Paradoxes and Games

What can Games Learn from Paradoxes?

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Slogan: Paraconsistency for Game Theory!

Paraconsistent Game Theory

helps us understand games

and paradoxes better!

Today’s Plan

1. Motivating Examples
2. Game Semantics for Paradoxes
3. A Game Theoretical Paradox
4. A Yabloesque Paradox in Epistemic Games
5. Conclusion

Motivating Examples

“Wardrobing”

“Wardrobing is buying an item of clothing, wearing it for a while, and then returning it in such a state that the store has to accept it but can no longer resell it. By engaging in wardrobing, consumers are not directly stealing money from the company; instead, it is a dance of buying and returning, with many unclear transactions involved. But there is at least one clear consequence — the clothing industry estimates that its annual losses from wardrobing are about $16 billion (about the same amount as the estimated annual loss from home burglaries and automobile theft combined) (...) [This is] the level of dishonesty practiced by individuals who want to be ethical and who want to see themselves as ethical — the so-called good people.”

Predictably Irrational, Dan Ariely, Harper, 2008.

Arrow's Theorem

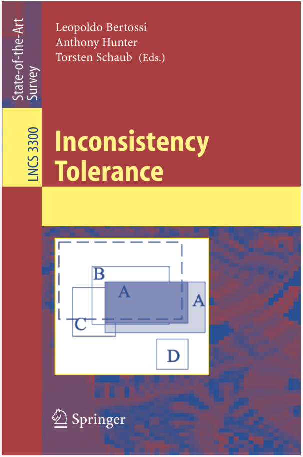
Paradoxes and impossibility results have shaped Social Choice Theory.

"The theorem states that no rank-order voting system can be designed that always satisfies these three 'fairness' criteria:

- If every voter prefers alternative X over alternative Y, then the group prefers X over Y.
- If every voter's preference between X and Y remains unchanged, then the group's preference between X and Y will also remain unchanged (even if voters' preferences between other pairs like X and Z, Y and Z, or Z and W change).
- There is no "dictator": no single voter possesses the power to always determine the group's preference.

[from *Wikipedia*]

Inconsistency Tolerance



How can you handle inconsistencies, besides belief revision or dynamic updates?

Inconsistency Tolerance, L. Bertossi, A. Hunter, T. Schaub (editors), Springer, 2005.

There are two different methodologies available.

Games vs Paradoxes

There are two different methodologies available.

We can

- devise games to express inconsistencies [**logic to games**]
- start with games and discover their paradoxes [**games to logic**]

Game Semantics for Paradoxes

Semantic Games

Hintikkan semantic games are determined, two-player, sequential, zero-sum, non-cooperative as they are *designed* for classical logic.

Semantic Games

Hintikkan semantic games are determined, two-player, sequential, zero-sum, non-cooperative as they are *designed* for classical logic.

- Can we design *logics for semantic games* with different properties? What is the logic of three-player, non-zero sum undetermined games?
- Can we design *games for non-classical logics*? How can we play semantic games for Belnap's B4 or da Costa Systems?
- How can we manipulate strategies in semantics games for other logics?

During the semantic verification game, the given formula is broken into subformulas by *two players* (Abelard and Heloise) *step by step*, and the game terminates when it reaches the propositional atoms.

If we end up with a propositional atom which is true, then Eloise the verifier wins the game. *Otherwise*, Abelard the falsifier wins. We associate *conjunction with Abelard, disjunction with Heloise*.

A win for the verifier is when the game terminates with a true statement. The verifier is said to have a winning strategy if she can force the game to her win, regardless of how her opponent plays.

Classical Semantic Games

Just because the game may end with a true/false atom does not necessarily suggest the truth/falsity of the given formula in general.

In classical logic, however, the major result of game theoretical semantics states that the verifier has a winning strategy if and only if the given formula is true in the model.

