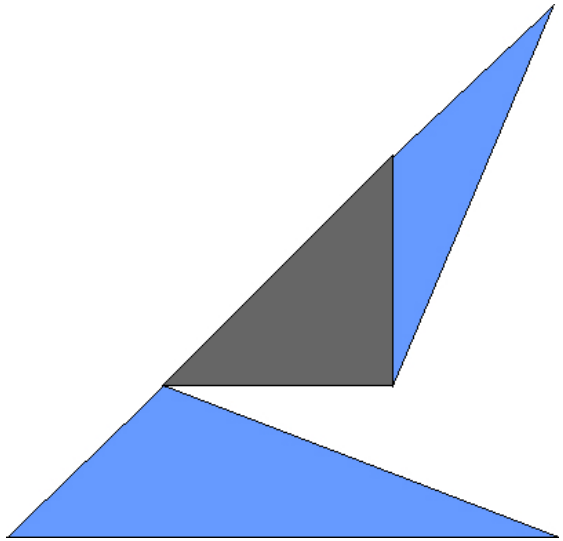
Against Lemma 3

First Modification of Lemma 3 Remove triangles in such a way that either one edge or two edges and a vertex will disappear.

Lemma 3 There are only two alternatives: the disappearance of one edge or else of two edges and a vertex - when one drops the triangles by one. Furthermore, one will end up with a single triangle at the end of this process.

Against Lemma 3

Remove a triangle by disconnecting the network.
So, two edges will be removed, and we will have a disconnected network.



Lemma 3 There are only two alternatives: the disappearance of one edge or else of two edges and a vertex - when one drops the triangles by one. Furthermore, one will end up with a single triangle at the end of this process.

Against Lemma 3

Second Modification of Lemma 3 Remove triangles in such a way that $V - E + F$ will not change.

Lemma 3 There are only two alternatives: the disappearance of one edge or else of two edges and a vertex - when one drops the triangles by one. Furthermore, one will end up with a single triangle at the end of this process.

Global Counterexamples

Directed towards the Conjecture!

Nested Cube

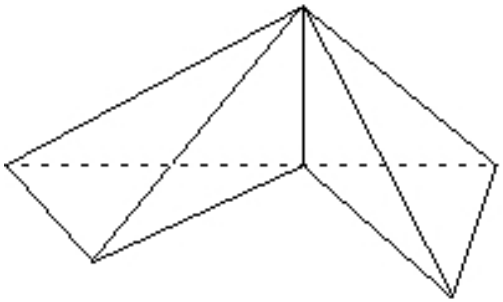
A pair of cubes, one of which is inside, but does not touch the other.
We have, in this case, $V - E + F = 4$, two times for each cube.
Is "nested cube" a polyhedron?

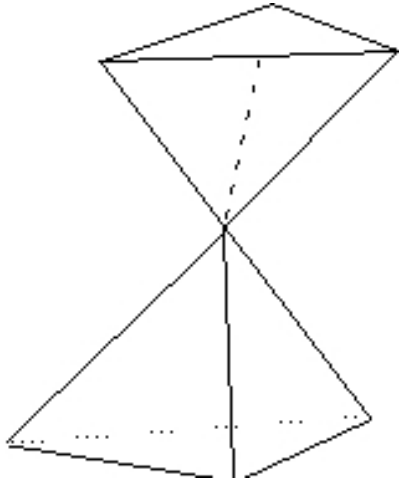
Definition of Polyhedron and Counterexamples

Definition 1 A polyhedron is a solid whose surface consists of polygonal faces.

Definition 2 A polyhedron is a surface consisting of a system of polygons.

Definition of Polyhedron and Conterexamples





Check that for both $V - E + F = 3$.

Exercise: See why the left-hand side "thing" is not a solid, and right-hand side "thing" is not a surface!

Definition 1 A polyhedron is a solid whose surface consists of polygonal faces.

Definition 2 A polyhedron is a surface consisting of a systems of polygons.

