A Survey of Topologic

Illuminating New Directions

Can Başkent

Department of Computer Science
Graduate Center, the City University of New York
cbaskent@gc.cuny.edu // www.canbaskent.net/logic

December 1st, 2011 - The Graduate Center

Contents

Epistemic notions vs Topological Notions I

"Most branches of mathematics (...) involve structures which give a mathematical content to the intuitive notions of *limit*, *continuity* and *neighborhood*. (...) Historically, the ideas of limit and continuity appeared very early in mathematics, notably in geometry, and their role has steadily increased with the development of analysis and its applications to the experimental science, since these ideas are closely related to those of *experimental determination* and *approximation*."

Epistemic notions vs Topological Notions II

"If we start from the physical concept of approximation, it is natural to say that a subset A of E is a neighborhood of an element a of A if, whenever we replace a by an element that 'approximates' a, this new element will also belong to A, provided of course that the 'error' involved is small enough, or, in other words, if all the points of E which are 'sufficiently near' a belong to A."

(Bourbaki, 1966, p. 11)

An Epistemic Example I

Suppose that a policeman uses radar to determine the speed of passing cars. At one instance, he reads that the speed of a car is 51mph in a 50 mile speed limit zone.

Question Is the car speeding?

If the error range of the radar is +/- 3 mph, then he does not *know* whether the car is speeding. If the policeman uses a more accurate radar with an error range, say +/- 0.5 mph then he *knows* that the car is speeding. Because, in that case, the car's speed is in the range $(50.5, 51.5)$ which is entirely contained in the speeding interval $(50, \infty)$.

An Epistemic Example II

So, we can represent these two situations as follows.
In the first case, the policeman *can* know that the car is speeding, and it is epistemically possible that the policeman *cannot* know that the car is speeding (due to the error range of the lousy radar). However, in the second case, the policeman *knows* that the car is speeding. (Parikh *et al.*, 2007)

Basics

Subset space logic (SSL or topologic) formalizes reasoning about sets and points with an underlying motivation of embedding the geometrical notion of *closeness* into epistemic logic (Moss & Parikh, 1992).

The key idea of topologic can be phrased as follows:
"In order to *get close*, one needs to spend some *effort*."
Therefore, in topologic, the knowledge is defined with respect to both a *point* and a *neighborhood* of that point.

Syntax and Semantics I

A subset space model is a triple $\langle S, \sigma, v \rangle$ where S is a set of points and $\sigma \subseteq \wp(S)$ and v is a valuation function. Notice that σ is **not** necessarily a mathematical structure (topology, lattice etc).

We have two modalities: Knowledge (K) and Effort ($\Box$) with the usual syntax with the countable set of propositional variables P:

Syntax

$$p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \mathsf{K}\varphi \mid \Box\varphi$$

Duals L and $\Diamond$ are defined in the usual sense, and $p \in P$.
Notice that this setting is for single agent.

Syntax and Semantics II

Research Direction!

Multi-agent subset space logic (Başkent, 2007)

Now, we can express variety of epistemic situations.
For example, consider the statement:
$\Diamond Kp$.

Research Direction!

Fitch's Paradox (Balbiani *et al.*, 2008)

Syntax and Semantics III

Here is the semantics:

$$
\begin{array}{lll}
s, U \models p & \text{iff} & s \in v(p) \\
s, U \models \varphi \wedge \psi & \text{iff} & s, U \models \varphi \text{ and } s, U \models \psi \\
s, U \models \neg \varphi & \text{iff} & s, U \not\models \varphi \\
s, U \models \mathsf{K}\varphi & \text{iff} & t, U \models \varphi \text{ for all } t \in U \\
s, U \models \Box \varphi & \text{iff} & s, V \models \varphi \text{ for all } V \subseteq U \text{ for } V \in \sigma
\end{array}
$$

Observe:
Semantics is defined with respect to a tuple (s, U) where $s \in U \in \sigma$. In this case, U is a neighborhood of s.
Truth of propositional variables is independent of the neighborhood.

Syntax and Semantics IV

We can think of U as the set of observations, measurements with respect to s.
Also, U is *kind of* set of accessible states for the points in it.
Therefore, topologic models can be translated into Kripke models.
This makes topologic a bit more explicit. Namely, in Kripke semantics, when you are given $w \models \varphi$ you don't really have the set of accessible states in the semantics. In $s, U \models \varphi$, you do have it in the semantics.

Research Direction!

