Topics in Subset Space Logic

MSc Thesis (*Afstudeerscriptie*)

written by

Can BAŞKENT
(born July 18th, 1981 in Istanbul, Turkey)

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Members of the Thesis Committee:
Johan van Benthem
Joel Hamkins
Dick de Jongh
Eric Pacuit

INSTITUTE FOR LOGIC, LANGUAGE AND COMPUTATION

To my father who taught me that "Məntıq elmlerin elmidir"
and to my mother

with the deepest gratitude...


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Contents

Chapter 1

Introduction to Subset Space Logic

1.1 A Bit of History

The topological interpretation of modal logics can be traced back to the 1930s. The first topological interpretation of modal logic was given by Tsao-Chen Tang published in *Bulletin of the American Mathematical Society*[1] in 1938 (Goldblatt, 2006). Tang proposed that "[sic]the algebraic postulates for the Lewis calculus of strict implication [sic] be the axioms for a Boolean algebra with an additional operation". The topological interpretation of this operation is the interior operator. However, technical details are not relevant for our discussions. So, the interested reader is referred to Goldblatt's expository paper on the mathematics of modal logic for more information (Goldblatt, 2006).

The most influential series of works on the topological interpretation of modal logic was initiated by J. C. C. McKinsey in 1941. He first started by considering the decidability problem for the classical modal logics S2 and S4 from an algebraic point of view. In other words, he wanted to show whether there is an algorithm for deciding if a given formula is a theorem of S2 or S4. His approach was to obtain a finite model which falsifies the formula, if the given formula is not a theorem of this logic. This is an example of what we now call "finite model property". As Goldblatt underlined, McKinsey's construction was an "important innovation" (Goldblatt, 2006). So, let us briefly review how he approached to the problem. McKinsey used the models which are of the form $(S, D, -,^*, \cdot)$. He then called them *ma-*

[1]Algebraic postulates and a geometric interpretation for the Lewis calculus of strict implication. *Bulletin of the American Mathematical Society*, 44:737744), 1938.

trices. $-,^*$ and $\cdot$ are the operations on a set S to evaluate the connectives $\neg, \Diamond$ and $\wedge$ and D is a set of designated elements of S. Then what does it mean that "$(S, D, -,^*, \cdot)$ satisfies a formula φ"? A formula φ is satisfied by a matrix of this form if each assignment of elements of S to the variables of φ results in φ being evaluated to a member of D. In this context, we define $x \Rightarrow y$ as $-^*(x \cdot . - y)$ which interprets the strict implication in S. A matrix is called normal if $x, y \in D$ implies $x \cdot y \in D$ and, $x, (x \Rightarrow y) \in D$ implies $y \in D$ and lastly $(x \Leftrightarrow y) \in D$ implies $x = y$. Recall further that a closure operation C on a set S is defined on the subsets X of S such that $\mathsf{C}\emptyset = \emptyset, \mathsf{C}(X \cup Y) = \mathsf{C}X \cup \mathsf{C}Y$ and $X \subseteq \mathsf{C}X = \mathsf{CC}X$.

Let us now see how the matrix construction works. If we take $(S, -,^*, \cdot)$ as the algebra of formulae, and let $-\alpha = \neg\alpha,^* \alpha = \Diamond\alpha$ and $\alpha\dot{\beta} = \alpha \wedge \beta$ together with the set of designated elements D as the set of S2 theorems; then the matrix $(S, D, \neg, \Diamond, \wedge)$ will be the characteristic matrix of S2. However, it will not satisfy the condition $(x \Leftrightarrow y) \in D$ implies $x = y$ for obvious reasons. Therefore in order to make our system a Boolean algebra, we have to identify the formulae $x, y \in D$ whenever $x \Leftrightarrow y \in D$. Then, we have a nice picture. The elements of D are the "equivalence classes of the theorems".

Now, after the very brief review of the preliminaries, let us go back to McKinsey's construction. He first considered the closure operator and its properties. McKinsey showed that any finite normal S4-matrix can be represented as the closure algebra of all subsets of some topological space. This construction uses the Stone representation theorem which states that any Boolean algebra is isomorphic to a set algebra. However, in our case, we use a corollary of the Stone's Theorem stating that a finite Boolean algebra can be represented as the power set algebra of its set of atoms (i.e. singletons). This result was later extended by McKinsey and Tarski to the arbitrary topological spaces in 1944. In their further work, they applied these results on the algebra of topology to the logic S4. Moreover, they gave several characterization of the logic S4 (McKinsey & Tarski, 1944). However, this discussion is not entirely relevant to our main concerns here. We invite the interested reader to consult the paper (Goldblatt, 2006) for a brief overview of the aforementioned constructions.

Let us now introduce the basics of the topological interpretation of modal logic.

1.2 Topological Semantics for Modal Logic

Before introducing the modal logics interpreted in topological spaces, let us start by recalling the definition of topological spaces.

Definition 1.2.1 (Topological Space). *A topological space $\mathcal{S} = \langle S, \sigma \rangle$ is a structure with a set S and a collection σ of subsets of S satisfying the following axioms:*

1. *The empty set and S are in σ.*
2. *The union of any collection of sets in σ is also in σ.*
3. *The intersection of a finite collection of sets in σ is also in σ.*

σ is said to be a *topology* on S. The elements of S are called *points* and the elements of σ are called *opens*. The complements of open sets are called *closed* sets.

The basic modal language $\mathcal{L}$ has a countable set of proposition letters P, a truth constant $\top$, the usual Boolean operators $\neg$ and $\wedge$, and a modal operator $\Box$. The dual of $\Box$ is denoted by $\Diamond$ and defined as $\Box\varphi \equiv \neg\Diamond\neg\varphi$.

A topological model $\mathcal{M}$ is a triple $\langle S, \sigma, v \rangle$ where $\mathcal{S} = \langle S, \sigma \rangle$ is a topological space, and v is a valuation function sending propositional letters to the subsets of S, i.e. $v : P \to \wp(S)$.

When we are in topological models, we will use the notation I for $\Box$ for intuitive reasons. Likewise, we will put C for $\Diamond$.

The notation $\mathcal{M}, s \models \varphi$ will read *the point s in the model $\mathcal{M}$ makes the formula φ true*. Consequently, we define the semantics as follows:

Definition 1.2.2 (Topological Semantics). *Truth of modal formulae in topological semantics is defined inductively at points s for a topological model $\mathcal{M} = \langle S, \sigma, v \rangle$:*

$$
\begin{array}{lll}
\mathcal{M}, s \models p & \textit{if and only if} & s \in v(p) \textit{ for } p \in P \\
\mathcal{M}, s \models \neg\varphi & \textit{if and only if} & \textit{not } \mathcal{M}, s \models \varphi \\
\mathcal{M}, s \models \varphi \wedge \psi & \textit{if and only if} & \mathcal{M}, s \models \varphi \textit{ and } \mathcal{M}, s \models \psi \\
\mathcal{M}, s \models \mathsf{I}\varphi & \textit{if and only if} & \exists U \in \sigma(s \in U \wedge \forall t \in U, \mathcal{M}, t \models \varphi)
\end{array}
$$

The C operator can then be defined accordingly

$$\mathcal{M}, s \models \mathsf{C}\varphi \quad \text{if and only if} \quad \forall U \in \sigma(s \in U \to \exists t \in U, \mathcal{M}, t \models \varphi)$$

$\mathcal{M} \models \varphi$ means $\mathcal{M}, s \models \varphi$ for all $s \in S$. $(\varphi)^{\mathcal{M}}$ denotes the points in the model $\mathcal{M}$ which satisfy φ.

In topological models, the I operator is interpreted as the topological *interior* operator $\mathbb{I}$. Recall that the interior of X $\mathbb{I}(X)$ is the largest open subset of X. We then define the valuation of I modality as follows: $v(\mathsf{I}\varphi) = \mathbb{I}(v(\varphi))$.

Recall now that the topological interior operator $\mathbb{I}$ satisfies the following properties for each $X, Y \subseteq S$: (i) $\mathbb{I}(S) = S$, (ii) $\mathbb{I}(X \cap Y) = \mathbb{I}(X) \cap \mathbb{I}(Y)$, (iii) $\mathbb{I}(\mathbb{I}(X)) = \mathbb{I}(X) \subseteq X$

There is a very well known connection between the topological spaces and the Kripke models. Namely, every S4 Kripke frame $\langle S, R\rangle$ gives rise to a topological space $\langle S, \sigma_R\rangle$, where σ_R is the set of all upward closed subsets of the given frame. It is easy to see that the empty set and S are in σ_R, and furthermore arbitrary unions and finite intersections of upward closed sets are still upward closed. Hence, σ_R is a topology.

The topology σ_R we obtained out of the Kripke frame $\langle S, R\rangle$ is a special one and is called Alexandroff topology. Alexandroff topologies are those in which each point has a *least* neighborhood. It is evident that, in Kripke frames the least neighborhood of a point is the set of points which are accessible in one step. In other words, the least neighborhood of a point s is the set $\{t \in W : sRt\}$.

However, Alexandroff topologies can be characterized in several ways. Another equivalent definition states that Alexandroff spaces are those topological spaces in which intersection of any family of opens is again an open. It is then a nice exercise to see the equivalence of these two definitions.

We just briefly explained how to obtain a topology out of a Kripke frame. We can also get a Kripke frame from a topological space. The reason for that is the fact that each topological space $\langle S, \sigma\rangle$ induces a partial order R_σ defined as $sR_\sigma t$ for $s \in \mathsf{C}(t)$. It is now straight forward to see that R_σ is reflexive and transitive. We then have $R = R_{\sigma_R}$ if and only if the topological space is Alexandroff (Benthem & Bezhanishvili, 2006).

1.2.1 Brief Overview of the Developments in the Topological Semantics of Modal Logic

First, we will give a broader picture of the recent developments in the topological semantics of modal logic. Then, we will look into the details of these recent developments.

The first major works on topological semantics for modal logic were carried out by McKinsey and Tarski as we already discussed (McKinsey & Tarski, 1944). The investigation of modal logics of geometric structures in the Euclidean spaces was first initiated by Dragalin in 1970s (Aiello *et al.* , 2003). Esakia, on the other hand, worked on the interpretation of $\Diamond$ as the topological *derivative operator*[2]. The important development of the 1970s was the neighborhood semantics of Scott and Montague which was introduced as a generalization of standard relational semantics. Neighborhood semantics has an indispensable topological flavor, yet the spatial aspect of this semantics was never prominent as Aiello et al. emphasized (Aiello *et al.* , 2003).

The works targeting the modal definability in topological spaces was recently completed by ten Cate, Gabelaia and Susterov (Cate *et al.* , 2006). They considered both the basic modal language and hybrid modal language in order to show the definability or non-definability of some topological properties. For example, they gave the precise modal formulae that defines the extremally disconnected topological spaces and the atomic spaces, the main result is a topological Goldblatt-Thomason theorem that states that the class $\mathbf{K}$ of topological spaces which is closed under formation of Alexandroff extensions is modally definable if and only if $\mathbf{K}$ is closed under taking open subspaces, interior images, topological sums and it reflects Alexandroff extensions (Cate *et al.* , 2006)[3].