More Definitions of Polyhedra

Definition 3 A polyhedron is a system of polygons arranged in such a way that (1) exactly two polygons meet at every edge and, (2) it is possible to get from inside of any polygon to the inside of any other polygon by a route which never crosses any edge at a vertex.

Clearly, in the first twin tetrahedra there is an edge where four polygons meet and in the second twins it is impossible to get from the inside of a polygon of the upper tetrahedron to the inside of the other polygon of the lower tetrahedron without a route that crosses some edges at a vertex.

Definition Perfect A polyhedron is a system of polygons for which the equation $V - E + F = 2$ holds.

More Definitions of Polyhedra

Definition 3 A polyhedron is a system of polygons arranged in such a way that (1) exactly two polygons meet at every edge and, (2) it is possible to get from inside of any polygon to the inside of any other polygon by a route which never crosses any edge at a vertex.

Clearly, in the first twin tetrahedra there is an edge where four polygons meet and in the second twins it is impossible to get from the inside of a polygon of the upper tetrahedron to the inside of the other polygon of the lower tetrahedron without a route that crosses some edges at a vertex.

Definition Perfect A polyhedron is a system of polygons for which the equation $V - E + F = 2$ holds.

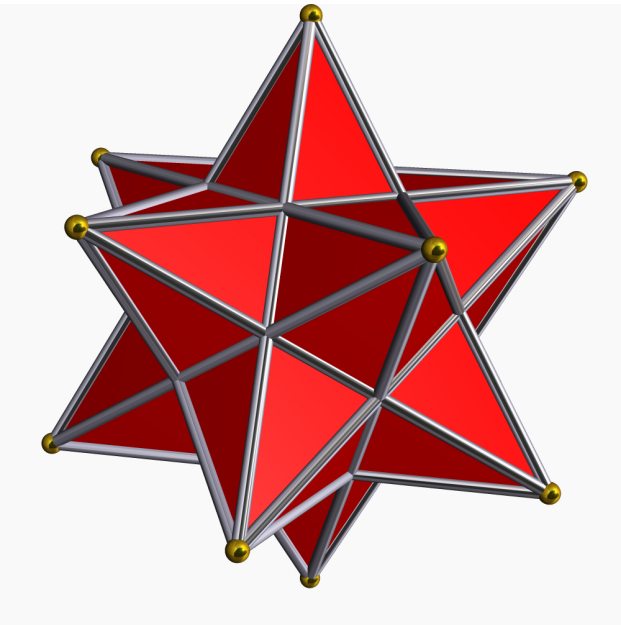
More Definitions of Polyhedra

Definition 3 A polyhedron is a system of polygons arranged in such a way that (1) exactly two polygons meet at every edge and, (2) it is possible to get from inside of any polygon to the inside of any other polygon by a route which never crosses any edge at a vertex.

Clearly, in the first twin tetrahedra there is an edge where four polygons meet and in the second twins it is impossible to get from the inside of a polygon of the upper tetrahedron to the inside of the other polygon of the lower tetrahedron without a route that crosses some edges at a vertex.

Definition Perfect A polyhedron is a system of polygons for which the equation $V - E + F = 2$ holds.

Urchin



For urchin $V - E + F = -6$, and it agrees with Definition 3.

Definition 3 A polyhedron is a system of polygons arranged in such a way that (1) exactly two polygons meet at every edge and (2) it is possible to get from inside of any polygon to the inside of any other polygon by a route which never crosses any edge at a vertex.

Defeat the Urchin

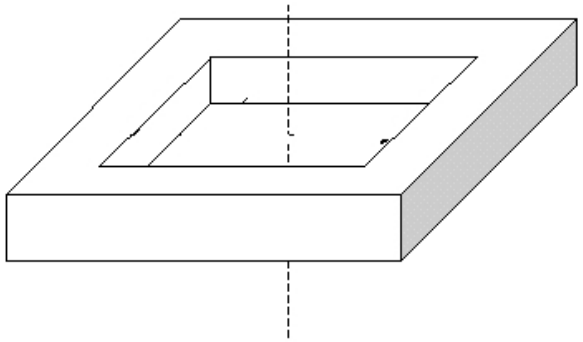
Definition 4 A polygon is a system of edges arranged is such a way that (1) exactly two edges meet at every vertex, and (2) the edges have no points in common except the vertices.