Non-Classical Semantic Games

Logic	Players	Inconsistent TV	Play
Logic of Paradox	3	Yes	Non-Sequential
First-Degree Entailment	2	Yes	Non-Sequential
Connexive Logic	2 Teams	No	Coalitional

"Game Theoretical Semantics for Some Non-Classical Logic", CB, *Journal of Applied Non-Classical Logics*, vol. 26, no. 3, pp. 208-39, 2016.

A Game Theoretical Paradox

The Brandenburger-Keisler paradox is a two-person self-referential paradox in epistemic game theory.

The following configuration of beliefs is impossible:

The Paradox

Ann believes that Bob assumes that Ann believes that Bob's assumption is wrong.

The paradox appears if you ask whether "Ann believes that Bob's assumption is wrong".

Notice that this is essentially a 2-person Russell's Paradox.

Brandenburger - Keisler Paradox

Brandenburger and Keisler showed that no belief model is complete for its (classical) first-order language: there are statements that cannot be modelled.

Therefore, ”not every description of belief can be represented” with belief structures.

“An Impossibility Theorem on Beliefs in Games”, A. Brandenburger and J. Keisler, *Studia Logica*, vol. 84, pp. 211-240, 2006.

Brandenburger - Keisler Paradox

This is a self-referential paradox. A two- (or n-) person generalization of the Liar Paradox, rendering it in an interactive and game theoretical setting.

Theoretical richness of self-referentiality relates it to fixed-points and a broad class of logics and models.

“Some Non-Classical Approaches to the Brandenburger-Keisler paradox”, CB, *Logic Journal of the IGPL*, vol. 23, pp. 533-552, 2015.

“From Lawvere to Brandenburger-Keisler: Interactive forms of diagonalization and self-reference ”, S. Abramsky and J. Zvesper, *Journal of Computer and System Sciences*, vol. 81, pp. 799-812, 2015.

The Model

The *assumption* modality is worth exploring:

The language is given as follows for a set of propositional variables P:

$$\varphi := p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Box^{ij}\varphi \mid \heartsuit^{ij}\varphi$$

where $p \in \mathsf{P}$ and $i \neq j$ for $i, j \in 2$. $\Box^{ij}$ denotes the knowledge modality whereas $\heartsuit^{ij}$ is the assumption modality.

The belief model is a tuple $M = (U^i, R^{ij}, V)$ where $R^{ij} \subseteq U^i \times U^j$ and V is a valuation function.

The expression $R^{ij}(x, y)$ represents that in state x, the player i believes that the state y is possible for player j.

The Model

The semantics for the modal operators is given as follows.

$$
\begin{array}{llllll}
x \models \Box^{ij}\varphi & \textit{iff} & \forall y \in U^j . R^{ij}(x,y) & \textit{implies} & y \models \varphi \\
x \models \heartsuit^{ij}\varphi & \textit{iff} & \forall y \in U^j . R^{ij}(x,y) & \textit{iff} & y \models \varphi
\end{array}
$$

Assumption is strong belief, perhaps the strongest belief.

A Paradox

Is there a *non*-self-referential paradox?

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Is there a *non*-self-referential paradox?

Is there a *non*-self-referential *game theoretical* paradox?

Yablo's Paradox

Yablo's Paradox, according to its author, is a non-self referential paradox.

Yablo considers the following sequence of sentences.

$$\begin{aligned} S_1 &: \forall k > 1, S_k \text{ is untrue,} \\ S_2 &: \forall k > 2, S_k \text{ is untrue,} \\ S_3 &: \forall k > 3, S_k \text{ is untrue,} \\ &\vdots \end{aligned}$$

"Paradox without Self-Reference", S. Yablo, *Analysis*, vol. 53, pp. 251-2, 1993.

By using *reductio*, Yablo argues that the above set of sentences is contradictory. Here, the infinitary nature of the paradox is essential as the each finite set of S_n is satisfiable.

The scheme of this paradox is not new. To the best of our knowledge, the first analysis of this paradox was suggested in 1953 by Yuting.

"Paradox of the Class of All Grounded Classes", Sh. Yuting, *The Journal of Symbolic Logic*, vol. 18, p. 114, 1953.