Furthermore, epistemic logic can also be interpreted in topological models (Benthem & Sarenac, 2004). In that paper, the behavior of epistemic modality and the common knowledge operator under topological products have been shown. The general investigation of modal logics for the product topologies has been carried out in the paper (Benthem *et al.* , 2006). For a more detailed survey of the topological and the geometric (such as between modality) semantics of modal logics, the interested reader is advised to refer to (Benthem & Bezhanishvili, 2006) and (Benthem *et al.* , 2006). We will give a brief overview of the above results with some rather technical remarks in the next section.

The topological interpretations of modal logics have been studied in AI in the field of temporal reasoning and its embedding into the spatial reasoning. One of the most inspiring work on this area was carried out by Kremer and Mints on the dynamic aspects of topological logic (Kremer & Mints, 2005). In this work, they considered the topological spaces with continuous functions. Their temporal modal operators were *next* and *henceforth* operators along side a topological interior modality.

[2]For each $A \subseteq S$ where S is the carrier set of a topological space, we recall that, $s \in d(A)$ if and only if $A \cap (U - \{s\}) \neq \emptyset$ for each open neighborhood U around s. The semantics of $\Diamond$ operator as derivative in the topological model $\langle S, \sigma \rangle$ is as follows.
$s \models \Diamond\varphi$ if and only if $\forall U \in \sigma (s \in U \rightarrow \exists t \in U - \{x\}$ such that $t \models \varphi)$

[3]We will define Alexandroff extensions in due time.

1.2.2 Recent Developments in the Topological Semantics of Modal Logic

In this section we will give an overview of the recent developments in the topological semantics of modal logic. First, we will spell out the topological version of the Goldblatt - Thomason Theorem without proof. Second, we will elaborate on the modal logic of topological products comparing it with the products in the Kripke semantics of modal logic. Lastly, we will briefly describe the dynamic topological logic.

Topological Goldblatt - Thomason Theorem Goldblatt - Thomason theorem of basic modal logic states precisely what kinds of first order definable frames are modally definable (Blackburn *et al.* , 2001). The theorem is stated as follows.

Theorem 1.2.1 (Goldblatt - Thomason Theorem)**.** *A first order definable class* ***K*** *of frames is modally definable if and only if it is closed under taking bounded morphic images, generated subframes, disjoint unions and reflects ultrafilter extensions.*

Proof. See (Blackburn *et al.* , 2001) for several proofs. ∎

Is it then possible to have a similar theorem for the topological interpretation of basic modal logic? Let us start with recalling some basic topological properties and their defining modal formulae. Extremally disconnected topological spaces and atomic topological spaces are definable in the basic language of topological interpretation of modal logic. However, the class of connected topological spaces and compact topological spaces are not definable as they are not closed under topological sums. For the technical details of the aforementioned results, we refer the reader to (Cate *et al.* , 2006).

Before stating the main theorem of this section, let us recall the definition of Alexandroff extensions. Given a topological space $\langle S, \sigma \rangle$, the Alexandroff extension of it is the topological space $\langle uf(S), \sigma^* \rangle$ where $uf(S)$ is the set of ultrafilters on S and σ^* is the topology generated by the collection of all sets of the form $\{U \in uf(S) : F \subseteq U\}$ where F is an open filter[4].

Now, in order to characterize the modally definable class of spaces in the basic modal language, a topological version of the Goldblatt - Thomason Characterization Theorem can be given as follows.

[4]A filter is an open filter if and only if for each $U \in F, \mathsf{I}(O) \in F$ as well. In this setting, we define $\mathsf{I}(O) = \bigcup_{U \in \sigma, U \subseteq O} U$

Theorem 1.2.2 (Topological Goldblatt - Thomason Theorem)**.** *The class* ***T*** *of topological spaces which is closed under formation of Alexandroff spaces is modally definable if and only if it is closed under taking open subspaces, interior images, topological sums and it reflects Alexandroff extensions.*

Proof. See (Cate *et al.* , 2006) for the proof. ■

Corollary 1.2.1. *The class of Alexandroff spaces is not modally definable.*

Proof. The class of Alexandroff extensions does not reflect Alexandroff extensions. Hence, the corollary follows. ■

Modal Logic of Products and Fusion of Topological Spaces In this section, we will review the basic topological constructions in modal logic. Our primary references in this topic are (Benthem & Sarenac, 2004) and (Benthem *et al.* , 2006). For simplicity, we will only consider the case for two agents.

Let $\mathbb{L}_1$ and $\mathbb{L}_2$ be two logics in the languages $\mathcal{L}_{\Box_1}$ and $\mathcal{L}_{\Box_2}$ respectively. The fusion $\mathbb{L}_1 \oplus \mathbb{L}_2$ is the smallest set of formulae in the language $\mathcal{L}_{\Box_1,\Box_2}$ containing the axioms for both $\Box_1$ and $\Box_2$ and closed under modus ponens, substitution, $\Box_1$-necessitation and $\Box_2$-necessitation.

Given two frames $\langle S, R_1\rangle$ and $\langle S, R_2\rangle$ from the logics $\mathbb{L}_1$ and $\mathbb{L}_2$ respectively, then the $\mathbb{L}_1 \oplus \mathbb{L}_2$ frames are triples $\langle S, R_1, R_2\rangle$.

In the topological semantics, given two topological space $\langle S, \sigma\rangle$ and $\langle S, \sigma'\rangle$, their fusion is the triple $\langle S, \sigma, \sigma'\rangle$. In other words, when we fuse topological spaces, we consider the given set with two different topologies on it; each corresponds to different agents.

On the other hand, we can also consider the products of two given topological spaces. Given two topological spaces $\langle S, \sigma\rangle$ and $\langle T, \tau\rangle$, the standard product topology π on $S \times T$ is defined by allowing the sets $U \times V$ to form a basis for π where U and V are opens in σ and τ respectively. We can now define two additional one dimensional topologies on $S \times T$ by lifting the topologies σ and τ. For a given set $U \subseteq S \times T$, we say U is horizontally open if for any $(s,t) \in U$, there exists an open $O \in \sigma$ such that $s \in O$ and $O \times \{t\} \in U$. In a similar manner, for a given set $V \subseteq S \times T$, we say V is vertically open if for any $(s,t) \in V$, there exists an open $Q \in \tau$ such that $t \in Q$ and $Q \times \{s\} \in V$. The complements of horizontally [vertically] open sets are called horizontally [vertically] closed sets. Then the collection of horizontally opens π_1 and the collection of vertically opens π_2 of $S \times T$ are topologies on $S \times T$. The basis for $\langle S \times T, \pi_1\rangle$ is $\{U \times \{t\} : U \in \sigma, t \in T\}$. Similarly, the basis for $\langle S \times T, \pi_2\rangle$ is $\{\{s\} \times V : s \in S, V \in \tau\}$.

We interpret the modalities $\Box_1$ and $\Box_2$ of $\mathcal{L}_{\Box_1,\Box_2}$ in $\langle S \times T, \pi_1, \pi_2 \rangle$ as follows.

$$\begin{array}{lll} (s,t) \models \Box_1\varphi & \text{iff} & \exists U \in \pi_1 \text{ s.t. } s \in U \text{ and } \forall s' \in U,\ (s',t) \models \varphi \\ (s,t) \models \Box_2\varphi & \text{iff} & \exists V \in \pi_2 \text{ s.t. } t \in V \text{ and } \forall t' \in V,\ (s,t') \models \varphi \end{array}$$

Duals of $\Box_1$ and $\Box_2$ are $\Diamond_1$ and $\Diamond_2$ respectively, and they are defined as follows.

$$\begin{array}{lll} (s,t) \models \Diamond_1\varphi & \text{iff} & \forall U \in \pi_1,\ s \in U \text{ implies } \exists s' \in U,\ (s',t) \models \varphi \\ (s,t) \models \Diamond_2\varphi & \text{iff} & \forall V \in \pi_2,\ t \in V \text{ implies } \exists t' \in V,\ (s,t') \models \varphi \end{array}$$

It is not difficult to see that a formula in the language $\mathcal{L}_{\Box_1}$ is valid in $\langle S \times T, \pi_1, \pi_2 \rangle$ if and only if it is valid in $\langle S, \sigma \rangle$ (Benthem *et al.* , 2006).

Apart from horizontal and vertical modalities, we can also define a product modality K interpreted directly on the product topology $\langle S \times T, \pi \rangle$. The semantics goes as follows.

$$\begin{array}{lll} (s,t) \models \Box\varphi & \text{iff} & \exists U \in \sigma \text{ with } s \in U \text{ and } \exists V \in \tau \text{ with } t \in V \text{ implies} \\ & & \forall (s',t') \in U \times V, \text{ we have } (s',t') \models \varphi \end{array}$$

The dual $\Diamond$ is defined as $\Diamond p \equiv \neg\Box\neg p$.

However, we will see in due time that although the standard product modality is definable in Kripke semantics (i.e. $\Box p \equiv \Box_1\Box_2 p$), it is not definable in topological semantics.

Kripke Semantics vs Topological Semantics As we already mentioned, the product modality $\Box$ is not definable in terms of $\Box_1$ and $\Box_2$ in topological semantics. The usual method for nondefinability results is to construct two bisimular models where $\Box p$ holds in one model, but not in the other. Therefore, we now need to introduce the definition of bisimulation in the product topologies. For the simplicity, we will consider the case for two topologies (Aiello *et al.* , 2003).

Definition 1.2.3 (Product Topo-Bisimulations). *Let $\mathcal{S} = \langle S, \sigma_1, \sigma_2, v \rangle$ and $\mathcal{S}' = \langle S', \sigma_1', \sigma_2', v' \rangle$ be topological models with equipped with two topologies each. A 2-topo-bisimulation is a nonempty relation $\rightleftarrows \subseteq S \times S'$ such that if $s \rightleftarrows s'$, then we have the following:*

1. *BASE CONDITION*

   $s \in v(p)$ *if and only if* $s \in v'(p)$ *for any propositional variable* p.

2. *FORTH CONDITION*

(a) *$s \in U \in \sigma_1$ implies that there exists $U' \in \sigma_1'$ such that $s' \in U'$ and for all $t' \in U'$ there exists $t \in U$ with $t \rightleftarrows t'$*

(b) *$s \in V \in \sigma_1$ implies that there exists $V' \in \sigma_2'$ such that $s' \in V'$ and for all $t' \in V'$ there exists $t \in V$ with $t \rightleftarrows t'$*

3. *BACK CONDITION*

(a) *$s' \in U' \in \sigma_1'$ implies that there exists $U \in \sigma_1$ such that $s \in U$ and for all $t \in U$ there exists $t' \in U'$ with $t \rightleftarrows t'$*

(b) *$s' \in V' \in \sigma_2'$ implies that there exists $V \in \sigma_2$ such that $s \in V$ and for all $t \in V$ there exists $t' \in V'$ with $t \rightleftarrows t'$*

In order to see that $\Box$ is not definable in the language $\mathcal{L}_{\Box_1,\Box_2}$ in terms of $\Box_1$ and $\Box_2$, we will consider two topological models on $\mathbb{Q} \times \mathbb{Q}$ (Benthem *et al.* , 2006). Let $v(p) = \{(\frac{1}{n}, \frac{1}{n}) : n \in \mathbb{N}\}$, and $v'(p) = \emptyset$. Observe that the closure of $v(p)$ is $(0,0)$ whereas clearly, the closure of $\emptyset$ is the empty set. Let us now define a product topo-bisimulation $\rightleftarrows$ on $\mathbb{Q} \times \mathbb{Q} - \{(\frac{1}{n}, \frac{1}{n}) : n \in \mathbb{N}\}$ as the identity relation. It is easy to verify that $\rightleftarrows$ is a bisimulation connecting the point $(0,0)$ to $(0,0)$. The base condition holds clearly as we already excluded the points in which p holds in the product space $\mathbb{Q} \times \mathbb{Q} - \{(\frac{1}{n}, \frac{1}{n}) : n \in \mathbb{N}\}$. As $\rightleftarrows$ is the identity relation, back and forth conditions are satisfied trivially. However, observe that $(0,0) \in v(\Diamond p)$ but $(0,0) \notin v'(\Diamond p)$. Therefore, $\langle \mathbb{Q} \times \mathbb{Q}, v\rangle, (0,0) \models \Diamond p$ but, $\langle \mathbb{Q} \times \mathbb{Q}, v'\rangle, (0,0) \not\models \Diamond p$. Hence, $\Box$ is not definable. Because if it were definable, then it should have been preserved under the bisimulation. The intuitive meaning of the above observation is that, $\Box_1$ and $\Box_2$ cannot cover the cases for diagonal geometric constructions as $\Box_1$ spans only horizontally whereas $\Box_2$ spans only vertically.

