Topics in Subset Space Logic

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Outlook of the Talk

- Introduction to SSL
- Truth Preserving Operations in SSL
- Public Announcement Logic in SSL
- Controlled Shrinking in SSL
- Multi-agent in SSL: an effort
- Conclusion and Future Work

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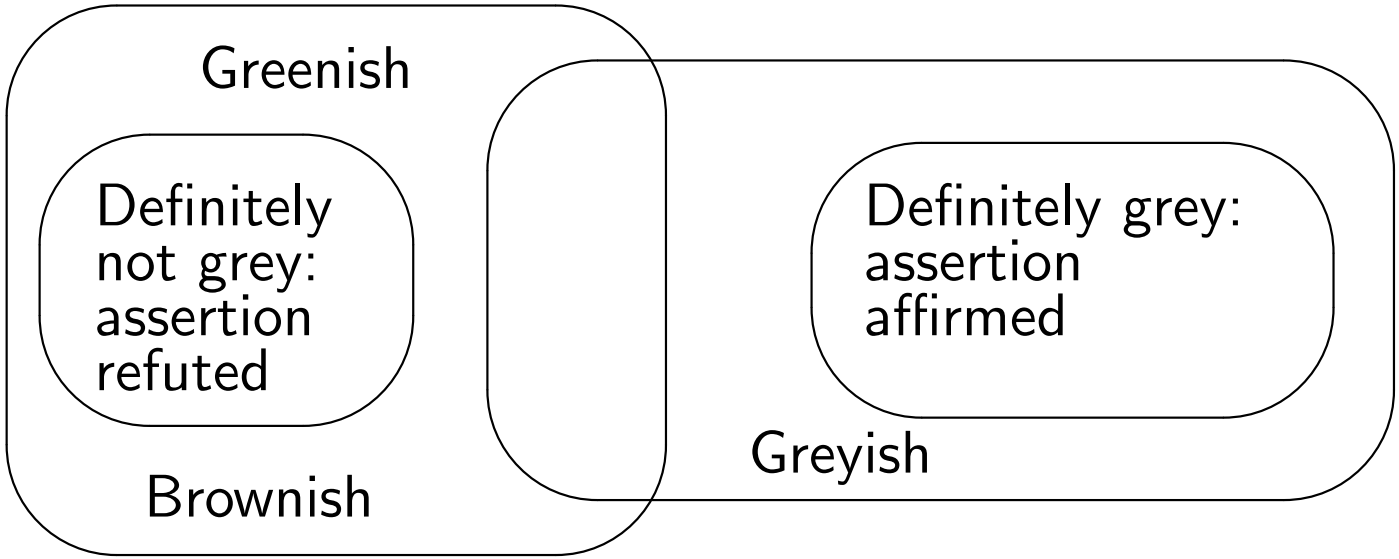
Vickers' Example

"My baby has green eyes."
The obvious question is, "Is this true or false?".
First, we may agree that her eyes really are green - we can *affirm* the assertion.
Second, we may agree that her eyes are some other colour, such as brown - we can *refute* the assertion.
Third, we may fail to agree; but perhaps if we hire a powerful enough colour analyser, that may decide us.
etc...

Vicker's Example

One can come up with the following diagram (Vickers).



Vickers' Example - Conclusion

What is crucial in Vickers' analysis is that statements are affirmable or refutable in a *finite* amount of time with spending *finite* amount of effort.

He defines: an assertion is *affirmative*, if and only if it is true precisely in the circumstances when it can be affirmed. Likewise, an assertion is *refutative* if and only if it is false precisely in the circumstances when it can be refuted.

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Vickers' Example - Conlusion

"[N]otion of *effort* enters in topology. Thus if we are at some point at s and make a measurement, we will then discover that we are in some neighborhood U of s, but not know where. If we make my measurement finer, then U will shrink, say, to a smaller neighborhood V." [Moss and Parikh]

Therefore, by spending some effort, we eliminate some of the possibilities, and finally obtain a smaller set of possibilities. The smaller the set of observation is, the larger the information we have.

Therefore, as it was also observed in the above example, to gain *knowledge*, we need to spend some *effort*.