Urchin was defeated: edges had common points except from vertices.

Save the Urchin

Definition 4' A polygon is a system of edges arranged in such a way that exactly two edges meet at every vertex.

Picture Frame



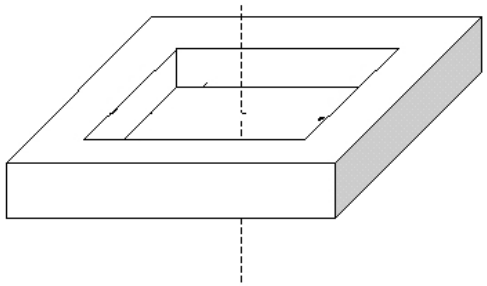
Refutes both Definition 4 and Definition 4'. Also, for picture frame $V - E + F = 0$.

Definition 4 A polygon is a system of edges arranged is such a way that (1) exactly two edges meet at every vertex, and (2) the edges have no points in common except the vertices.

Definition 4' A polygon is a system of edges arranged in such a way that exactly two edges meet at every vertex.

Defeat the Picture Frame

Definition 5 In the case of genuine polyhedron, through any arbitrary points in the space there will be at least one plane whose cross-section with the polyhedron will consist of one single polygon.



Take a point from inside the frame.

Cylinder

Refutes Definition 5: $V - E + F = 1$.

Definition 5 In the case of genuine polyhedron, through any arbitrary points in the space there will be at least one plane whose cross-section with the polyhedron will consist of one single polygon.

What is an Edge?

Definition 6 An edge has two vertices.

The Method of Monster-Barring "Using this method one can eliminate any counterexample to the original conjecture by a sometimes deft but always ad hoc redefinition of the polyhedron, [or] of its defining terms, or of the defining terms of its defining terms."

The Essence of Monster-Barring By suitably restricting both conjecture and the proof to the proper domain, the conjecture, which is now true, will be perfected, and the basically sound proof, which is now rigorous, will be perfected and obviously will contain no more false lemmas.

What is an Edge?

Definition 6 An edge has two vertices.

The Method of Monster-Barring "Using this method one can eliminate any counterexample to the original conjecture by a sometimes deft but always ad hoc redefinition of the polyhedron, [or] of its defining terms, or of the defining terms of its defining terms."

The Essence of Monster-Barring By suitably restricting both conjecture and the proof to the proper domain, the conjecture, which is now true, will be perfected, and the basically sound proof, which is now rigorous, will be perfected and obviously will contain no more false lemmas.

A New Statement of the Theorem

A New Statement of the Theorem For all polyhedra that have no cavities, no tunnels or no 'multiple structures' $V - E + F = 2$.

- Excludes all monsters, i.e. exceptions.
- Ad hoc.
- How can you be sure that you have enumerated all exceptions?

A New Statement of the Theorem

A New Statement of the Theorem For all polyhedra that have no cavities, no tunnels or no 'multiple structures' $V - E + F = 2$.

- Excludes all monsters, i.e. exceptions.
- Ad hoc.
- How can you be sure that you have enumerated all exceptions?

A New Statement of the Theorem

A New Statement of the Theorem For all polyhedra that have no cavities, no tunnels or no 'multiple structures' $V - E + F = 2$.

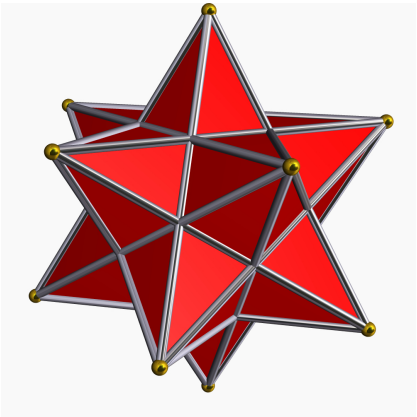
- Excludes all monsters, i.e. exceptions.
- Ad hoc.
- How can you be sure that you have enumerated all exceptions?