Moreover, it is a very well known fact that the product logics in Kripke semantics validate the axioms COM and CHR where COM is the commutativity principle ($\Box_1\Box_2 p \equiv \Box_2\Box_1 p$) and CHR is the Church - Rosser property ($\Diamond_1\Box_2 p \to \Box_2\Diamond_1 p$) (Gabbay *et al.* , 2003). However, the topological products refute both COM and CHR (Benthem *et al.* , 2006). The counterexample given is the product $\mathbb{R} \times \mathbb{R}$.

In order to refute COM, let the valuation of the propositional variable p be as follows (See the Figure 1.2.2 taken from (Benthem *et al.* , 2006), left hand side).

$v(p) = (\bigcup_{x \in (-1,0)} \{x\} \times (x, -x)) \cup (\{0\} \times (-1,1)) \cup (\bigcup_{x \in (0,1)} \{x\} \times (-x, x))$.

Then, it is easy to see that $(0,0) \models \Box_1\Box_2 p$ but, $(0,0) \not\models \Box_2\Box_1 p$. Because, observe that when you first move along the horizontal axis, there is always an open set which is entirely contained in the vertical axis. On the contrary,

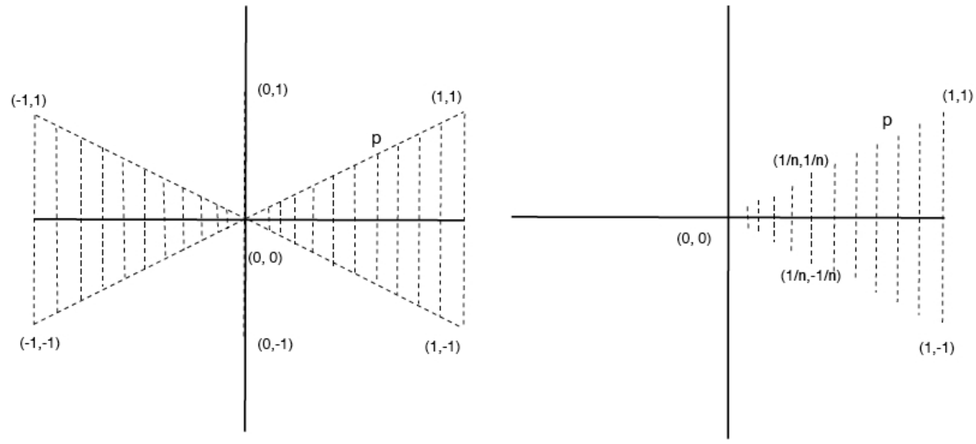


Figure 1.1: Counterexamples for COM and CHR

when you first move along the vertical axis starting from $(0, 0)$, there is no horizontal open contained in the set. Hence, COM fails.

In order to refute CHR, let the valuation of the propositional variable p as follows (See the above Figure, right hand side). $v(p) = \bigcup\{\{1/n\} \times (-1/n, 1/n) : n \in \mathbb{N}\}$.

Then, it is easy to see that $(0, 0) \models \Diamond_1\Box_2 p$ but, $(0, 0) \not\models \Box_2\Diamond_1 p$. Similar reasoning to the above also applies to this case as well.

However, although COM and CHR fail in topological products in general, there is a certain class of topological spaces in which COM and CHR are valid.

Theorem 1.2.3. *For the topological space $\mathcal{S}$, $\mathcal{S}$ is Alexandroff if and only if $\mathcal{S} \times \mathcal{S} \models \mathrm{COM}$.*

Proof. See (Benthem *et al.* , 2006). ■

Theorem 1.2.4. *For the topological spaces $\mathcal{S}$ and $\mathcal{T}$, if either $\mathcal{S}$ or $\mathcal{T}$ is Alexandroff, then $\mathcal{S} \times \mathcal{T} \models \mathrm{CHR}$.*

Proof. See (Benthem *et al.* , 2006). ■

Theorem 1.2.5. *For the topological space $\mathcal{S}$, if $\mathcal{S}$ is not Alexandroff, then $\mathcal{S} \times \mathcal{S} \not\models \mathrm{CHR}$.*

Proof. See (Benthem *et al.* , 2006). ■

In conclusion, the topological interpretation of modal logic diverges from the Kripke interpretation of modal logic. We remarked the several differences for product logics such as the invalidity of COM and CHR in topological products. The crucial reason for that is the fact that the concept of product in topological spaces is more involved that that of in Kripke frames. Therefore, we need additional restrictions for the topological spaces (like being Alexandroff) in order to make them satisfy COM and CHR.

Dynamic Topological Logic As Kremer and Mints put in their paper, dynamic topological logic "provides a context for studying the confluence of three research areas: the topological semantics of S4, topological dynamics, and temporal logic" (Kremer & Mints, 2005).

Dynamic topological logic is a trimodal logic with topological interior modality $\Box$, and two temporal modalities next $\bigcirc$ and henceforth $*$. The formula $\bigcirc\varphi$ is true at the moment m if and only if the formula φ is true in the next moment $m+1$. The formula $*\varphi$ is true at the moment m if and only if the formula φ is true at the each moment n for each $n \geq m$. Observe that, $*\varphi$ is equivalent to the infinite conjunction $\varphi \wedge \bigcirc\varphi \wedge \bigcirc^2\varphi \wedge \bigcirc^3\varphi \wedge \dots$. Then a dynamic topological model $\langle S, f, v\rangle$ is an ordered triple where S is a topological space, f is a continuous function and, v is a valuation function which assigns to each propositional variable a subset of S. The function f is used to interpret the temporal modalities $\bigcirc$ and $*$ in the following way. In order to use the functions, we will interpret the propositions as the subsets of S. Then, trivially, we have, for instance, $\Box\varphi = \mathsf{I}(\varphi)$ for the proposition φ. Now, assume that, at the moment m, the formula φ is true at the point fs. As φ is considered as a subset of S, then we observe that $fs \in \varphi$ as φ is true at the point fs. $fs \in \varphi$ means that after f acted on s once, then φ will be true at s. Hence $s \in \bigcirc\varphi$. Thus, it is not difficult to see that $s \in \bigcirc\varphi$ if and only if $fs \in \varphi$. We leave it to the reader to convince herself that the other direction is also valid. In conclusion, with some simple mathematical manipulations, we see that the *next* modality is interpreted as $\bigcirc\varphi = f^{-1}\varphi$. As we already pointed out, $*\varphi$ is equal to the infinite conjunction $\varphi \wedge \bigcirc\varphi \wedge \bigcirc^2\varphi \wedge \bigcirc^3\varphi \wedge \dots$. Hence, the interpretation of $*\varphi$ is given as follows. $*\varphi = \bigcap_{n\geq 0} f^{-n}\varphi$.

Also note that it is not known whether or not the logic of all dynamic topological systems $\langle S, f\rangle$ is axiomatizable. We refer the reader to the aforementioned paper for the axiomatization of the several fragments of dynamic topological logics (Kremer & Mints, 2005).

On the other hand, it was shown that dynamic topological logic over arbitrary topological spaces as well as those over $\mathbb{R}^n$ for each $n \geq 1$, are undecidable (Konev *et al.* , 2006). For the rather sophisticated and mathematically involved approach to the aforementioned problem, we again refer the reader to the cited paper.

1.3 Subset Space Logic

In the early 90s, Moss and Parikh presented a bimodal logic called *subset space logic* to formalize the reasoning about sets and points. One of the modal operators was intended to quantify *over* the sets ($\Box$) whereas the other modal operator was intended to quantify *in* the sets (K). The sets in Moss and Parikh's structure are called *observation* or *measurement* sets. The underlying motivation for the introduction of these two modalities is to be able to speak about the notion of *closeness*. The key idea of Moss and Parikh's approach to the closeness can be formulazied as follows:

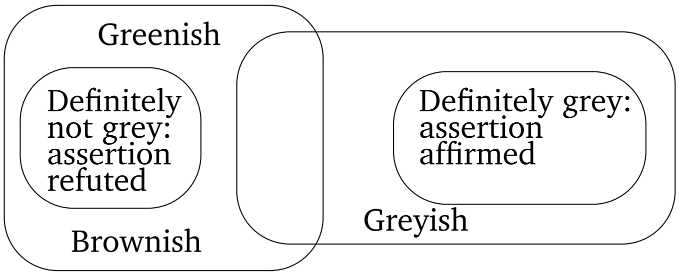
> In order to *get close*, one needs to spend some *effort*.

The first investigation of the logical meaning of "effort" goes back to late 80s. Vickers, in his 1989 book *Topology via Logic* introduced the logic of affirmative and refutative statements (Vickers, 1989). In this logic, the notion of effort had a significant role. In order to illustrate the role of effort in Vickers' logic let us consider the following example he posed.

> "My baby has green eyes."
>
> The obvious question is, "Is this true or false?".
>
> (...)
>
> First, we may agree that her eyes really are green - we can *affirm* the assertion.
>
> Second, we may agree that her eyes are some other colour, such as brown - we can *refute* the assertion.
>
> Third, we may fail to agree; but perhaps if we hire a powerful enough colour analyser, that may decide us.
>
> Fourth, the baby may be away at her grandparents' today, so that we just have to wait.
>
> Fifth ... the diligent reader will think of many more.
>
> (Vickers, 1989, p. 5)

What is crucial in Vickers's analysis is that statements are affirmable or refutable in a *finite* amount of time with spending *finite* amount of effort.

Consequently, Vickers suggests to draw a Venn diagram. In this diagram, each point will correspond to an actual color. However, as all borderline cases arguable, affirmations will be opens[5]. To illustrate the above discussions we can come up with the following Venn diagram (Vickers, 1989).



Therefore, as it was also observed in the above example, to gain *knowledge*, we need to spend some *effort*. In due course, we will also establish the formal connection between effort and knowledge. Let us now recall Vickers' definitions of affirmative and refutative statements. We call an assertion *affirmative*, if and only if it is true precisely in the circumstances when it can be affirmed. Likewise, an assertion is *refutative* if and only if it is false precisely in the circumstances when it can be refuted.

Based on the above observations and remarks, Moss and Parikh set out to formalize the intuitions about knowledge and effort. They also noted that they are interested in the project of "accounting for mathematical practice in weak logical systems whose primitives are appropriately chosen (...) for the easy parts of topological reasoning" (Dabrowski *et al.* , 1996).