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SSL: Model and Language

A subset space model is a triple $\mathcal{S} = \langle S, \sigma, v \rangle$ where $\langle S, \sigma \rangle$ is a subset frame, $v : P \to \wp(S)$ is a valuation function for the countable set of propositional variables P

The language $\mathcal{L}_S$ of SSL is:
$p \mid \top \mid \neg\varphi \mid \varphi \wedge \psi \mid \mathrm{K}\varphi \mid \Box\varphi$

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SSL: Semantics

$s, U \models p$	if and only if	$s \in v(p)$	
$s, U \models \varphi \wedge \psi$	if and only if	$s, U \models \varphi$	and $s, U \models \psi$
$s, U \models \neg\varphi$	if and only if	$s, U \not\models \varphi$	
$s, U \models \mathrm{K}\varphi$	if and only if	$t, U \models \varphi$	for all $t \in U$
$s, U \models \Box\varphi$	if and only if	$s, V \models \varphi$	for all $V \in \sigma$ such that $s \in V \subseteq U$

Neighborhood Situation

(s, U) is called a *neighborhood situation* if U is a neighborhood of s, i.e. if $s \in U$.

$\mathcal{L}_0$: be the propositional language generated by the set of propositional letters P.
Then, for the subset space frame $\mathcal{S} = \langle S, \sigma \rangle$, if $\varphi \in \mathcal{L}_0$, then we have $(\varphi)^{\mathcal{S}} \subseteq S$ whereas if $\varphi \in \mathcal{L}_S - \mathcal{L}_0$, we then have $(\varphi)^{\mathcal{S}} \subseteq S \times \sigma$.

Axioms

1. All the substitutional instances of the tautologies of the classical propositional logic
2. $(A \to \Box A) \wedge (\neg A \to \Box \neg A)$ for atomic sentence A
3. $\mathsf{K}(\varphi \to \psi) \to (\mathsf{K}\varphi \to \mathsf{K}\psi)$
4. $\mathsf{K}\varphi \to (\varphi \wedge \mathsf{K}\mathsf{K}\varphi)$
5. $\mathsf{L}\varphi \to \mathsf{K}\mathsf{L}\varphi$ Euclidean
6. $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$
7. $\Box\varphi \to (\varphi \wedge \Box\Box\varphi)$
8. $\mathsf{K}\Box\varphi \to \Box\mathsf{K}\varphi$ Cross-Axiom

K is S5 and $\Box$ is S4.

Cross Axiom Models: An alternative interpretation

A cross axiom frame is a tuple $\mathcal{J} = \langle J, \xrightarrow{\mathsf{L}}, \xrightarrow{\Diamond} \rangle$, such that J is a non-empty set, $\xrightarrow{\mathsf{L}}$ is an equivalence relation on J and $\xrightarrow{\Diamond}$ is a preorder on J where the following property holds: If $i \xrightarrow{\Diamond} j \xrightarrow{\mathsf{L}} k$, then there is some l such that $i \xrightarrow{\mathsf{L}} l \xrightarrow{\Diamond} k$.
A cross axiom model is a cross axiom frame together with an interpretation I of the atomic propositions of the language of subset spaces. I must satisfy the condition that if $i \xrightarrow{\Diamond} j$, then $i \in I(A)$ iff $j \in I(A)$.

Proposition

Every subset space has a corresponding cross axiom model.

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Completeness

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Completeness: what then?

To obtain the subsets in the collection of maximally consistent sets, an auxiliary pre-order and an anti-tone mapping is needed.

$\langle Q, \leq, \perp \rangle$ is the pre-order with the least element and $f : \langle Q, \leq, \perp \rangle \to \langle \wp(S)^*, \supseteq, S \rangle$. For $p, q \in P$, $p \leq q$ iff $f(p) \supseteq f(q)$.

Decidability

Finite model property fails in SSL.
Consider $\Box(\Diamond\varphi \wedge \Diamond\neg\varphi)$ at (s, U) where U is the minimal open about s.

Decidability then can be shown on Cross Axiom models by filtration as Cross Axiom models has a finite model property.

Directed, Intersection and Lattice Frames

A subset frame $\mathcal{S}$ is called a directed frame if for every $s \in S$ and $U, V \in \sigma$ with $s \in U$ and $s \in V$, there exists a $W \in \sigma$ such that $s \in W$ and $W \subseteq U \cap V$.

A subset frame $\mathcal{S}$ is called an intersection frame if whenever $U, V \in \sigma$ and $U \cap V \neq \emptyset$, then $U \cap V \in \sigma$ as well.