A New Statement of the Theorem

A New Statement of the Theorem For all polyhedra that have no cavities, no tunnels or no 'multiple structures' $V - E + F = 2$.

- Excludes all monsters, i.e. exceptions.
- Ad hoc.
- How can you be sure that you have enumerated all exceptions?

Monster Adjustment on Urchin



There is no star-polygons in urchin, but only triangular faces. 60 faces, 90 edges and 32 vertices give the Euler characteristics 2.

A Bit of Geometric Topology

- Picture frame cannot be inflated into a sphere or a plane.
- Because genus (number of "holes") of a sphere is zero.
- It is topologically the 'same' to stretch the polyhedra onto the [Euclidean] plane and onto the sphere.
- Picture frame can be inflated into a torus as its genus is one.
- So the general formula of Euler characteristics in manifolds is:

$$V - E + F = 2 - 2.g(S)$$

  where S is the surface we consider to be inflated on, and $g(S)$ is the genus of that surface.
- In this way, the domain of the conjecture and lemma (i) is restricted. But, the proof still remains the same.

A Bit of Geometric Topology

- Picture frame cannot be inflated into a sphere or a plane.
- Because genus (number of "holes") of a sphere is zero.
- It is topologically the 'same' to stretch the polyhedra onto the [Euclidean] plane and onto the sphere.
- Picture frame can be inflated into a torus as its genus is one.
- So the general formula of Euler characteristics in manifolds is:

$$V - E + F = 2 - 2.g(S)$$

  where S is the surface we consider to be inflated on, and $g(S)$ is the genus of that surface.
- In this way, the domain of the conjecture and lemma (i) is restricted. But, the proof still remains the same.

A Bit of Geometric Topology

- Picture frame cannot be inflated into a sphere or a plane.
- Because genus (number of "holes") of a sphere is zero.
- It is topologically the 'same' to stretch the polyhedra onto the [Euclidean] plane and onto the sphere.
- Picture frame can be inflated into a torus as its genus is one.
- So the general formula of Euler characteristics in manifolds is:

$$V - E + F = 2 - 2.g(S)$$

  where S is the surface we consider to be inflated on, and $g(S)$ is the genus of that surface.
- In this way, the domain of the conjecture and lemma (i) is restricted. But, the proof still remains the same.

A Bit of Geometric Topology

- Picture frame cannot be inflated into a sphere or a plane.
- Because genus (number of "holes") of a sphere is zero.
- It is topologically the 'same' to stretch the polyhedra onto the [Euclidean] plane and onto the sphere.
- Picture frame can be inflated into a torus as its genus is one.
- So the general formula of Euler characteristics in manifolds is:

$$V - E + F = 2 - 2.g(S)$$

  where S is the surface we consider to be inflated on, and $g(S)$ is the genus of that surface.
- In this way, the domain of the conjecture and lemma (i) is restricted. But, the proof still remains the same.

A Bit of Geometric Topology

- Picture frame cannot be inflated into a sphere or a plane.
- Because genus (number of "holes") of a sphere is zero.
- It is topologically the 'same' to stretch the polyhedra onto the [Euclidean] plane and onto the sphere.
- Picture frame can be inflated into a torus as its genus is one.
- So the general formula of Euler characteristics in manifolds is:

$$V - E + F = 2 - 2.g(S)$$

  where S is the surface we consider to be inflated on, and $g(S)$ is the genus of that surface.
- In this way, the domain of the conjecture and lemma (i) is restricted. But, the proof still remains the same.