On the other hand, the introduction of the notion of effort to the context of epistemic logic brings along a paradigm shift in epistemic logic. The traditional understanding of knowledge has been defined "in terms of agent(s)'s *view*" (Dabrowski *et al.* , 1996). In Kripke models, the knowledge is defined from the actual state within the scope of accessible states. The agent is said to know a formula φ, if φ holds at the each accessible state. In Hintikka's terms, an agent i is said to know φ if and only if in all worlds compatible with what i knows, it is the case that φ (Hintikka, 1962). Therefore, we also observe that each ascription of knowledge divides the set of possible worlds into two: compatible (i.e. accessible) and

[5]Consequently refutations will be closed sets. However, one can very well take affirmations statements as closed and refutations as opens.

incompatible (i.e. inaccessible) states. Therefore, the knowledge is strictly depending on agent's view (i.e. the states that an agent considers possible) at the *actual* state. However, in the bimodal language developed by Moss and Parikh, the knowledge was defined with respect to both a *point* and a *neighborhood* of this point. This is where the effort enters to the game. They note the above considerations and discussions as follows.

> (...) [N]otion of *effort* enters in topology. Thus if we are at some point at s and make a measurement, we will then discover that we are in some neighborhood U of s, but not know where. If we make my measurement finer, then U will shrink, say, to a smaller neighborhood V.
>
> (Dabrowski *et al.* , 1996)
>
> [The original notation of the quotation was edited in order to make it compatible with our notation. - C.B.]

Therefore, by spending some effort, we eliminate some of the possibilities, and finally obtain a smaller set of possibilities. The smaller the set of observation is, the larger the information we have. However, note that, there is no given formal way for *making the observations finer*. We will elaborate more on our last remark later.

1.4 In This Thesis

In this thesis, we will first provide a comprehensive outlook of subset space logic in detail in order to set the basis for our future discussions of the subject.

Then, we will import some simple truth preserving operations which are familiar from (basic) modal logic and provide their definitions in our new language. Furthermore, we will observe that these operations are valid in subset space logic as well. As expected, validity preserving operations will enable us to point out definable and non-definable properties in the language of subset space logic.

Furthermore, we will discuss several important extensions of subset space logic. These extensions will be indispensably significant in our future discussion. In addition to that, we will follow the tradition and present a game theoretical semantics for subset space logic. We will therefore introduce evaluation games and bisimulation games. Moreover, we will even present bisimulation games and evaluation games for the extended languages. This can be seen as a continuation of topological games.

Equipped with all these tools, we will observe that the subset space logic is strong enough to axiomatize the dynamic aspects of knowledge change, in particular, the public announcement logic. We will then provide the full axiomatization of subset space public announcement logic and its then straightforward completeness proof. As long as the research area of "geometry of knowledge" is considered, we believe, it is significant to see that public announcement logic works well in the subset space language.

All these discussions will lead us to take a closer look at the notion of *shrinking* - which can be considered as the temporal and perhaps the dynamic operator of the subset space logic. We will observe that, in fact, the shrinking operator is not a remote concept in formal sciences. We will motivate our point with several examples chosen from the broader research areas in various branches of logic such as methodology and philosophy of mathematics, belief revision etc. These considerations yet will not able us to formalize the improved concept of shrinking. However, we will suggest one approach to analyze the conceptual framework for the shrinking operator - which is unfortunately far from being complete and precise. However, we believe, this initiation of discussions on the shrinking operator will emphasize the significance of the aforementioned operator.

After that as a third point, we will consider the multi-agent version of subset space logic. However, it will turn out that it is not as nice as it is expected to be. We will suggest several methods to formalize the concept. Thereon we will import some basic results from modal logic and observe that they are valid in the subset space logic as well.

Last, but not least, we will recall the concept of common knowledge, and point out the definition of common knowledge in the language of basic and extended subset space logic. As another contribution, we will consider the extensions of public announcement logic with an additional operator together with a general notion of common knowledge (called *relativized common knowledge*). We will then easily prove the completeness of public announcement logic extended with these aforementioned operators in the extended language of subset space logic by reducing it to already known completeness results.

Finally, we will conclude with some open problems and future work ideas that might bring some light to the shaded areas in the subset space logic - the logic which we believe has the necessary tools *per se* to analyze many conceptual frameworks in logic.

Chapter 2

Basic Results of Subset Space Logic

2.1 Subset Space Models

We will consider two different approaches to formalize the subset space logics. The first one is the subset space models whereas the second one is a bi-modal Kripke model.

The language of subset space logic $\mathcal{L}_S$ has a countable set P of proposition letters, a truth constant $\top$, the usual Boolean operators $\neg$ and $\wedge$, and two modal operators K and $\Box$. The formulae of $\mathcal{L}_S$ are obtained from atomic propositions by closing under $\neg$, $\wedge$, K and $\Box$.

A *Subset frame* is a pair $\mathcal{S} = \langle S, \sigma \rangle$ where S is a set of points and σ is a set of subsets of S. However, note that σ is not necessarily a topology. The elements of σ are called *observations*.

Definition 2.1.1 (Subset Space Models)**.** *A subset space model is a triple $\mathcal{S} = \langle S, \sigma, v \rangle$ where $\langle S, \sigma \rangle$ is a subset frame, $v : P \to \wp(S)$ is a valuation function for the countable set of propositional variables P.*

We can now define the semantics of subset spaces.

Definition 2.1.2 (Semantics of Subset Spaces)**.** *For $s \in S$ and $s \in U \in \sigma$, we define the satisfaction relation $\models_{\mathcal{S}}$ on $(S \times \sigma) \times \mathcal{L}$ by induction on the length of the formulae. We drop the subscript $_{\mathcal{S}}$ when the space we are in is obvious.*

$$\begin{array}{llll} s, U \models p & \textit{if and only if} & s \in v(p) & \\ s, U \models \varphi \wedge \psi & \textit{if and only if} & s, U \models \varphi & \textit{and } s, U \models \psi \\ s, U \models \neg\varphi & \textit{if and only if} & s, U \not\models \varphi & \\ s, U \models \mathsf{K}\varphi & \textit{if and only if} & t, U \models \varphi & \textit{for all } t \in U \\ s, U \models \Box\varphi & \textit{if and only if} & s, V \models \varphi & \textit{for all } V \in \sigma \\ & & & \textit{such that } s \in V \subseteq U \end{array}$$

We call $\Box$ *shrinking* operator and K *knowledge* operator. The duals of $\Box$ and K are $\Diamond$ and L respectively and defined as follows $\mathsf{L}\varphi \equiv \neg\mathsf{K}\neg\varphi$ and $\Box\varphi \equiv \neg\Diamond\neg\varphi$. Consequently, their semantics are defined as follows.

$$\begin{array}{llll} s, U \models \mathsf{L}\varphi & \text{if and only if} & t, U \models \varphi & \text{for some } t \in U \\ s, U \models \Diamond\varphi & \text{if and only if} & s, V \models \varphi & \text{for some } V \in \sigma, \\ & & & \text{such that } s \in V \subseteq U. \end{array}$$

(s, U) is called a *neighborhood situation* if U is a neighborhood of s, i.e. if $s \in U \in \sigma$. If at (s, U) we *know* φ, this then means that we can move from the given reference point s to any other point t in the given neighborhood situation (s, U). Likewise, by the shrinking modality, we shrink the neighborhood around the given point. However, neither by the knowledge nor by the shrinking modality we can leave the initial neighborhood U. Thus,Moss & Parikh concluded that "the intuition behind this logic with its two modalities is that knowledge is affected not only by the *situation* we are in, but also by the amount of *effort* we have put in" (Dabrowski *et al.* , 1996).

It is also worthwhile to remark that, subset space logic has two disjoint sets of formulae with regards to the extensions of the formulae. Considering the semantics of the subset spaces, it is not difficult to see that the truth of propositional variables and Boolean formulae do not depend on the neighborhood situations but rather depend on the points. Therefore, the extensions of these formulae are simply the points of the space. However, the extensions of the modal formulae are the neighborhood situations.

Let $\mathcal{L}_0$ be the propositional language generated by the set of propositional letters P. Then, for the subset space frame $\mathcal{S} = \langle S, \sigma \rangle$, if $\varphi \in \mathcal{L}_0$, then we have $(\varphi)^{\mathcal{S}} \subseteq S$ whereas if $\varphi \in \mathcal{L}_S - \mathcal{L}_0$, we then have $(\varphi)^{\mathcal{S}} \subseteq S \times \sigma$.

2.2 Axioms

Let us now give the axioms for subset space logic. The axioms simply reflects the fact that the K modality is S5-like whereas the $\Box$ modality is S4-like. Moreover, we need additional axioms to state the interaction between those two modalities.

The basic axioms of subset space logic are given as follows.

1. All the substitutional instances of the tautologies of the classical propositional logic
2. $(A \to \Box A) \wedge (\neg A \to \Box\neg A)$ for atomic sentence A
3. $\mathsf{K}(\varphi \to \psi) \to (\mathsf{K}\varphi \to \mathsf{K}\psi)$
4. $\mathsf{K}\varphi \to (\varphi \wedge \mathsf{KK}\varphi)$
5. $\mathsf{L}\varphi \to \mathsf{KL}\varphi$
6. $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$
7. $\Box\varphi \to (\varphi \wedge \Box\Box\varphi)$
8. $\mathsf{K}\Box\varphi \to \Box\mathsf{K}\varphi$

The rules of inferences of subset space logic are given as follows.

Modus ponens $\varphi \to \psi, \varphi \therefore \psi$

K**-Necessitation** $\varphi \therefore \mathsf{K}\varphi$

$\Box$**-Necessitation** $\varphi \therefore \Box\varphi$

The last axiom is called *the cross axiom* and establishes the connection between shrinking and knowledge modalities. Let us see that the cross axiom in fact holds. Let us consider its dual $\Diamond\mathsf{L}\varphi \to \mathsf{L}\Diamond\varphi$. Assume that $s, U \models \Diamond\mathsf{L}\varphi$. So, there is a subset $V \subseteq U$ such that $s, V \models \mathsf{L}\varphi$. Then, again by definition there exists some t in V such that $t, V \models \varphi$. As $V \subseteq U$, we derive that $t, U \models \Diamond\varphi$. Recall that $t \in V$, hence $t \in U$. Thus, $s, U \models \mathsf{L}\Diamond\varphi$.

The axiom $\mathsf{L}\varphi \to \mathsf{KL}\varphi$ is also worth considering. Consider its contrapositive $\mathsf{LK}\varphi \to \mathsf{K}\varphi$. Assume that $s, U \models \mathsf{LK}\varphi$. Then, for some $t \in U$, we have $t, U \models \mathsf{K}\varphi$. So, for each $u \in U$ we conclude $u, U \models \varphi$. However, this means $s, U \models \mathsf{K}\varphi$.

The axiom $(A \to \Box A) \wedge (\neg A \to \Box\neg A)$ for an atomic sentence A is called the axiom of *atomic permanence*. Intuitively, it says that, the truth of an atomic sentence A at a point s is independent from its neighborhood. In other words, whichever neighborhood of s we consider, the truth of atomic sentences remains intact.

Then, the soundness of the axioms can easily be shown by an induction on the length of derivations.

Theorem 2.2.1 (Soundness of Subset Space Axioms)**.** *The axioms of subset space are sound.*

Proof. Straightforward. ∎

2.3 Cross Axiom Frames vs. Subset Space Frames

The subset space logic can be interpreted both in the cross axiom frames and the subset frames although they are rather different structures. In this section, we will focus on the differences between cross axiom frames and subset space frames.