A subset frame $\mathcal{S}$ is called a lattice frame if it is an intersection frame which is also closed under finite unions; and is called a complete lattice frame if it is closed under arbitrary intersections and unions.

Defining Properties

WDA	$\Diamond\Box\varphi \to \Box\Diamond\varphi$
	sound for weakly directed spaces
UA	$\Diamond\varphi \wedge \mathsf{L}\Diamond\psi \to \Diamond(\Diamond\varphi \wedge \mathsf{L}\Diamond\psi \wedge \mathsf{K}\Diamond\mathsf{L}(\varphi \vee \psi))$
	sound for subset spaces closed under binary unions
WUA	$\mathsf{L}\Diamond\varphi \wedge \mathsf{L}\Diamond\psi \to \mathsf{L}\Diamond(\mathsf{L}\Diamond\varphi \wedge \mathsf{L}\Diamond\psi \wedge \mathsf{K}\Diamond\mathsf{L}(\varphi \vee \psi))$
	weaker than UA
CI	$\Box\Diamond\varphi \to \Diamond\Box\varphi$
	sound for subset spaces closed under all intersections
M_n	$(\Box\mathsf{L}\Diamond\varphi \wedge \Diamond\mathsf{K}\psi_1 \wedge \cdots \wedge \psi_n)$
	$\to \mathsf{L}(\Diamond\varphi \wedge \Diamond\mathsf{K}\psi_1 \wedge \cdots \wedge \Diamond\mathsf{K}\psi_n)$
	WD and all M_n are complete for directed spaces

Table: Additional properties in subset spaces and their respective defining formulae (from Moss et al.).

Overlap Modality

$$s, U \models \mathsf{O}\varphi \quad \textit{iff} \quad \forall U' \in \sigma : (s \in U' \rightarrow s, U' \models \varphi)$$

$\Box$ is a special case of O.

Overlap operator was designed to enable us to quantify "not only downwards, but also diagonally" among the set of observations (Heinemann).

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Some Basic Topological Properties in SSL

Proposition

φ is open if and only if $\varphi \to \Diamond \mathsf{K}\varphi$ is valid.

Proposition

Dually, φ is closed if and only if $\Box \mathsf{L}\varphi \to \varphi$.

Proposition

$v(p)$ is dense if and only if $\Box \mathsf{L}p$ holds. Similarly, $v(p)$ is nowhere dense if and only if $\Diamond \mathsf{L}\neg p$ is valid.

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Topological Interpretation vs. SSL

Define a translation map $^{\mathbf{t}} : \mathcal{L} \to \mathcal{L}_S$

$p^{\mathbf{t}} = p$	for propositional letter p
$(\varphi \wedge \psi)^{\mathbf{t}} = \varphi^{\mathbf{t}} \wedge \psi^{\mathbf{t}}$	for the formulae φ, ψ in $\mathcal{L}$
$(\neg\varphi)^{\mathbf{t}} = \neg(\varphi^{\mathbf{t}})$	for the formulae φ in $\mathcal{L}$
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Disjoint Unions

Definition

Two subset space models are disjoint if their domain contains no common element. For disjoint subset space models $\mathcal{S}_i = \langle S_i, \sigma_i, v_i \rangle$, *for* $i \in I$ *their disjoint union is the structure* $\mathcal{S} = \biguplus_{i \in I} \mathcal{S}_i = \langle S, \sigma, v \rangle$ *where* $S = \bigcup_{i \in I} S_i$, $\sigma = \bigcup_{i \in I} \sigma_i$ *and* $v(p) = \bigcup_{i \in I} v_i(p)$.

Theorem

For disjoint subset space models $\mathcal{S}_i$ *for* $i \in I$ *and for each neighborhood situation* (s, U) *in* $\mathcal{S}_i$, *we have* $s, U \models_{\mathcal{S}} \varphi$ *if and only if* $s, U \models_{\mathcal{S}_i} \varphi$, *for each formula* φ *in the language of subset space logic* $\mathcal{L}_{\mathcal{S}}$.

Note that this definition does not preserve topological properties. However, another definition which preserves topological properties can be given.

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Generated Subset Spaces

We can throw away the points at which we do not have any observations.