A Bit of Geometric Topology

- Picture frame cannot be inflated into a sphere or a plane.
- Because genus (number of "holes") of a sphere is zero.
- It is topologically the 'same' to stretch the polyhedra onto the [Euclidean] plane and onto the sphere.
- Picture frame can be inflated into a torus as its genus is one.
- So the general formula of Euler characteristics in manifolds is:

$$V - E + F = 2 - 2.g(S)$$

  where S is the surface we consider to be inflated on, and $g(S)$ is the genus of that surface.
- In this way, the domain of the conjecture and lemma (i) is restricted. But, the proof still remains the same.

It is time for heuristics!

What about the heuristic roles of those counterexamples?

Simple Connectedness and its Heuristics

Lemma 3 There are only two alternatives: the disappearance of one edge or else of two edges and a vertex - when one drops the triangles by one. Furthermore, one will end up with a single triangle at the end of this process.

Remove triangles from inside!
Remove triangles in such way that the map will be disconnected!

Second Modification of Lemma 3 Remove triangles in such a way that $V - E + F$ will not change.

Simple Connectedness and its Heuristics

Lemma 3 There are only two alternatives: the disappearance of one edge or else of two edges and a vertex - when one drops the triangles by one. Furthermore, one will end up with a single triangle at the end of this process.

Remove triangles from inside!

Positive Heuristics!

Remove triangles in such way that the map will be disconnected!

Second Modification of Lemma 3 Remove triangles in such a way that $V - E + F$ will not change.

Nested Cube

Negative Heuristics!

These definitions tell us what is not a polyhedron.

Definition 1 A polyhedron is a solid whose surface consists of polygonal faces.

Definition 2 A polyhedron is a surface consisting of a systems of polygons.

Definition 3 A polyhedron is a system of polygons arranged in such a way that (1) exactly two polygons meet at every edge and, (2) it is possible to get from inside of any polygon to the inside of any other polygon by a route which never crosses any edge at a vertex.

Definition Perfect A polyhedron is a system of polygons for which the equation $V - E + F = 2$ holds.

Nested Cube

Negative Heuristics!

These definitions tell us what is not a polyhedron.

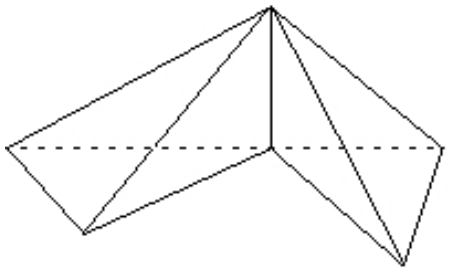
Definition 1 A polyhedron is a solid whose surface consists of polygonal faces.

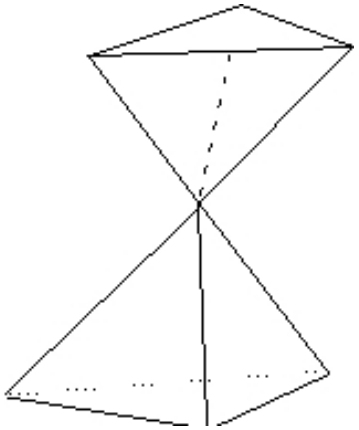
Definition 2 A polyhedron is a surface consisting of a systems of polygons.

Definition 3 A polyhedron is a system of polygons arranged in such a way that (1) exactly two polygons meet at every edge and, (2) it is possible to get from inside of any polygon to the inside of any other polygon by a route which never crosses any edge at a vertex.

Definition Perfect A polyhedron is a system of polygons for which the equation $V - E + F = 2$ holds.

Twin Polyhedra

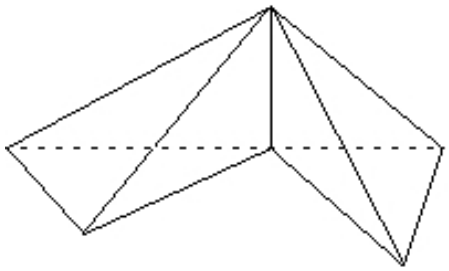


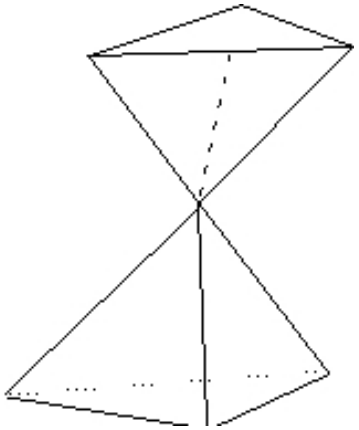


Negative Heuristics

There are still polyhedra satisfying $V - E + F = 2$.