Definition 2.3.1 (Cross Axiom Models)**.** *A cross axiom frame is a tuple $\mathcal{J} = \langle J, \xrightarrow{\mathsf{L}}, \xrightarrow{\Diamond} \rangle$, such that J is a non-empty set, $\xrightarrow{\mathsf{L}}$ is an equivalence relation on J and $\xrightarrow{\Diamond}$ is a preorder on J where the following property holds: If $i \xrightarrow{\Diamond} j \xrightarrow{\mathsf{L}} k$, then there is some l such that $i \xrightarrow{\mathsf{L}} l \xrightarrow{\Diamond} k$.*

A cross axiom model is a cross axiom frame together with an interpretation I of the atomic propositions of the language of subset spaces. I must satisfy the condition that if $i \xrightarrow{\Diamond} j$, then $i \in I(A)$ iff $j \in I(A)$.

Several observations about cross axiom models can be noted here. First, observe that cross axiom models only consider the points and completely ignore the observation sets. Second, L and $\Diamond$ modalities have an interaction in this model; namely the cross axiom $\Diamond\mathsf{L}\varphi \to \mathsf{L}\Diamond\varphi$ is valid in cross axiom models.

Therefore, it should now be clear that we can interpret the subset space logic both in subset space frames and cross axiom frames.

Now, several questions can be raised about the relation between these two frames. The first consideration is whether or not we can obtain a subset space frame from a cross axiom frame. Likewise, the second consideration of ours is whether or not we can obtain a subset space from a cross axiom frame.

Recall that in the Section 1.2, we presented the methods for obtaining a topological frame out of a Kripke frame and vice versa. This method was rather intuitive and straight forward. Can we then have a similar method for cross axiom frames and subset spaces? Unfortunately, the answer to this question is partially negative.

We can get a cross axiom frame from a subset space frame. We will observe in due time that the canonical model of subset spaces is a cross axiom frame. This is the straight forward method for obtaining the canonical models which is utilized frequently, for instance in (Blackburn *et al.* , 2001). Yet, still, the canonical model of the subset space logic is a cross axiom frame but not a subset space.

However, a more general and intuitive method also exists to obtain a cross axiom model out of a subset space model (Dabrowski *et al.* , 1996).

Lemma 2.3.1. *Every subset space has a corresponding cross axiom model.*

Proof. Given $\langle S, \sigma \rangle$ consider $J = \{(s, U) \in S \times \sigma : s \in U\}$. Clearly, this is a subset space frame. The relations of the cross axiom frame is then defined as follows. $(s, U) \xrightarrow{\mathsf{K}} (t, V)$ if and only if $U = T$ and $(s, U) \xrightarrow{\Diamond} (s, V)$ if and only if $V \subseteq U$. The valuation v_J of propositional variables then is defined as $v_J(p) = \{(s, U) : s \in v(p) \cap U\}$.

It is easy to check that this construction gives a cross axiom model which is equivalent to the subset space we started with. ■

In the above proof, observe that the states in the cross axiom model are neighborhood situations.

Can we then come back? In other words, given a cross axiom model, can we construct the corresponding subset space?

This is not a trivial question and we do not know the answer. As the cross axiom frames dismiss the neighborhood component of neighborhood situations, the critical point is to construct a neighborhood for each point in question. However, it is not trivial and there is no known way to achieve this.

2.4 Completeness and Decidability Results

2.4.1 Completeness

Theorem 2.4.1 (Completeness). *The basic axioms are strongly complete for subset space models.*

Proof. We will now sketch the completeness proof of subset space logic. First of all, note that the proof of completeness is not entirely similar to the standard modal completeness proofs[1]. The reason for that is the fact that at the level of maximally consistent theories, there is no known way to define a corresponding subset space structure.

Therefore, to obtain the subsets in the collection of maximally consistent sets, an auxiliary pre-order and an antitone mapping (from the carrier set of the pre-order to the collection of non-empty subsets of the carrier set of the subset space) were used to obtain the collection of subsets.

Let us now see step by step how the proof goes. In the proof, we will use the theories which are the maximal consistent subsets of the language $\mathcal{L}_S$. Th will then denote the set of theories in $\mathcal{L}_S$.

In order to see that we have a complete proof system, we need to show that for each theory T in Th, there exists a subset space model $\mathcal{S} = \langle S, \sigma, v \rangle$, a point s in S and a subset U with $s \in U$ such that $s, U \models_{\mathcal{S}} T$.

[1]For many "standard" completeness proof, see (Blackburn *et al.* , 2001)

Now we will construct a cross axiom model for theories in Th following Definition 2.3.1. We set $U \xrightarrow{\mathsf{L}} V$ if and only if whenever we have $\varphi \in V$, then we have $\mathsf{L}\varphi \in U$. Similarly, $U \xrightarrow{\Diamond} V$ if and only if whenever we have $\varphi \in V$ then we have $\Diamond\varphi \in U$ as well. Several observations might be of necessity regarding the relations $\xrightarrow{\mathsf{L}}$ and $\xrightarrow{\Diamond}$. We then have the following.

Lemma 2.4.1.

1. $\xrightarrow{\mathsf{L}}$ *is an equivalence relation.*

2. $\xrightarrow{\Diamond}$ *is reflexive and transitive.*

3. *If* $\mathsf{L}\varphi \in T$*, there is some* U *so that* $\varphi \in U$ *and* $T \xrightarrow{\mathsf{L}} U$

4. *If* $\Diamond\varphi \in T$*, there is some* U *so that* $\varphi \in U$ *and* $T \xrightarrow{\Diamond} U$

Proof. First two propositions are the results of the S4-ness of $\Diamond$ and S5-ness of L. The last two propositions are so called *Existence Lemmas*. For the not-so-exciting details of this proof, the interested reader is referred to (Blackburn *et al.* , 2001, p. 198). ∎

Now, it is not difficult to see that $\langle \mathsf{Th}, \xrightarrow{\mathsf{L}}, \xrightarrow{\Diamond} \rangle$ is a cross axiom frame. We only need to see that the cross axiom is valid in $\langle \mathsf{Th}, \xrightarrow{\mathsf{L}}, \xrightarrow{\Diamond} \rangle$.

Proposition 2.4.1. *Let* U *and* V *be in* Th *and suppose further that there is a theory* $W \in \mathsf{Th}$ *such that* $U \xrightarrow{\Diamond} W \xrightarrow{\mathsf{L}} V$*. Then, there is a theory* $T \in \mathsf{Th}$ *such that* $U \xrightarrow{\mathsf{L}} T \xrightarrow{\Diamond} V$*.*

Proof. We need to construct such T. Let us start with considering the following set.

$$S = \{\Diamond\varphi : \varphi \in V\} \cup \{\psi : \mathsf{K}\psi \in U\}$$

Let us call $S_1 = \{\Diamond\varphi : \varphi \in V\}$ and $S_2 = \{\psi : \mathsf{K}\psi \in U\}$. We claim S is consistent. In order to see that, assume not. Therefore, there is a finite subset of S which is inconsistent. Now, observe that, S_1 and S_2 are closed under conjunction. In order to see that S_1 is closed under conjunction take $\Diamond\varphi_1$ and $\Diamond\varphi_2$ in S_1. Then by the definition of S_1 we have φ_1 and φ_2 in V. As V is maximally consistent $\varphi_1 \wedge \varphi_2 \in V$. Then, by the definition of S_1, we have $\Diamond(\varphi_1 \wedge \varphi_2) \in S_1$. Hence, $\Diamond\varphi_1 \wedge \Diamond\varphi_2 \in S_1$. In a similar fashion, take ψ_1 and ψ_2 from S_2. Then, we will have $\mathsf{K}\psi_1 \in U$ and $\mathsf{K}\psi_2 \in U$. As K commutes with $\wedge$ we have $\mathsf{K}(\psi_1 \wedge \psi_2)$ in U. Hence $\psi_1 \wedge \psi_1 \in S_2$. Therefore, there are $\Diamond\varphi \in S_1$ and $\psi \in S_2$ such that $\vdash \Diamond\varphi \to \neg\psi$ where $\varphi \in V$ and $\mathsf{K}\psi \in U$. Then, $\vdash \mathsf{L}\Diamond\varphi \to \mathsf{L}\neg\psi$ by Lemma 2.4.1. Thus, the sentence $\mathsf{L}\Diamond\varphi \to \mathsf{L}\neg\psi$ belongs to U. Recall that we have $\varphi \in V$. Then, as $U \xrightarrow{\Diamond} W \xrightarrow{\mathsf{L}} V$ is assumed, we further have $\mathsf{L}\varphi \in W$ and finally $\Diamond\mathsf{L}\varphi \in U$. By the Cross Axiom for subset spaces, we conclude $\mathsf{L}\Diamond\varphi \in U$. Together with $\vdash \mathsf{L}\Diamond\varphi \to \mathsf{L}\neg\psi$, we conclude by modus ponens that $\mathsf{L}\neg\psi$, that is $\neg\mathsf{K}\psi \in U$. However, we already had $\mathsf{K}\psi$ in U and U is assumed to be a maximally consistent set. The contradiction shows that S is consistent. By the Lindenbaum Lemma [(Blackburn *et al.* , 2001, Lemma 4.17, p. 197)], let $S \subseteq T$ be the maximally consistent extension of S. Then by construction, we conclude $U \xrightarrow{\mathsf{L}} T \xrightarrow{\Diamond} V$. ■

Note that, the canonical cross axiom frame $\langle \mathsf{Th}, \xrightarrow{\mathsf{L}}, \xrightarrow{\Diamond} \rangle$ is not a subset frame. The interested reader is advised to the counterexample given in (Moss *et al.* , 2007, Example B).

We will now build the space S of "abstract" points. We will also obtain the subsets by using a antitone map f defined from a poset P to the nonempty subsets of S, denoted by $\wp(S)^*$. For each s in S and for each p in P with $t \in f(p)$, we will have a "target" theory $t(s,p)$. The main aim of the construction is to arrange $th(s, f(p)) = t(s,p)$.

Let us now see how one can obtain the subset space out of the above constructions. Take a set S with a designated element s_0 and a poset with least element $\langle P, \leq, \perp \rangle$ and finally a homomorphism from $\langle P, \leq, \perp \rangle$ to $\langle \wp(S)^*, \supseteq, S \rangle$, that is for p and q in P, we have $p \leq q$ iff $f(p) \supseteq f(q)$ and $f(\perp) = S$. Lastly, we define a partial function $t : S \times P \to \mathsf{Th}$ such that $t(s,p)$is defined when $s \in f(p)$. The following properties will also be required for each p in P, s in $f(p)$ and φ.

- If $u \in f(p)$ then $t(s,p) \xrightarrow{\mathcal{L}} t(u,p)$
- If $\mathsf{L}\varphi \in t(s,p)$ then for some $u \in f(p)$ we have $\varphi \in t(u,p)$
- If $q \leq p$ then $t(s,p) \xrightarrow{\Diamond} t(s,q)$
- If $\Diamond\varphi \in t(s,p)$ then for some $q \leq p$, we have $\varphi \in t(s,q)$
- $t(s_0, \perp) = T$ where T is the theory from above which we want to model.