Proposition

For $\mathcal{S} = \langle S, \sigma, v \rangle$, *let* $S' = S - \{s : s \notin \cup\sigma\}$ *and* $v'(p) = v(p) \cap S'$. *Then* $\mathcal{S}' = \langle S', \sigma, v' \rangle$ *and* $\mathcal{S} = \langle S, \sigma, v \rangle$ *satisfy the same formulae.*

Generated Subset Spaces

Definition

Let $\mathcal{S} = \langle S, \sigma, v \rangle$ be a subset space model. Let (s, U) be the designated neighborhood situation. Then we obtain the generated subset space $\mathcal{S}' = \langle S', \sigma', v' \rangle$ of $\mathcal{S}$ as follows.

- $\sigma' := \sigma - \{V \in \sigma : V \not\subseteq U\}$
- $S' := S - \cup\sigma'$
- $v'(p) := v(p) \cap S'$ *for each propositional letter* p.

Proposition

For each $s \in S'$, we have $s, U \models_{\mathcal{S}} \varphi$ if and only if $s, U \models_{\mathcal{S}'} \varphi$.

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Bisimulation

Definition

For, $\mathcal{S} = \langle S, \sigma, u \rangle$ and $\mathcal{T} = \langle T, \tau, v \rangle$ a topologic bisimulation is a non-empty relation $\rightleftarrows$ for neighborhood situations in $(S \times \sigma) \times (T \times \tau)$ such that if $(s, U) \rightleftarrows (t, V)$, then we have:

1. **Base Condition**

   $s \in u(p)$ *if and only if* $t \in v(p)$ *for each* p

2. **Back Conditions**

   2.1 $\forall t' \in V$ *there exists* $s' \in U$ *with* $(s', U) \rightleftarrows (t', V)$.

   2.2 $\forall V' \subseteq V$ *such that* $t \in V'$, *there is* $U' \subseteq U$ *with* $s \in U'$ *such that* $(s, U') \rightleftarrows (t, V')$

3. **Forth Conditions**

   3.1 $\forall s' \in U$ *there exists* $t' \in V$ *with* $(s', U) \rightleftarrows (t', V)$.

   3.2 $\forall U' \subseteq U$ *such that* $s \in U'$, *there is* $V' \subseteq V$ *with* $t \in V'$ *such that* $(s, U') \rightleftarrows (t, V')$.

Bisimulation Invariance

Theorem (Bisimulation Invariance for Subset Spaces)

If $(s, U) \rightleftarrows (t, V)$ *then they satisfy the same formulae.*

Converse is true only under the special conditions.

Theorem

Let $\mathcal{S} = \langle S, \sigma, u \rangle$ *and* $\mathcal{T} = \langle T, \tau, v \rangle$ *be two finite subset space. Then for each neighborhood situations* (s, U) *in* $S \times \sigma$ *and* (t, V) *in* $T \times \tau$*; we have* $(s, U) \rightleftarrows (t, V)$ *if and only if* $(s, U) \leftrightsquigarrow (t, V)$.

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Evaluation and Bisimulation Games

Position	Player	Admissible Moves
$(\bot, (s, U))$	$\exists$	$\emptyset$
$(\top, (s, U))$	$\forall$	$\emptyset$
$(p, (s, U))$ with $s \in v(p)$	$\forall$	$\emptyset$
$(p, (s, U))$ with $s \notin v(p)$	$\exists$	$\emptyset$
$(\psi_1 \wedge \psi_2, (s, U))$	$\forall$	$\{(\psi_1, (s, U)), (\psi_2, (s, U))\}$
$(\psi_1 \vee \psi_2, (s, U))$	$\exists$	$\{(\psi_1, (s, U)), (\psi_2, (s, U))\}$
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Adequacy Theorems for Evaluation and Bisimulation games follow.

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$(\Diamond\psi, (s, U))$	$\exists$	$\{(\psi, (s, V)) : s \in V \subseteq U\}$
$(\Box\psi, (s, U))$	$\forall$	$\{(\psi, (s, V)) : s \in V \subseteq U\}$
$(\mathsf{P}\psi, (s, U))$	$\exists$	$\{(\psi, (s, U')) : s \in U'\}$
$(\mathsf{O}\psi, (s, U))$	$\forall$	$\{(\psi, (s, U')) : s \in U'\}$

Adequacy Theorems for Evaluation and Bisimulation games follow.