How to translate subset frames to Kripke frames? What is the complexity of this translation?

Axioms

The axioms of SSL simply reflect the fact that the K modality is S5-like whereas the $\Box$ modality is S4-like. Moreover, we need an additional axiom to state the interaction between the two modalities: $\mathsf{K}\Box\varphi \to \Box\mathsf{K}\varphi$ (perfect recall/cross axiom).
Yet another important fact is that the atomic sentences are independent from their neighborhoods, thus the following axiom for atomic sentence F is valid in SSL: $(F \to \Box F) \wedge (\neg F \to \Box\neg F)$.
The rules of inference for SSL is as expected: modus ponens and necessitation for both modalities.
Moreover, SSL is sound and complete with respect to the aforementioned axiomatization.
Furthermore, it is decidable (without finite model property).

Tree-Like Structures I

Georgatos considered topologic in tree-like spaces. A tree-like space $\langle S, \sigma \rangle$ is a subset space where for all $U, V \in \sigma$, either $U \subseteq V$, or $V \subseteq U$ or $U \cap V = \emptyset$.
Clearly, in the countable case, the set of subsets of a treelike space forms a tree under subset ordering (Georgatos, 1997).

Tree-Like Structures II

Tree-like models are axiomatized with the following two-additional axioms:

- $\Box(\Box\varphi \to \psi) \vee \Box(\Box\psi \to \varphi)$
- $\Box \mathsf{K}\varphi \wedge \mathsf{K}(\Box\varphi \to \Box\psi) \to \Box \mathsf{K}(\Box\varphi \to \Box\psi)$

The first axiom characterizes the reflexive, transitive and connected frames.
Validity of the second axiom is left as an exercise.

Tree-Like Structures III

Research Direction!

Can we express compactness in topologic?

We can add several (long and a bit complicated) axioms to subset spaces to express **topological** spaces, and a get a complete and finitely decidable system (Georgatos, 1994).

Directed Frames I

If for all $s \in S$ and $U, V \in \sigma$ whenever $s \in U$ and $s \in V$, there exists $W \in \sigma$ so that $s \in W \subseteq (U \cap V)$, the the frame $\langle S, \sigma \rangle$ is called directed (Weiss & Parikh, 2002).
To formalize directed spaces within the language of SSL, we need to add the following two axioms:

- $\Diamond \Box \varphi \rightarrow \Box \Diamond \varphi$
- $(\Box \mathsf{L} \Diamond \varphi \wedge \Diamond \mathsf{K} \psi_1 \wedge \ldots \wedge \Diamond \mathsf{K} \psi_n) \rightarrow \mathsf{L}(\Diamond \varphi \wedge \Diamond \mathsf{K} \psi_1 \wedge \ldots \wedge \Diamond \mathsf{K} \psi_n)$

Communication Graphs I

One of the immediate applications of SSL is the communication graphs. In an earlier paper, Pacuit and Parikh assumed that some agents are connected by a communication graph. In the communication graph, an edge from agent i to agent j means that agent i can directly receive information from agent j. Agent i can then refine its own information by learning information that j has, including information acquired by j from another agent, k. (Pacuit & Parikh, 2007)

Communication Graphs II

Let $\Diamond p$ mean that p becomes true after a sequence of communications that respects the communication graph - namely, it becomes true after some effort has been spent.
Let $\Box p$ mean that p becomes true after every sequence of communications that respects the communication graph (after any effort/communication, it is true).

Communication Graphs III

Clearly, every communication necessitates a common language. So, their idealization has two assumptions:

- All the agents share a common language.
- The agents make available all possible pieces of (purely propositional) information which they know and which are expressible in this common language.

The technical details of this logic uses history based structures and semantics of messages as they are natural tools for graphs with epistemic reading (Parikh & Ramamujam, 2003).

Additional Modalities: Overlap I

Heinemann wrote variety of papers on the subject (Heinemann, 1999a; Heinemann, 2003a; Heinemann, 2003b; Heinemann, 2005a; Heinemann, 2005b; Heinemann, 2003c; Heinemann, 2005c; Heinemann, 2006a; Heinemann, 1999b; Heinemann, 2006b; Heinemann, 2009c; Heinemann, 2008; Heinemann, 2009b; Heinemann, 2009a) (and counting...).

I will discuss some of his work here.