For S, P, f and t constructed with these properties, we will then have the following subset space: $\mathcal{S} = \langle S, \{f(p) : p \in P\}, f \rangle$, where $f(P) = \{s : A \in t(x,p)$ for propositional letter $A\}$.

Then the truth lemma is given as follows: $th_{\mathcal{S}}(s, f(p)) = t(s,p)$. The proof of the truth lemma goes by induction on the length of the formulae φ.

Then by the truth lemma and the last bulleted property above, we see $th(s_0, \perp) = T$. So, the theory we started with has a model.

This concludes the proof. We refer the reader to (Dabrowski *et al.* , 1996) for technical details. ■

2.4.2 Decidability

After observing that the subset spaces are complete, one can wonder it they are decidable. First of all, note that the subset spaces do not satisfy the finite model property. Let us consider the counterexample from (Moss *et al.* , 2007).

Lemma 2.4.2 (Failure of Finite Model Property). *Subset space logic does not have the finite model property.*

Proof. Let $\psi \equiv \Box(\Diamond\varphi \wedge \Diamond\neg\varphi)$. Assume, $s, U \models \psi$. Assume further that, we are in a finite space (with (s, U)), and U is a $\subseteq$-minimal open set about s. We can make this assumption since we are in a finite space. However, as U is the minimal neighborhood around s, we then obtain $s, U \models \varphi \wedge \neg\varphi$. Contradiction shows that finite model property does not hold in subset spaces. ■

Although the finite model property fails in subset space, it turns out that the subset space is decidable. To achieve this, we will use Cross Axiom Models (see Definition 2.3.1). The proof we will present is due to Krommes (Krommes, 2003).

Remark that, as we emphasized earlier, now we are in a bimodal Kripke structure. We construct the Kripke model in such a way that the states in the Kripke model will be neighborhood situations, that is, $J = \{(s, U) \in S \times \sigma : s \in U\}$. In a similar fashion, the relations $\xrightarrow{\mathsf{L}}$ and $\xrightarrow{\Diamond}$ and the valuation are defined as before.

Then, are the cross axiom model and subset space model equivalent?

Lemma 2.4.3. *For cross axiom model $\mathcal{J}$ of subset space model $\mathcal{S}$, we have $s, U \models_{\mathcal{J}} \varphi$ if and only if $s, U \models_{\mathcal{S}} \varphi$.*

Proof. The proof is by induction on the length of φ. The case for propositional variables and Boolean formulae are easy. Let us consider the case $\varphi \equiv \mathsf{L}\psi$. Assume $s, U \models_{\mathcal{S}} \mathsf{L}\psi$. Then for some $t \in U$, we have $t, U \models_{\mathcal{S}} \psi$. Then by definition $(s, U) \xrightarrow{\mathsf{L}} (t, U)$. By induction hypothesis $t, U \models_{\mathcal{J}} \psi$. As (t, U) is $\xrightarrow{\mathsf{L}}$-accessible from (s, U) in $\mathcal{J}$, we conclude $s, U \models_{\mathcal{J}} \mathsf{L}\psi$. Converse is also similar.

To complete the induction, assume that $\varphi = \Diamond\psi$. Let $s, U \models_{\mathcal{S}} \Diamond\psi$. Then, for some $V \subseteq U$ we have $s, V \models_{\mathcal{S}} \psi$. Then by definition, $(s, U) \xrightarrow{\Diamond} (s, V)$. By induction hypothesis, $s, V \models_{\mathcal{J}} \psi$. As (s, V) is $\xrightarrow{\Diamond}$-accessible from (s, U) in $\mathcal{J}$, we conclude that $s, U \models_{\mathcal{J}} \Diamond\psi$. Converse direction is also similar.

Hence the induction is complete. ■

Corollary 2.4.1 (Completeness with respect to Cross Axiom Models). *Subset space logic is complete for interpretations in cross axiom models.*

Proof. Follows directly from the previous lemma. ■

Instead of worrying about the fact that subset spaces do not enjoy the finite model property, we will construct a finite cross axiom model for any formula in the language of subset space logic.

We first start with the canonical model $\mathcal{C} = \langle \mathsf{Th}, \xrightarrow{\mathsf{L}}, \xrightarrow{\Diamond} \rangle$ for subset space logic obtained in a usual way. Then, for a fixed formula φ we need to find a quotient $\mathcal{C}'$ of $\mathcal{C}$ such that $\mathcal{C}'$ is a cross axiom model which is also a model of φ. To achieve this we use a filtration to get a finite quotient structure $\mathcal{C}'$ from $\mathcal{C}$. For fixed φ, define the following sets:

$\Sigma^{\neg}$ = all subformulas of φ and their negations.

Σ' = all conjunctions and disjunctions of finite subsets of $\Sigma^{\neg}$.

Σ'' = all conjunctions and disjunctions of finite subsets of Σ'.

$\Sigma^{\mathsf{KL}} = \{\mathsf{L}\psi : \psi \in \Sigma''\} \cup \{\mathsf{K}\psi : \psi \in \Sigma''\}$

$\Sigma = \Sigma'' \cup \Sigma^{\mathsf{KL}}$

Note that all sets are finite and subformula closed and solely depend on φ. Now, we need to define the equivalence class of theories in $\mathcal{C}$. For any theory T in $\mathcal{T}$, we define the equivalence class $[T]$ of T as follows: $[T] = \{S \in \mathcal{T} : T \cap \Sigma = S \cap \Sigma\}$. Then the minimal filtration $\xrightarrow{[\mathsf{L}]}$ of $\xrightarrow{\mathsf{L}}$ is defined as

$$[S] \xrightarrow{[\mathsf{L}]} [T] \text{ iff } \exists S' \in [S], \exists T' \in [T] \text{ such that } S' \xrightarrow{\mathsf{L}} T'$$

Minimal filtration $\xrightarrow{[\Diamond]}$ of $\xrightarrow{\Diamond}$ is defined in a similar fashion. Hence, it can be shown that $\mathcal{C}'$ is a filtration of $\mathcal{C}$. After dealing with some technical details, it can be observed that $\mathcal{C}'$ is a cross axiom model.

Then the decidability result follows.

Theorem 2.4.2 (Decidability). *The subset space logic is decidable.*

2.5 Additional Properties

As we already emphasized, subset spaces need *not* be topological spaces. However, they might very well enjoy several additional properties. In this section, we will focus on directed frames, intersection frames and lattice frames and provide the formulae in the language $\mathcal{L}_S$ of subset space logic which characterize them. Let us first give the definitions of the aforementioned frames.

Definition 2.5.1 (Directed Space). *A subset frame $\mathcal{S}$ is called a directed frame if for every $s \in S$ and $U, V \in \sigma$ with $s \in U$ and $s \in V$, there exists a $W \in \sigma$ such that $s \in W$ and $W \subseteq U \cap V$.*

Definition 2.5.2 (Intersection frame). *A subset frame $\mathcal{S}$ is called an intersection frame if whenever $U, V \in \sigma$ and $U \cap V \neq \emptyset$, then $U \cap V \in \sigma$ as well.*

Definition 2.5.3 (Lattice frame). *A subset frame $\mathcal{S}$ is called a lattice frame if it is an intersection frame which is also closed under finite unions; and is called a complete lattice frame if it is closed under arbitrary intersections and unions.*

Let us now have a closer look at the properties we just defined above.

Weak Directedness Axiom We first observe that the following *Weak Directedness* (WDA) axiom is sound in directed frames.

WDA: $\Diamond\Box\varphi \to \Box\Diamond\varphi$.

To see that assume $s, U \models \Diamond\Box\varphi$. So, there exists $V \subseteq U$ such that $s, V \models \Box\varphi$. Now, take an arbitrary subset U' of U with $s \in U'$ and show $s, U' \models \Diamond\varphi$. As both U' and V are in σ and we are in a directed space, there exists $W \in \sigma$ such that $W \subseteq U' \cap V$ with $s \in W$. As $W \subseteq V$, we conclude $s, W \models \varphi$. Thus, $s, U' \models \Diamond\varphi$. Recall that U' was an arbitrary subset of U. Hence, $s, U \models \Box\Diamond\varphi$.

Note that, although WDA is sound for directed spaces, it is not complete for intersection spaces (Dabrowski *et al.* , 1996, Example C). The reason for that is the fact that there are models which validate the WDA but are not closed under intersection. Any subset model with finite observation sets which are not closed under intersection will work. Why is that? Because in a subset frame with finitely many observation sets, we can shrink each observation set U around a point s to a minimal observation set V about the same point s. Therefore, $s, V \models \varphi \to \Box\varphi$ for each φ. Hence, in this case, we have $s, U \models \Diamond\Box\varphi$ since there exists the minimal neighborhood $V \subseteq U$. Now, take an arbitrary subset $U' \subseteq U$ of U. Then, V will be a subset of U' as well. Thus, $s, U' \models \Diamond\varphi$. Recall that U' was an arbitrary subset of U. So, we conclude $s, U \models \Box\Diamond\varphi$. Hence, the WDA follows.

Union Axiom Similarly, we can also consider the frames which are closed under finite unions. The corresponding axiom scheme is called *Union Axiom* (UA).

UA: $\Diamond\varphi \wedge \mathsf{L}\Diamond\psi \rightarrow \Diamond(\Diamond\varphi \wedge \mathsf{L}\Diamond\psi \wedge \mathsf{K}\Diamond\mathsf{L}(\varphi \vee \psi))$

Let us now see the soundness of UA. Take arbitrary U and V from σ with $s \in U$ and $t \in V$. Such U and V exist due to the assumption. Suppose now $s, U \models \varphi$ and $t, V \models \psi$. Suppose, for some $X \in \sigma$ we have $s, X \models \Diamond\varphi \wedge \mathsf{L}\Diamond\psi$. Let $W = U \cup V$. As the space we are in is closed under union $W \in \sigma$ and furthermore $W \subseteq X$. Then, we have, $s, W \models \Diamond\varphi \wedge \mathsf{L}\Diamond\psi$. Moreover, every point in W has a neighborhood in which φ or ψ hold somewhere in this neighborhood, as each point in W either is in U or V. Hence, $s, X \models \Diamond(\Diamond\varphi \wedge \mathsf{L}\Diamond\psi \wedge \mathsf{K}\Diamond\mathsf{L}(\varphi \vee \psi))$.

We will call the system whose axioms are the basic axioms we presented in Section 2.2 together with the WD axiom and the UA **topologic**.

WDA	$\Diamond\Box\varphi \rightarrow \Box\Diamond\varphi$
	sound for weakly directed spaces
UA	$\Diamond\varphi \wedge \mathsf{L}\Diamond\psi \rightarrow \Diamond(\Diamond\varphi \wedge \mathsf{L}\Diamond\psi \wedge \mathsf{K}\Diamond\mathsf{L}(\varphi \vee \psi))$
	sound for subset spaces closed under binary unions
WUA	$\mathsf{L}\Diamond\varphi \wedge \mathsf{L}\Diamond\psi \rightarrow \mathsf{L}\Diamond(\mathsf{L}\Diamond\varphi \wedge \mathsf{L}\Diamond\psi \wedge \mathsf{K}\Diamond\mathsf{L}(\varphi \vee \psi))$
	weaker than UA
CI	$\Box\Diamond\varphi \rightarrow \Diamond\Box\varphi$
	sound for subset spaces closed under all intersections
M_n	$(\Box\mathsf{L}\Diamond\varphi \wedge \Diamond\mathsf{K}\psi_1 \wedge \cdots \wedge \psi_n)$
	$\rightarrow \mathsf{L}(\Diamond\varphi \wedge \Diamond\mathsf{K}\psi_1 \wedge \cdots \wedge \Diamond\mathsf{K}\psi_n)$
	WD and all M_n are complete for directed spaces

Table 2.1: Additional properties in subset spaces and their respective defining formulae taken from (Moss *et al.* , 2007).