Semantics

Definition

Let $\mathcal{M} = \langle W, R, V \rangle$ be a model and i be an agent. For atomic propositions, negations and conjunction the definition is as usual. For modal operators, we have the following semantics:

$$\mathcal{M}, w \models \mathsf{K}_i\varphi \quad \textit{iff} \quad \mathcal{M}, v \models \varphi \textit{ for each } v \textit{ such that } (w, v) \in R_i$$

$$\mathcal{M}, w \models [\varphi]\psi \quad \textit{iff} \quad \mathcal{M}, w \models \varphi \textit{ implies } \mathcal{M}|\varphi, w \models \psi$$

Here the updated model $\mathcal{M}|\varphi = \langle W', R', V' \rangle$ is defined by restricting $\mathcal{M}$ to those states where φ holds.

Reduction Axioms

The proof system of public announcement logic is the proof system of multi-modal S5 epistemic logic with the following additional axioms.

Atoms	$[\varphi]p \leftrightarrow (\varphi \to p)$
Partial Functionality	$[\varphi]\neg\psi \leftrightarrow (\varphi \to \neg[\varphi]\psi)$
Distribution	$[\varphi](\psi \wedge \chi) \leftrightarrow ([\varphi]\psi \wedge [\varphi]\chi)$
Knowledge Announcement	$[\varphi]\mathsf{K}_i\psi \leftrightarrow (\varphi \to \mathsf{K}_i[\varphi]\psi)$

The rule of inference for $[*]$ is called the *announcement generalization* and is described as follows.

From $\vdash \psi$, *derive* $\vdash [\varphi]\psi$.

Semantics

The semantics for topologic PAL differs only on public announcement operator whose semantics is given as follows:

$$s, U \models [\varphi]\psi \quad \textit{if and only if} \quad s, U \models \varphi \textit{ implies } s, U_\varphi \models \psi$$

where $U_\varphi = U \cap (\varphi)$

Compare: $\mathcal{M}, w \models [\varphi]\psi$ iff $\mathcal{M}, w \models \varphi$ implies $\mathcal{M}|\varphi, w \models \psi$

Axioms

Therefore, it is easy to see that the following axiomatize the topologic-PAL:

Atoms	$[\varphi]p \leftrightarrow (\varphi \to p)$
Partial Functionality	$[\varphi]\neg\psi \leftrightarrow (\varphi \to \neg[\varphi]\psi)$
Distribution	$[\varphi](\psi \wedge \chi) \leftrightarrow ([\varphi]\psi \wedge [\varphi]\chi)$
Knowledge Announcement	$[\varphi]\mathsf{K}\psi \leftrightarrow (\varphi \to \mathsf{K}[\varphi]\psi)$
Shrinking Reduction	$[\varphi]\Box\psi \leftrightarrow (\varphi \to \Box[\varphi]\psi)$

Completeness

Theorem (Completeness of Topologic PAL)

Topologic PAL is complete with respect to the axiom system given above.

Proof.

By reduction axioms we can reduce each formula in the language of topologic PAL to a formula in the language of (basic) topologic. As topologic is complete, so is topologic PAL. □

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PAL with Overlap Operator

Theorem (Reduction Axiom for Overlap Operator)

$[\varphi]\mathsf{O}\varphi \leftrightarrow (\varphi \rightarrow \mathsf{O}[\varphi]\psi)$ *is sound.*

Theorem (Completeness of Topologic PAL with Overlap)

Topologic public announcement logic with overlap operator is complete.

PAL with Overlap Operator

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Theorem (Completeness of Topologic PAL with Overlap)

Topologic public announcement logic with overlap operator is complete.

Philosophy of Science: Lakatos

Proofs and Refutations gives a rationally reconstructed account of the methodological evaluation of Euler's formula for polyhedra: $V - E + F = 2$.
Starting from a collection of observations (or assertions) about some peculiar properties of polyhedron, the arguments proceed by reducing these observations (or assertions) by some mathematical *thought experiments* as Lakatos himself called.

Philosophy of Science: Lakatos

Let us see an example.
Let us assume $(polyhedron, U) \models V_{torus} - E_{torus} + F_{torus} = 2$ where U is the collection of observed polyhedral objects. Some may be genuine polyhedra, some not.
Clearly, $E(torus) = 0$. Contradiction.
Then we need to get rid of some objects in U we previously thought of genuine polyhedra. For example, we need to get rid of torus, Klein bottle, Mobiüs strip etc. to get $U' \subset U$.
The formal way of achieving that is to introduce the Euler characteristic function for both oriented and non-oriented objects.