A Yabloesque Paradox in Epistemic Games

Is there a *non*-self-referential *game theoretical* paradox?

A Yabloesque Paradox in Epistemic Games

Consider the following sequence of assumptions where numerals represent game theoretical players.

$$
\begin{aligned}
A_1 &: 1 \text{ believes that } \forall k > 1, k\text{'s assumption } A_l \text{ about } \forall l > k \text{ is untrue,}\\
A_2 &: 2 \text{ believes that } \forall k > 2, k\text{'s assumption } A_l \text{ about } \forall l > k \text{ is untrue,}\\
A_3 &: 3 \text{ believes that } \forall k > 3, k\text{'s assumption } A_l \text{ about } \forall l > k \text{ is untrue,}\\
&\vdots
\end{aligned}
$$

"A Yabloesque Paradox in Epistemic Game Theory", CB, *Synthese*, 2016, forthcoming (online first access).

An Interpretation

Imagine a queue of players, where players are conveniently named after numerals, holding beliefs about each player behind them, but not about themselves. In this case, each player i believes that each player $k > i$ behind them has an assumption about each other player $l > k$ behind them and i believes that each k's assumption is false.

This statement is perfectly perceivable for games, and involves a specific configuration of players' beliefs and assumptions, which can be expressible in the language.

However, similar to Yablo's paradox and the Brandenburger - Keisler paradox, this configuration of beliefs is impossible.

The Yabloesque Brandenburger - Keisler paradox requires ω-many players $i \in I$. The language is given as follows for a set of propositional variables P:

$$\varphi := p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Box^{ij}\varphi \mid \heartsuit^{ij}\varphi$$

where $p \in \mathsf{P}$ and $i \neq j$ for $i, j \in I$ with $|I| = \omega$.

The extended belief model is a tuple $M = (\{U^i\}_{i \in I}, \{R^{ij}\}_{i \neq j \in I}, V)$ where $R^{ij} \subseteq U^i \times U^j$ and V is a valuation function.

The expression $R^{ij}(x, y)$ represents that in state x, the player i believes that the state y is possible for player j.

The semantics for the modal operators is given as follows.

$$x \models \Box^{ij}\varphi \quad \textit{iff} \quad \forall y \in U^j.R^{ij}(x,y) \textit{ implies } y \models \varphi$$
$$x \models \heartsuit^{ij}\varphi \quad \textit{iff} \quad \forall y \in U^j.R^{ij}(x,y) \textit{ iff } y \models \varphi$$

Theorem

The Yabloesque Brandenburger - Keisler sentence is inconsistent.

A Paraconsistent Model

For the same syntax, it is possible to give a topological semantics for epistemic game models that is *inconsistency-friendly*.

Therefore, there exists a *paraconsistent* model in which the yabloesque Brandenburger - Keisler paradox is satisfiable!

A Paraconsistent Model

Using topological products, it is possible to construct interactive, multi-agent models for epistemic game theory that can allocate non-classical logics.

"Some Non-Classical Approaches to the Brandenburger-Keisler paradox", CB, *Logic Journal of the IGPL*, vol. 23, pp. 533-552, 2015.

"A Yabloesque Paradox in Epistemic Game Theory", CB, *Synthese*, 2016, forthcoming (online first access).

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- Postulate an interactive theory of Tarskian truth, using Yablo's Paradox,
- Generalize Liar-like paradoxes,
- Study the paradoxes of intelligent interaction,
- Put *Logical Pluralism* at work,
- Present a broader paradigm to understand paradoxes,

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Paraconsistent games

- Postulate an interactive theory of Tarskian truth, using Yablo's Paradox,
- Generalize Liar-like paradoxes,
- Study the paradoxes of intelligent interaction,
- Put *Logical Pluralism* at work,
- Present a broader paradigm to understand paradoxes,
- Identify the limitations of epistemic games.

Conclusion

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and paradoxes better!

Thank you!

Talk slides and the papers are available at

`CanBaskent.net/Logic`