Additional Modalities: Overlap II

He introduced an additional overlap modality O to the syntax of subset space logic (Heinemann, 2006b).
The semantics of O is as follows:

$$s, U \models \mathsf{O}\varphi \quad \text{iff} \quad \forall V \in \sigma . s \in V \rightarrow s, V \models \varphi$$

Now observe the following validities:

- $\mathsf{O}\varphi \rightarrow \Box\varphi$
- $\mathsf{O}\varphi \rightarrow \varphi \wedge \mathsf{OO}\varphi$
- $\varphi \rightarrow \mathsf{OP}\varphi$

where P is the dual of O.

Additional Modalities: Overlap III

Research Direction!

Is O definable in topologic? Why/why not?

We can add several axioms for the overlap operator to get a complete system: normativity for O, atomic permanence for O, reflexivity, transitivity, symmetry, and O implies $\Box$.
For completeness, add the necessitation proof rule for O.
Heinemann showed that his system is complete (Heinemann, 2006b).

A Brief Summary of Other Extensions

- Hybrid nominals i
- Temporal next time operators $\bigcirc$
- Universal modality A
- Controlled subset spaces and functions (based on (Başkent, 2007))
- Continuous functions
- Disjoint neighborhoods

and counting...

Dynamic Epistemology on Subset Spaces

We can now discuss a dynamic epistemic take on subset spaces.
Public announcement logic (PAL) deals with knowledge updates with a state elimination based paradigm (Plaza, 1989).
Consider $[\varphi]\psi$ with the intended meaning that *after the public announcement of* φ, ψ *holds*.
Announcements are external and truthful (i.e. *by God*).

Public Announcement in SSL I

After a public announcement, agents receive new information and update their information set by eliminating the states that do not agree with the announcements (as the announcements are truthful).
Public announcements in SSL simply shrinks the neighborhood.
After the announcement φ which is true at the neighborhood situation, we obtain a smaller neighborhood U_φ which can be defined as $U_\varphi = U \cap (\varphi)_2$ where
$(\varphi)_2 = \{U : (s, U) \in (\varphi) \text{ for some } s\}$ for the extension (φ).

Public Announcement in SSL II

Similarly, for a given subset space model $\mathcal{S} = \langle S, \sigma, V \rangle$, we get the updated model $\mathcal{S}_\varphi = \langle S_\varphi, \sigma_\varphi, V_\varphi \rangle$ after the announcement φ. In this context, $S_\varphi = S \cap (\varphi)_1$ where $(\varphi)_1 = \{s : (s, U) \in (\varphi) \text{ for some } U\}$, and $\sigma_\varphi = \{U \cap S_\varphi : U \in \sigma :\}$, and $V_\varphi = V \cap S_\varphi$, as expected.

Axioms

The following axiomatizes the PAL in SSL.

Atoms	$[\varphi]p \leftrightarrow (\varphi \rightarrow p)$
Partial Functionality	$[\varphi]\neg\psi \leftrightarrow (\varphi \rightarrow \neg[\varphi]\psi)$
Distribution	$[\varphi](\psi \wedge \chi) \leftrightarrow ([\varphi]\psi \wedge [\varphi]\chi)$
Knowledge Announcement	$[\varphi]K\psi \leftrightarrow (\varphi \rightarrow K[\varphi]\psi)$
Effort Announcement	$[\varphi]\Box\psi \leftrightarrow (\varphi \rightarrow \Box[\varphi]\psi)$

Theorem ((Başkent, to appear))

PAL in SSL is sound and complete.

Topological Spaces

It is an easy and nice exercise to see that public announcement logic also works for topological spaces (Başkent, to appear).

Research Direction!

What's the connection between common knowledge and public announcements?

Knowability and Subset Spaces I

We can now take several further steps:

- Quantify over announcements (Balbiani *et al.*, 2007; Balbiani *et al.*, 2008): Which formulas are preserved after arbitrary announcements?
- Quantify over models (Wen *et al.*, 2011): Which submodels can be considered as epistemic updates?

Knowability and Subset Spaces II

In a recent paper, Wen et al. considered downward closed subset space models to describe an alternative logic for epistemic updates (Wen *et al.*, 2011).
In their system, the updates can be any subset of the accessibility relation, and they show that downward closed subset space logic is expressive enough for that logic.

Research Ideas

- Game Logic has a similar semantics to subset spaces, so it is possible to merge subset spaces and game logic
- Relationship between different multi-agent versions of subset spaces
- More defining properties for different spaces (Cantor spaces were investigated already)
- Applying such ideas to philosophy proper (Başkent, 2009; Başkent, 2012 to appear).