Philosophy of Science: Lakatos

The effort in this context corresponds to some mathematical calculations or suggesting a counter example or even refuting a counterexample.
For example, if we establish that the Euler formula holds for simply connected polyhedra, then, we will discard some observations about the polyhedra which are not simply connected - such as torus. Hence, without changing our point of view, we changed our neighborhood situation by considering some smaller set around the reference point we are occupying.

Semantics

Let $\mathcal{F}$ be a collection of functions from S to S, and further let $F \subseteq \mathcal{F}$. Take two subset spaces $\mathcal{S} = \langle S, \sigma, v \rangle$ and $\mathcal{S}_F = \langle S, \sigma_F, v \rangle$. Here, σ_F is the image of each $U \in \sigma$ under each function $f \in F$. In other words, $\sigma_F := \{fU : f \in F, U \in \sigma\}$. We will call $\mathcal{S}_F$ *the image space of* $\mathcal{S}$ *under* F.

Each function $f \in F$ are contracting mappings intended to represent the increase in the information. Hence, $fU \subseteq U$ should hold for each function f and for each observation set U

Semantics

$s, U \models_{\mathcal{S}} [F]\varphi$ iff $s, fU \models_{\mathcal{S}_F} \varphi$ for each $f \in F$

The dual of $[F]$ will be defined as follows:

$s, U \models_{\mathcal{S}} \langle F \rangle \varphi$ iff $s, fU \models_{\mathcal{S}_F} \varphi$ for some $f \in F$

Some Observations

1. $[F](\varphi \to \psi) \to ([F]\varphi \to [F]\psi)$
   It is easy to see that $[F]$ modality realizes the **K** axiom
2. $[F][F]\varphi \to [F]\varphi$
   This axiom is valid if F is closed under function decomposition.
3. $[F]\varphi \to [F][F]\varphi$
   This axiom is valid if F is closed under function composition.
4. $[F]\varphi \to \varphi$
   This axiom is valid if the identity function id_F is in F.
5. $\Box\varphi \to [F]\varphi$
6. $\mathrm{K}[F]\varphi \to [F]\mathrm{K}\varphi$
   This is the cross axiom for $[F]$ and K

Intersection of Observations

The intersection subset frame $\mathcal{T} = \mathcal{S} \cap \mathcal{S}'$ is the frame $\mathcal{T} = \langle S, \tau \rangle$ where $\tau = \{U : U \in \sigma_1 \cap \sigma_2\}$.

Lemma

For downward closed set U, if $s, U \models_{\mathcal{T}} \varphi$, then $s, U \models_{\mathcal{S}} \varphi$ and $s, U \models_{\mathcal{S}'} \varphi$ for each formula φin the language of subset space logic.

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Direct Intersections

Definition (Direct Intersections)

Given two subset space frames $\mathcal{S}_1 = \langle S, \sigma_1 \rangle$ *and* $\mathcal{S}_2 = \langle S, \sigma_2 \rangle$*, the direct intersection of* $\mathcal{S}_1$ *and* $\mathcal{S}_2$ *is* $\mathcal{S}_1 \sqcap \mathcal{S}_2 = \langle S, \tau \rangle$ *where* $\tau = \{X : X = U \cap V$ *where* $U \in \sigma_1$ *and* $V \in \sigma_2\}$.

Proposition

If $s, U \models_{\mathcal{S}_1} \varphi$ *and* $s, V \models_{\mathcal{S}_2}$*, then* $s, U \cap V \models_{\mathcal{S}_1 \sqcap \mathcal{S}_2} \varphi$ *for each* $\varphi \in \mathcal{L}_S$.

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Product

A *product multi-agent subset frame* is a tuple $\mathcal{S} = \langle S, \prod_i \sigma_i \rangle$ where $\langle S, \sigma_i \rangle$ is a subset space frame for each agent i in the set of agents I.

$(s, \vec{U})$ is a *neighborhood situation* if $s \in U_i$ for each i.

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$(s, \vec{U})$ is a *neighborhood situation* if $s \in U_i$ for each i.