Weak Union Axiom Consider the following axiom schema which is called *Weak Union Axiom* as it is the weaker form of the UA.

WUA: $\mathsf{L}\Diamond\varphi \wedge \mathsf{L}\Diamond\psi \to \mathsf{L}\Diamond(\mathsf{L}\Diamond\varphi \wedge \mathsf{L}\Diamond\psi \wedge \mathsf{K}\Diamond\mathsf{L}(\varphi \vee \psi))$

In order to observe the weakness of WUA the reader is referred to (Dabrowski *et al.* , 1996, Example C). In that example the authors consider a case in which they have a space which is not closed under intersection. Then they observe that WDA and WUA hold in that space although UA fails.

Closed Under Intersection Consider the axiom scheme CI.

CI: $\Box\Diamond\varphi \to \Diamond\Box\varphi$

We will now observe that CI is sound for subset spaces which are closed under arbitrary intersections. Given a subset space $\langle S, \sigma\rangle$, let s be a point in it. Assume $s, U \models \Box\Diamond\varphi$. Let U_s be the intersection of all opens containing s. Hence, $s, U_s \models \Box\varphi \vee \Box\neg\varphi$. Due to the fact that in the case of arbitrary intersections the neighborhoods of s stabilize either on φ or $\neg\varphi$, we conclude $s, U_s \models \Box\varphi$. Then, as $U_s \subseteq U$ we conclude $s, U \models \Diamond\Box\varphi$.

On the other hand, CI is *not* sound for finite intersection spaces. For a counterexample, refer to (Dabrowski *et al.* , 1996, Example B). As expected, in that example, authors consider a subset space which is closed under arbitrary intersections but not closed for finite intersections. For the technical details, see the aforementioned example.

More on Directedness Consider the axiom scheme M_n

M_n: $(\Box\mathsf{L}\Diamond\varphi \wedge \Diamond\mathsf{K}\psi_1 \wedge \cdots \wedge \mathsf{K}\psi_n) \to \mathsf{L}(\Diamond\varphi \wedge \Diamond\mathsf{K}\psi_1 \wedge \cdots \wedge \Diamond\mathsf{K}\psi_n)$

It has been shown in (Weiss & Parikh, 2002) that the subset space axioms together with WDA and the axioms M_n are complete for directed spaces. For the detailed treatment of the axiom schema M_n, we refer the reader to the aforementioned article.

2.5.1 Concluding Remarks

We will call the system whose axioms are the basic axioms given in Section 2.2 together with WDA and UA *topologic*. The underlying idea behind topologic is to create a system which is "strong enough to support elementary topological reasoning" (Moss *et al.* , 2007).

The axioms of topologic hold in all topological spaces, but there is more. Georgatos showed the following (Georgatos, 1997).

Theorem 2.5.1. *Topologic axioms are complete for topological spaces, and moreover for complete lattice spaces. Furthermore, any sentence satisfiable in any topological space is also satisfiable in a finite topological space.*

The proof uses some sophisticated mathematical techniques (such as splittings) and hence is omitted here.

2.6 Extending the Subset Spaces

In this section, we will briefly mention some further works in subset space logic which have been carried out by Georgatos and Heinemann. These works extended the conceptual framework of subset space logic either by introducing some restrictions to the subset spaces or by extending the language $\mathcal{L}_S$ of subset space logic. In this respect, Georgatos considered some special class of subset spaces. However, Heinemann contributed to subset space logics by introducing additional operators. Among others, Heinemann introduced the nominals, and the temporal and the topological operators to the subset space logics. Here, we will consider the *nextstep* and the *seperation* operators together with the *overlap* operator. Lastly, we will briefly mention the *hybridization* of the subset space logic.

Georgatos's Treelike Structures Georgatos's CUNY PhD. dissertation and his related papers (Georgatos, 1994) and (Georgatos, 1997) consider subset spaces in a treelike space setting, that is, for each pair of subsets U, V in σ, either $U \subseteq V$ or $V \subseteq U$ or $U \cap V \neq \emptyset$. He further showed that the topologic axioms are complete for topological spaces and indeed for complete lattice spaces. Moreover, any sentence satisfiable in a topological space is also satisfiable in a finite topological space. Furthermore, he also showed that the theory of treelike spaces is decidable as it has the finite model property.

His constructions utilized sophisticated mathematical tools such as "stable splittings" and hence we will not go into the details.

Heinemann's Separation Modality In (Heinemann, 1999), Heinemann introduced nextstep $\bigcirc$ and separation S operators to the basic framework of subset space logic and gave the completeness and decidability proofs of his extended topologic logic. Let us briefly mention the semantics of this logic. Heinemann starts with defining subset frames. For a nonempty set S let $d = (E_j)_{j \in \mathbb{N}}$ be a sequence of equivalence relations on S such that every class of E_j is the union of some classes of E_{j+1}, for all natural numbers j.

Then we call the pair $\mathcal{F} = \langle S, d \rangle$ a subset tree frame. Also note that the subset tree models are treelike in the sense of Georgatos' utilization.

U_j^s will denote the equivalence class of s with respect to the relation E_j. Then the neighborhood situations will be of the form (s, U_j^s). The semantics of the $\bigcirc$ and the S operators are given as follows.

$$\begin{array}{lll} s, U_j^s \models \bigcirc\varphi & \text{iff} & s, U_{j+1}^s \models \varphi \\ s, U_j^s \models \mathsf{S}\varphi & \text{iff} & \text{for each } t \in U_j^s, \text{ if } U_{j+1}^t \cap U_{j+1}^s = \emptyset, \text{ then } t, U_{j+1}^t \models \varphi \end{array}$$

Observe that the $\bigcirc$ operator is self dual.

After giving the semantics, Heinemann proceeds into the axiomatization and suggests a sound axiom system. Then, as we pointed out earlier, he proceeds to prove the completeness and the decidability.

The basic idea of the completeness proof is to use a Henkin-like construction. Because there is no known method to obtain the separation property for sets on the canonical model.

Therefore, in order to achieve this, one needs to define mappings to obtain subsets which comply with the requirements for the separation and next time operators. This approach is familiar from the completeness proof of the basic language of subset space logic. As we remarked already, we will not go into the technical details and refer the reader to the aforementioned paper.

Heinemann's Overlap Modality Recall that the semantics of subset space logic does not let us leave the particular neighborhood occupied. We can either move to a point in the neighborhood or else shrink the neighborhood keeping the reference point intact. In (Heinemann, 2006b), Heinemann introduced the overlap operator O with the intention of leaving the current neighborhood and moving to another neighborhood of the same point. As he pointed out, the overlap operator was designed to enable us to quantify "not only downwards, but also diagonally" among the set of observations (Heinemann, 2006b). Therefore, overlap operators will also include shrinking cases (but not the other way around, obviously). Hence, we can say that shrinking is a special case of overlapping.

The semantics of the overlap operator O is given as follows:

$$s, U \models \mathsf{O}\varphi \quad \text{iff} \quad \forall U' \in \sigma : (s \in U' \rightarrow s, U' \models \varphi)$$

The dual of O is denoted by P, and defined in the usual way. As we already pointed out, $\Box$ is a special case of O. This observation is captured by the following statement: $\models \mathsf{O}\varphi \rightarrow \Box\varphi$.

The complete axiomatization of the topologic with the additional overlap operator was given by Heinemann in (Heinemann, 2006b). The additional axioms for the overlap operator are as follows:

1. $\mathsf{O}(\varphi \to \psi) \to (\mathsf{O}\varphi \to \mathsf{O}\psi)$
2. $(A \to \mathsf{O}A) \wedge (\mathsf{P}A \to A)$ for atomic A
3. $\mathsf{O}\varphi \to \mathsf{O}\mathsf{O}\varphi$
4. $\varphi \to \mathsf{O}\mathsf{P}\varphi$
5. $\mathsf{O}\varphi \to \Box\varphi$

The proof theory of subset space logic extended with the overlap operator uses Modus Ponens, K-necessitation, $\Box$-necessitation and O-necessitation in its proof theory. Moreover, together with the axioms of topologic, they completely axiomatize the topologic with overlap operator. The completeness and decidability proofs of the extended subset space logic can be found in (Heinemann, 2006b).

The important observation that Heinemann notes is that the overlap operator *dismisses* the set component of the neighborhood situation. A similar observation was made in (Dabrowski *et al.* , 1996) as well. They defined bi-persistent sentences as the sentences whose satisfaction in a neighborhood situation (s, U) depends only on the point s. So, a sentence φ is bi-persistence if $\vdash \Diamond\varphi \to \Box\varphi$. It is then easy to observe that each sentence $\mathsf{O}\varphi$ is bi-persistence. We leave this simple manipulation to the reader.

Although the overlap operator is bi-persistent, as Heinemann remarked, it is not definable in the language of topologic. The simple reason is that the overlap operator enables us to *enlarge* the neighborhood we are in. However, the basic language of subset space logic cannot express the expansion of the neighborhood. To illustrate this fact, consider the two very simple subset spaces $\mathcal{S} = \langle \{s\}, \{s\}, v \rangle$ and $\mathcal{S}' = \langle \{s', t'\}, \{\{s'\}, \{s', t'\}\}, v' \rangle$. Assume we have only two propositional variables in the language: p and q. Let $v(p) = \{s\}, v(q) = \emptyset$ and $v'(p) = \{s'\}, v'(q) = \{t'\}$. In order to establish the nondefinability result here, we obviously need the concept of bisimulation. As we have not defined and not discussed bisimulations yet, we leave it to the reader to establish this result after we introduce the bisimulations. As usual, in nondefinability the proof, we will find a formula (which is $\mathsf{P}q$, for example) which does not hold in bisimular neighborhood situations.

Completeness of the extension of subset space logic with overlap operator is rather similar to that of basic subset space logic. One only needs to introduce the necessary properties which comply with the overlap operator to the completeness proof. We refer the reader to the aforementioned article for rather straight forward construction (Heinemann, 2006b).

Hybridization Without introducing additional modalities, we can also extend the language $\mathcal{L}_S$ of subset space logic by adding *names* (Heinemann, 2003a) and (Heinemann, 2003b).

There are two approaches to the hybridization of subset space logic. We can either treat the logic as having two distinct set of "components" which are points and subsets; or we can treat the logic as having one set of "component" which is the neighborhood situations.

For the first approach, we will add two disjoint sets of names to the language: names for points and names for sets (Heinemann, 2003b). We will denote the set of names for points by $N_P = \{i, j, \dots\}$ and the set of names for sets by $N_S = \{A, B, \dots\}$.

Then the semantics of the hybrids is as expected and is given as follows.

$$\begin{array}{lll} s, U \models i & \text{iff} & s = v(i) \\ s, U \models A & \text{iff} & U = v(A) \end{array}$$

For obvious reasons, introduction of nominals increases the expressivity of the logic. For instance, *linear*, *seperated* and *connected* frames can be defined in the extended language. See the table to check the defining formulae and its corresponding property. The details of the mathematical construction of these correspondence theorems can be found in (Heinemann, 2003b).