Research Ideas

- Game Logic has a similar semantics to subset spaces, so it is possible to merge subset spaces and game logic
- Relationship between different multi-agent versions of subset spaces
- More defining properties for different spaces (Cantor spaces were investigated already)
- Applying such ideas to philosophy proper (Başkent, 2009; Başkent, 2012 to appear).

Research Ideas

- Game Logic has a similar semantics to subset spaces, so it is possible to merge subset spaces and game logic
- Relationship between different multi-agent versions of subset spaces
- More defining properties for different spaces (Cantor spaces were investigated already)
- Applying such ideas to philosophy proper (Başkent, 2009; Başkent, 2012 to appear).

Research Ideas

- Game Logic has a similar semantics to subset spaces, so it is possible to merge subset spaces and game logic
- Relationship between different multi-agent versions of subset spaces
- More defining properties for different spaces (Cantor spaces were investigated already)
- Applying such ideas to philosophy proper (Başkent, 2009; Başkent, 2012 to appear).

Thanks for your attention!

Talk slides are available at:

`www.canbaskent.net/logic`

References I

Balbiani, Philippe, Baltag, Alexandru, van Ditmarsch, Hans, Herzig, Andreas, Hoshi, Tomohiro, & de Lima, Tiago. 2007.
What Can We Achieve by Arbitrary Announcements? A Dynamic Take on Fitch's Knowability.
In: Samet, Dov (ed), *Procedings of the 11th Conference on Theoretical Aspects of Rationality and Knowledge (TARK-2007)*.

Balbiani, Philippe, Baltag, Alexandru, van Ditmarsch, Hans, Herzig, Andreas, & de Lima, Tiago. 2008.
'Knowable' as 'known after an announcement'.
Review of Symbolic Logic, **1**(3), 305–334.

Başkent, Can. 2007 (July).
Topics in Subset Space Logic.
M.Phil. thesis, Institute for Logic, Language and Computation, Universiteit van Amsterdam.

Başkent, Can. 2009.
A Geometrical - Epistemic Approach to Lakatosian Heuristics.
Pages 8–14 of: Drossos, C., Peppas, P., & Tsinakis, C. (eds), *Proocedings of Seventh Panhellenic Logic Symposium*.
Patras University Press.

Başkent, Can. 2012 to appear .
A Formal Approach to Lakatosian Heuristics.
Logique et Analyse.

References II

BAŞKENT, CAN. to appear .
Public Announcement Logic in Geometric Frameworks.
Fundamenta Infomaticae.

BOURBAKI, NICOLAS. 1966.
General Topology.
Elements of Mathematics, vol. 1.
Addison-Wesley.

GEORGATOS, KONSTANTINOS. 1994.
Knowledge Theoretic Properties of Topological Spaces.
Pages 147–159 of: MASUCH, M., & POLOS, L. (eds), *Knowledge Representation and Uncertainty*.
Lecture Notes in Artificial Intelligence, vol. 808.
Springer-Verlag.

GEORGATOS, KONSTANTINOS. 1997.
Knowledge on Treelike Spaces.
Studia Logica, **59**(2), 271–301.

HEINEMANN, BERNHARD. 1999a.
Seperating Sets by Modal Formulas.
Pages 140–53 of: HAERBERER, A. M. (ed), *Algebraic Methodology and Software Technology, AMAST'98*.
Lecture Notes in Computer Science, vol. 1548.
Springer.

References III

HEINEMANN, BERNHARD. 1999b.
Temporal Aspects of the Modal Logic of Subset Spaces.
Theoretical Computer Science, **224**(1-2), 135–155.

HEINEMANN, BERNHARD. 2003a.
An Application of Monodic First-Order Temporal Logic to Reasoning about Knowledge.
Pages 10–6 of: REYNOLDS, M., & SATTAR, A. (eds), *Proceedings TIME-ICTL 2003*.
IEEE Computer Society Press.

HEINEMANN, BERNHARD. 2003b.
Extended Canonicity of Certain Topological Properties of Set Spaces.
Pages 135–49 of: VARDI, M., & VORONKOV, A. (eds), *Logic for Programming, Artificial Intelligence, and Reasoning, LPAR 2003*.
Lecture Notes in Artificial Intelligence, vol. 2850.
Springer.