Twin Polyhedra

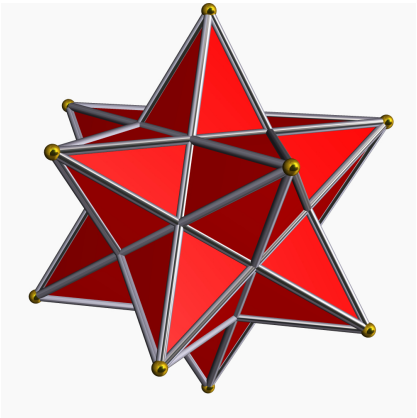




Negative Heuristics

There are still polyhedra satisfying $V - E + F = 2$.

Urchin

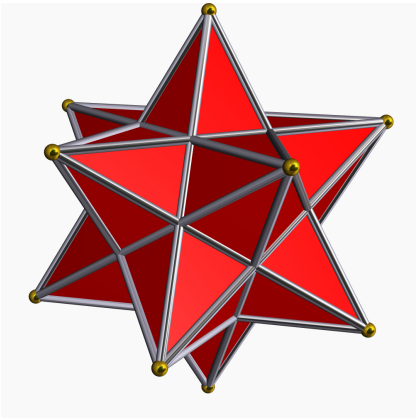


Negative Heuristics!

Revise the definitions to exclude urchin.

Definition 4 A polygon is a system of edges arranged is such a way that (1) exactly two edges meet at every vertex, and (2) the edges have no points in common except the vertices.

Urchin



Negative Heuristics!

Revise the definitions to exclude urchin.

Definition 4 A polygon is a system of edges arranged is such a way that (1) exactly two edges meet at every vertex, and (2) the edges have no points in common except the vertices.

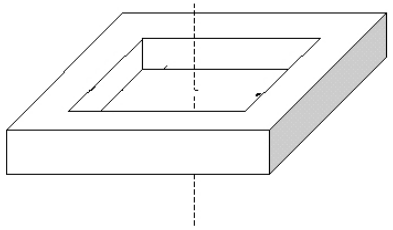
Picture Frame

Positive Heuristics!

Revise the definitions to include only spherical polyhedra.

Negative Heuristics!

Revise the definitions to exclude non-spherical polyhedra.



Definition 5 In the case of genuine polyhedron, through any arbitrary points in the space there will be at least one plane whose cross-section with the polyhedron will consist of one single polygon.

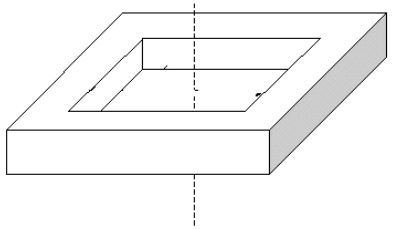
Picture Frame

Positive Heuristics!

Revise the definitions to include only spherical polyhedra.

Negative Heuristics!

Revise the definitions to exclude non-spherical polyhedra.



Definition 5 In the case of genuine polyhedron, through any arbitrary points in the space there will be at least one plane whose cross-section with the polyhedron will consist of one single polygon.

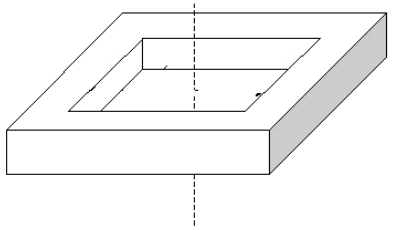
Picture Frame

Positive Heuristics!