Product: Semantics

$s, \vec{U} \models p$	*if and only if*	$p \in v(A)$
$s, \vec{U} \models \varphi \wedge \psi$	*if and only if*	$s, \vec{U} \models \varphi$ *and* $s, \vec{U} \models \psi$.
$s, \vec{U} \models \neg\varphi$	*if and only if*	$s, \vec{U} \not\models \varphi$.
$s, \vec{U} \models \mathsf{K}_i\varphi$	*if and only if*	$t, \vec{U} \models \varphi$ *for all* $t \in \cap_i U_i \in \sigma_i$.
$s, \vec{U} \models \Box_i\varphi$	*if and only if*	$s, \vec{V} \models \varphi$ *for all* $\vec{V}$ *where* $U_j = V_j$ *for* $j \neq i$, *and* $V_i \subseteq U_i$.

Proposition

COM_K and $COM_\Box$ together with CHR_K and $CHR_\Box$ are valid in product subset spaces.

COM_K and $COM_\Box$ will denote the commutativity property for the knowledge operator and for the shrinking operator respectively. In a similar manner, CHR_K will denote the Church-Russer property for the knowledge operator, that is $K_1L_2\varphi \leftrightarrow L_2K_1\varphi$ and $CHR_\Box$ is defined likewise.

Product: Semantics

$$
\begin{array}{lll}
s, \vec{U} \models p & \textit{if and only if} & p \in v(A) \\
s, \vec{U} \models \varphi \wedge \psi & \textit{if and only if} & s, \vec{U} \models \varphi \textit{ and } s, \vec{U} \models \psi. \\
s, \vec{U} \models \neg\varphi & \textit{if and only if} & s, \vec{U} \not\models \varphi. \\
s, \vec{U} \models \mathsf{K}_i\varphi & \textit{if and only if} & t, \vec{U} \models \varphi \textit{ for all } t \in \cap_i U_i \in \sigma_i \ . \\
s, \vec{U} \models \Box_i\varphi & \textit{if and only if} & s, \vec{V} \models \varphi \textit{ for all } \vec{V} \textit{ where } U_j = V_j \\
 & & \textit{for } j \neq i, \textit{ and } V_i \subseteq U_i.
\end{array}
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Proposition

COM_{K} and $COM_{\Box}$ together with CHR_{K} and $CHR_{\Box}$ are valid in product subset spaces.

$\mathrm{COM}_{\mathsf{K}}$ and $\mathrm{COM}_{\Box}$will denote the commutativity property for the knowledge operator and for the shrinking operator respectively. In a similar manner, $\mathrm{CHR}_{\mathsf{K}}$ will denote the Church-Russer property for the knowledge operator, that is $\mathsf{K}_1\mathsf{L}_2\varphi \leftrightarrow \mathsf{L}_2\mathsf{K}_1\varphi$ and $\mathrm{CHR}_{\Box}$ is defined likewise.

Multi Direct Products: Semantics

The product of the frames $\mathcal{S}_1 = \langle S_1, \sigma_1 \rangle$ and $\mathcal{S}_2 = \langle S_2, \sigma_2 \rangle$ is the frame $\mathcal{F}_1 \times \mathcal{F}_2 = \langle S_1 \times S_2, \sigma_h, \sigma_v \rangle$ in which for each $u, u' \in U$ and $v, v' \in V$ and for each $U' \subseteq U$ and $V' \subseteq V$, we then have,

$$
\begin{array}{lll}
(u,v), U \times V \models \mathrm{K}_1\varphi & \text{iff} & \text{for all } u' \in U : (u',v), U \times V \models \varphi \\
(u,v), U \times V \models \Box_1\varphi & \text{iff} & \text{for all } u \in U' \subseteq U : (u,v), U' \times V \models \varphi
\end{array}
$$

$$
\begin{array}{lll}
(u,v), U \times V \models \mathrm{K}_2\varphi & \text{iff} & \text{for all } v' \in V : (u,v'), U \times V \models \varphi \\
(u,v), U \times V \models \Box_2\varphi & \text{iff} & \text{for all } v \in V' \subseteq V : (u,v), U \times V' \models \varphi
\end{array}
$$

Multi Direct Products: Observations

Proposition

COM_{K} $(\mathsf{K}_1\mathsf{K}_2\varphi \leftrightarrow \mathsf{K}_2\mathsf{K}_1\varphi)$ and $COM_{\Box}$ $(\Box_2\Box_1\varphi \leftrightarrow \Box_1\Box_2\varphi)$ are valid in topologic products. Moreover, they can be generalized to n-agents case.