HEINEMANN, BERNHARD. 2003c.
Towards Uniform Reasoning via Structured Subset Spaces.
Pages 185–203 of: BALBIANI, P., SUZUKI, N.-Y., WOLTER, F., & ZAKHARYASCHEV, M. (eds), *Advances in Modal Logic*, vol. 4.
King's College Publications.

References IV

HEINEMANN, BERNHARD. 2005a.
Algebras as Knowledge Structures.
Pages 471–482 of: JEDRZEJOWICZ, J., & SZEPIETOWSKI, A. (eds), *Mathematical Foundations of Computer Science, MFCS 2005.*
Lecture Notes in Computer Science, vol. 3618.
Springer.

HEINEMANN, BERNHARD. 2005b.
A Spatio-temporal View of Knowledge.
Pages 703–8 of: RUSSELL, I., & MARKOV, Z. (eds), *Proceedings 18th Int. FLAIRS Conference.*
AAAI Press.

HEINEMANN, BERNHARD. 2005c.
The Topological Effect of Improving Knowledge Acquisition.
Pages 21–30 of: A. GELBUKH, A. DE ALBORNOZ, H. TERASHIMA-MARIN (ed), *MICAI 2005: Advances in Artificial Intelligence.*
Lecture Notes in Artificial Intelligence, vol. 3789.
Springer.

HEINEMANN, BERNHARD. 2006a.
Reasoning about Knowledge and Continuity.
Pages 37–42 of: GOEBEL, R., & SUTCLIFFE, G. (eds), *Proceedings 19th International FLAIRS Conference.*
AAAI Press.

References V

Heinemann, Bernhard. 2006b.
Regarding Overlaps in Topologic.
Pages 259–277 of: Governatori, Guido, Hodkinson, Ian, & Venema, Yde (eds), *Advances in Modal Logic*, vol. 6.
College Publications.

Heinemann, Bernhard. 2008.
Topology and Knowledge of Multiple Agents.
Pages 1–10 of: Geffner, H. (ed), *Proceedings of IBERAMIA*.
Lecture Notes in Artificial Intelligence, vol. 5290.

Heinemann, Bernhard. 2009a.
Modelling Uniformity and Control during Knowledge Acquisition.
Pages 65–70 of: Wilson, David, & Lane, H. Chad (eds), *Proceedings of the Twenty-First International Florida Artificial Intelligence Research Society Conference*.
AAAI Press.

Heinemann, Bernhard. 2009b.
Observational Effort and Formally Open Mappings.
Pages 197–208 of: Ono, H., Kazanawa, M., & de Queiroz, R. (eds), *WoLLIC*.
LNAI 5514, vol. 5514Springer-Verlag, for WoLLIC.

Heinemann, Bernhard. 2009c.
Using Hybrid Logic for Reasoning About Functions in Spaces of Sets.
???

References VI

Moss, Lawrence S., & Parikh, Rohit. 1992.
Topological Reasoning and the Logic of Knowledge.
Pages 95–105 of: Moses, Yoram (ed), *Proceedings of TARK IV*.

Pacuit, Eric, & Parikh, Rohit. 2007.
Reasoning About Communication Graphs.
In: van Benthem, Johan, Löwe, Benedikt, & Gabbay, Dov (eds), *Proceedings of Augustus de Morgan Workshop: Interactive Logic: Games and Social Software*.

Parikh, Rohit, & Ramamujam, R. 2003.
A Knowledge Based Semantics of Messages.
Journal of Logic, Language and Information, **12**(4), 453 – 467.

Parikh, Rohit, Moss, Lawrence S., & Steinsvold, Chris. 2007.
Topology and Epistemic Logic.
In: Aiello, Marco, Pratt-Hartman, Ian E., & van Benthem, Johan (eds), *Handbook of Spatial Logics*.
Springer.

Plaza, Jan A. 1989.
Logic of Public Communication.
Pages 201–216 of: Emrich, M. L., Pfeifer, M. S., Hadzikadic, M., & Ras, Z. W. (eds), *4th International Symposium on Methodologies for Intelligent Systems*.

References VII

WEISS, M. ANGELA, & PARIKH, ROHIT. 2002.
Completeness of Certain Bimodal Logics for Subset Spaces.
Studia Logica, **71**(1), 1–30.

WEN, XUEFENG, LIU, HU, & HUANG, FAM. 2011.
An Alternative Logic for Knowability.
Pages 342–355 of: VAN DITMARSCH, HANS, LANG, JEROME, & JU, S. (eds), *Proceedings of LORI 2011*.
LNAI 6953.
Springer-Verlag.