Revise the definitions to include only spherical polyhedra.

Negative Heuristics!

Revise the definitions to exclude non-spherical polyhedra.



Definition 5 In the case of genuine polyhedron, through any arbitrary points in the space there will be at least one plane whose cross-section with the polyhedron will consist of one single polygon.

Cylinder

Negative Heuristics!

Exclude cylinder by redefining the edge!

Definition 6 An edge has two vertices.

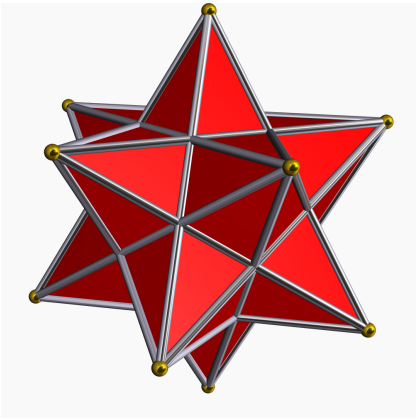
Cylinder

Negative Heuristics!

Exclude cylinder by redefining the edge!

Definition 6 An edge has two vertices.

Urchin

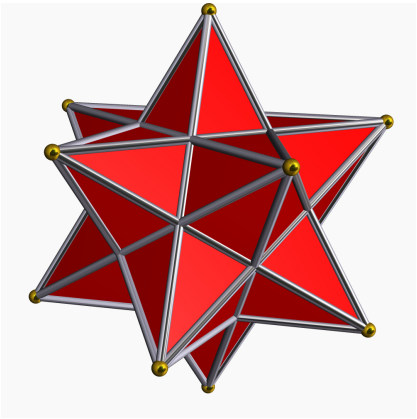


Positive Heuristics!

Reinterpret the terms!

There is no star-polygons in urchin, but only triangular faces. 60 faces, 90 edges and 32 vertices give the Euler characteristics 2.

Urchin



Positive Heuristics!

Reinterpret the terms!

There is no star-polygons in urchin, but only triangular faces. 60 faces, 90 edges and 32 vertices give the Euler characteristics 2.

A Bit of Topology

Positive Heuristics!

Introduce the concept of genus!

$$V - E + F = 2 - 2.g(S)$$

where S is the surface we consider to be inflated on, and $g(S)$ is the genus of that surface.

A Bit of Topology

Positive Heuristics!

Introduce the concept of genus!

$$V - E + F = 2 - 2.g(S)$$

where S is the surface we consider to be inflated on, and $g(S)$ is the genus of that surface.

Obstacles!

What are the problematic points in *Proofs and Refutations*?

Are all these true?

Was the history of Euler formula for polyhedra so smooth and "rational"?

Rational Reconstruction

- Not correct simply because, PR diverges from the real history.
- Re-constructed. Real history was not constructed like that.
- Only a case study. How can we generalize?

Rational Reconstruction

- Not correct simply because, PR diverges from the real history.
- Re-constructed. Real history was not constructed like that.
- Only a case study. How can we generalize?

Rational Reconstruction

- Not correct simply because, PR diverges from the real history.
- Re-constructed. Real history was not constructed like that.
- Only a case study. How can we generalize?

On the Horizon

- Game theoretic semantics for heuristics.
- Modal logic of heuristics.

... as illustrated in *Proofs and Refutations*.

On the Horizon

- Game theoretic semantics for heuristics.
- Modal logic of heuristics.

... as illustrated in *Proofs and Refutations*.

Happy Birthday Imre

Lakatos was born on November 9, 1922 as Imre Lipschitz.

Happy Birthday Imre

Lakatos was born on November 9, 1922 as Imre Lipschitz.

Thanks for your attention!

Talk slides are available at:
`www.illc.uva.nl/~cbaskent/mat/tagung.pdf`

More details (full paper) at:
`www.illc.uva.nl/~cbaskent/mat/proofs.pdf`

The help and support of Dr. Samet Bağce of Middle East Technical University, Ankara is kindly acknowledged.