Proposition

CHR_{K} $(\mathsf{K}_1\mathsf{L}_2\varphi \leftrightarrow \mathsf{L}_2\mathsf{K}_1\varphi)$ and $CHR_{\Box}$ $(\Box_1\Diamond_2\varphi \leftrightarrow \Diamond_2\Box_1\varphi)$ are valid in topologic products.

Common Knowledge in SSL

Simplest definition:

$$\mathsf{C}\varphi \equiv \varphi \wedge \Diamond\mathsf{K}\varphi \wedge \Diamond\mathsf{K}\Diamond\mathsf{K}\varphi \ldots$$

$s, U \models \mathsf{C}\varphi :=$
$\forall n \in \mathbb{N}$ and $t \in S$, we then have:
if $U_0, U_1, \ldots, U_n \in \sigma$ satisfy $U_0 = U$ and $U_i \cap U_{i+1} \neq \emptyset$
for $i = 0, \ldots, n-1$ and, $t \in U_n$, then $t, U_n \models \varphi$

The following is the iteration definition of common knowledge.

$$s, U \models \mathsf{C}\varphi \equiv s, U \models \underbrace{\mathsf{KO} \ldots \mathsf{KO}}_{n-\text{times}} \varphi$$

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Relativized Common Knowledge in SSL

$s, U \models \mathsf{C}(\varphi, \psi) :=$
$\forall n \in \mathbb{N}$ and $t \in S$, we then have:
if $U_{0_\varphi}, \ldots, U_{n_\varphi} \in \sigma$ satisfy $U_{0_\varphi} = U$ and $U_{i_\varphi} \cap U_{(i+1)_\varphi} \neq \emptyset$
for $i = 1, \ldots, n-1$, and $t \in U_{n_\varphi}$, then $t, U_{n_\varphi} \models \psi$
where U_{i_φ} is $U_i \cap (\varphi)$.

More intuitively:

$$s, U \models \mathsf{C}(\varphi, \psi) \equiv s, U_\varphi \models \underbrace{\mathsf{KO} \ldots \mathsf{KO}}_{n-times} \psi$$

Relativized Common Knowledge in SSL

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More intuitively:

$$s, U \models \mathsf{C}(\varphi, \psi) \equiv s, U_\varphi \models \underbrace{\mathsf{KO} \ldots \mathsf{KO}}_{n-times} \psi$$

Common Knowledge in Topologic PAL

Theorem

$[\varphi]\mathsf{C}(\psi, \chi) \leftrightarrow (\varphi \rightarrow \mathsf{C}(\varphi \wedge [\varphi]\psi, [\varphi]\chi))$ *is sound.*

Theorem

Topologic public announcement logic with overlap and relativized common knowledge operators is complete.

Common Knowledge in Topologic PAL

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$[\varphi]\mathsf{C}(\psi, \chi) \leftrightarrow (\varphi \rightarrow \mathsf{C}(\varphi \wedge [\varphi]\psi, [\varphi]\chi))$ *is sound.*

Theorem

Topologic public announcement logic with overlap and relativized common knowledge operators is complete.

Obstacles!

Multi-agent: not very clear how the agents should merge their information. The behavior of neighborhood situations in multi-agent case is not as straight forward as in the case of Kripke structures.

Cross-Axiom Frames: the relation between cross axiom frames and subset frames is not clear. It is especially vague how to obtain a subset frame from cross axiom frame.

Observation Sets: it is not clear if we are supposed to consider all the possible observations or some selected or given collection of observations for a given set. This does not change anything as the technical results do not depend on this. However, from a semantical point of view, we believe, an agent cannot possibly consider the all possible observations in any case.

Recap of the Results

- We imported some simple truth preserving operations.
- Introduced the game theoretical semantics
- Considered PAL in SSL
- Introduced controlled shrinking motivated by philosophy of science
- Considered multi-agent semantics in SSL

Future Work

Complexity of various multi-agent subset spaces: with or without common knowledge operator.

Universal Modalities: it is also possible to extend the language with the universal modalities E and A in order to increase the expressivity.

Thanks for your attention!

Talk slides and the thesis are available at:

www.illc.uva.nl/∼cbaskent