Linear	$\mathsf{K}\Box(i \to \mathsf{L}j) \vee \mathsf{K}\Box(j \to \mathsf{L}i)$ for each $i, j \in N_P$
Seperated	$i \wedge \mathsf{L}(j \wedge \neg i) \to \Diamond\mathsf{K}\neg j \wedge \mathsf{K}(j \to \Diamond\mathsf{K}\neg i)$ for each $i, j \in N_P$
Connected	$\mathsf{K}\Diamond(A \vee B) \to \mathsf{L}(\Diamond A \wedge \Diamond B)$

Table 2.2: Hybrid defining formulae for several properties

Heinemann further presents the sound and complete axiomatization of the above hybrid version of subset space logic. We refer the interested reader to the aforementioned article.

The second approach to hybridize the subset space logic is carried out by introducing names for neighborhood situations and the hybrid operator

$@_N$ (Heinemann, 2003a). Therefore, $N_{NS} = \{N, M, \dots\}$ will be the set of the names designated for neighborhood situations. The intended semantics is then as follows.

$$s, U \models @_N\varphi \quad \text{iff} \quad v(N) \models \varphi$$

Heinemann further presents the sound and complete axiomatization of the second hybrid version of subset space logic. The important point, this time, in the completeness proof is to take care of the valuation of names in the canonical model. In order to achieve this, Heinemann shows that every maximal consistent set can be extended to a *named* maximal consistent set[2]. We refer the interested reader to the aforementioned article.

2.6.1 Concluding Remarks

Subset space logic has an intrinsic temporal flavor. The temporal nature is visible when one looks closer to the function of the shrinking operator. As Heinemann constructed, each shrinking cases can be treated as discrete time units and thus this justifies the temporal logical approach to the subset space logic. On the other hand, hybridization enables us to name the sets and points. Therefore, the hybrid language makes it easier to define additional properties.

However, they all come with a price. For obvious reasons, the extended languages make it slightly more difficult to show the completeness of the respective logics which requires additional mappings in the canonical models.

[2]A maximal consistent set is named if and only if it contains some $i \in N_P$ and some $A \in N_S$

Chapter 3

Expressivity of Subset Space Logic

The language of the subset space logic is strong enough to express some basic topological properties. In this section, we will recall some formulae in the language of subset space logic and their corresponding topological properties. Moreover, we will also consider the expressive power of subset space logic by comparing it to the topological semantics of basic modal logic.

In the second section of this chapter, we will observe that validity preserving operations of the basic modal language also work in subset space logic with slight modifications. Consequently, we will consider the notion of bisimulation and its game theoretical semantics.

3.1 Definability Results

The motivation behind the subset space logic and hence behind the topologic is that it must be strong enough to "support elementary topological reasoning" (Dabrowski *et al.* , 1996). Therefore, one can expect that the language of subset space logic is powerful enough to express some basic topological notions.

Let us here now note some simple definability results in the language of subset space logic. In order to talk about topological properties, throughout this section we will assume that we are in a topological space - not only a subset space.

Recall that a set $v(p)$ is open if each point in this set has a neighborhood which is entirely contained in $v(p)$ for propositional letter p. We observe that $v(p)$ is open if and only if at every s in $v(p)$, the sentence $\Diamond \mathsf{K}p$ holds.

Let us now see why.

Lemma 3.1.1. *$v(p)$ is open if and only if at every s in $v(p)$, the sentence $\Diamond \mathsf{K}p$ holds for propositional letter p.*

Proof. Assume $v(p)$ is open. Take an arbitrary point s from $v(p)$. Then, it has a neighborhood, say U, which is entirely in $v(p)$. Hence, we can shrink it and then move in the smaller neighborhood freely; and we will still be in $v(p)$; that is p will still hold. Hence, $\Diamond \mathsf{K}p$ will hold.

For the converse direction, assume that an arbitrary s from $v(p)$ satisfies $\Diamond \mathsf{K}p$. Then, s has a neighborhood, and wherever we move in this neighborhood we are still in the set of states that satisfy p which is $v(p)$. Hence, $v(p)$ is open. ■

In a similar manner, we say that a formula φ is open if $v(\varphi)$ is an open set.

Proposition 3.1.1. *φ is open if and only if $\varphi \to \Diamond \mathsf{K}\varphi$ is valid.*

Proof. Straight forward ■

Proposition 3.1.2. *Dually, φ is closed if and only if $\Box \mathsf{L}\varphi \to \varphi$.*

Proof. Similar to above. ■

Proposition 3.1.3. *$v(p)$ is dense if and only if $\Box \mathsf{L}p$ holds.*

Proof. Assume $v(p)$ is dense. Then any neighborhood of any point s in $v(p)$ contains at least one point from $v(p)$. So, for all neighborhoods, there exists a point in them in which p holds, i.e. $\Box \mathsf{L}p$ holds.

Conversely, assume $\Box \mathsf{L}p$. Assume now that, for some point s in $v(p)$, s has a neighborhood in which no point satisfies p. So, for some neighborhood obtained by $\Diamond$, no points in this neighborhood satisfy p. So, $\mathsf{K}\neg p$ holds in this neighborhood and this neighborhood was picked by $\Diamond$. Hence $\Diamond \mathsf{K}\neg p$ is valid. But, the last statement is contradictory with $\Box \mathsf{L}p$. Hence, the contradiction shows that $v(p)$ is dense. ■

By similar motivations, we say φ is dense if $\mathsf{L}\varphi$ is valid and it is nowhere dense if $\Diamond \mathsf{L}\neg\varphi$ is valid.

However, several notions are undefinable in basic topologic language. For example, to define continuity, we need additional operators (Heinemann, 2006a).

Topological Interpretation of Modal Logic vs Subset Space Logic In this small section, we will simply note the expressive power of subset space logic over topological interpretation of modal logic. In order to achieve this, we will define a translation map $^{\mathbf{t}} : \mathcal{L} \to \mathcal{L}_S$ from the basic modal language $\mathcal{L}$ to the basic subset space language $\mathcal{L}_S$ as follows:

$p^{\mathbf{t}} = p$	for propositional letter p
$(\varphi \wedge \psi)^{\mathbf{t}} = \varphi^{\mathbf{t}} \wedge \psi^{\mathbf{t}}$	for the formulae φ, ψ in $\mathcal{L}$
$(\neg\varphi)^{\mathbf{t}} = \neg(\varphi^{\mathbf{t}})$	for the formulae φ in $\mathcal{L}$
$(\mathsf{I}(\varphi))^{\mathbf{t}} = \Diamond\mathsf{K}\varphi^{\mathbf{t}}$	for the formulae φ in $\mathcal{L}$

Observe that, in the translation map, Proposition 3.1.1 is used implicitly. The translation map $^{\mathbf{t}}$ shows that $\mathcal{L}$ is a fragment of the bimodal language $\mathcal{L}_S$. Furthermore, the lemma below shows that $\mathcal{L}_S$ is more expressive by showing that $^{\mathbf{t}}$ is not onto $\mathcal{L}_S$.

Proposition 3.1.4. *The sentence $\mathsf{L}p$ for the propositional letters p, is not equal to the translation of any formula φ from the basic modal language $\mathcal{L}$.*

Proof. Consider two models with $\mathbb{R}$ as their universe. In the first model $\mathcal{M}_1$ suppose p does not hold anywhere whereas in the second model $\mathcal{M}_2$, p holds only at the point π. It is not difficult to see that the interpretation of any basic modal formula φ are the same in $\mathcal{M}_1$ and $\mathcal{M}_2$, except possibly at the point π. In particular, for instance, for $0 \in \mathbb{R}$, we have $0 \models_{\mathcal{M}_1} \varphi$ if and only if $0 \models_{\mathcal{M}_2} \varphi$ for each $\varphi \in \mathcal{L}$. Now substitute, $\mathsf{L}p$ for φ. In $\mathcal{M}_1$ there is no point in which p holds, so $\mathsf{L}p$ cannot be true anywhere. However, in $\mathcal{M}_2$, p is true at the point π. Therefore, we have $0 \not\models_{\mathcal{M}_1} \mathsf{L}p$ and $0 \models_{\mathcal{M}_2} \mathsf{L}p$. Contradiction shows that, $\mathsf{L}p$ cannot be definable in the basic modal language $\mathcal{L}$. ■

Hence, the subset space language $\mathcal{L}_S$ is more expressive than basic modal language $\mathcal{L}$ as there are formulae in $\mathcal{L}_S$ which are not expressible in $\mathcal{L}$ under standard translation. Hence, $^{\mathbf{t}}$ is not onto $\mathcal{L}_S$.

3.2 Validity Preserving Operations

Recall that compact topological spaces and continuous topological spaces are not modally definable (Cate *et al.* , 2006). But are they definable in subset space logic? In order to be able to discuss the definability of compactness and connectedness, we first need to consider the validity preserving operations in subset space logics. We again note that, we will consider the topologic subset spaces in order to be able to talk about the notions of compactness and connectedness. Otherwise, these notions do not make sense at all in an non-topological subset space.

Some basic validity preserving operations can easily be defined in the context of subset space logic. Let us start with the simplest one.

Disjoint Unions Disjoint unions are perhaps the most intuitive way to obtain larger structures. We start by defining the disjoint unions for the general case as follows.

Definition 3.2.1 (Disjoint Unions). *Two subset space models are disjoint if their domain contains no common element. For disjoint subset space models $\mathcal{S}_i = \langle S_i, \sigma_i, v_i \rangle$, for $i \in I$ their disjoint union is the structure $\mathcal{S} = \biguplus_{i \in I} \mathcal{S}_i = \langle S, \sigma, v \rangle$ where $S = \bigcup_{i \in I} S_i$, $\sigma = \bigcup_{i \in I} \sigma_i$ and $v(p) = \bigcup_{i \in I} v_i(p)$.*

If the subset spaces we want to put together for disjoint unions have some common elements, the easiest way to overcome this problem is to index these members in order to make them distinct.

The details for the construction of disjoint unions for basic modal language with some easy examples can be found in (Blackburn *et al.* , 2001). The key proposition for disjoint unions is about the truth invariance of modal formulae under the construction of disjoint unions. The proof of the aforementioned proposition can also be found in (Blackburn *et al.* , 2001). Furthermore, we can easily import this result to the subset spaces.

Theorem 3.2.1 (Invariance Under Disjoint Unions for Subset Spaces). *For disjoint subset space models $\mathcal{S}_i$ for $i \in I$ and for each neighborhood situation (s, U) in $\mathcal{S}_i$, we have $s, U \models_{\mathcal{S}} \varphi$ if and only if $s, U \models_{\mathcal{S}_i} \varphi$, for each formula φ in the language of subset space logic $\mathcal{L}_S$.*

Proof. The proof is by induction on the length of the formulae. The case for propositional variables and Boolean cases are easy and hence skipped.

Let us consider the case for $\varphi \equiv \mathsf{K}\psi$. Assume $s, U \models_{\mathcal{S}_i} \mathsf{K}\psi$. Then for each $t \in U$, we have $t, U \models_{\mathcal{S}_i} \psi$. However, by induction hypothesis $t, U \models_{\mathcal{S}} \psi$. As $\sigma = \bigcup_{i \in I} \sigma_i$, (t, U) is a neighborhood situation in $\mathcal{S}$. So, $s, U \models_{\mathcal{S}} \mathsf{K}\psi$.

Converse direction is similar.

The case for $\varphi \equiv \Box\psi$ is also very similar. ∎

However, if we want to discuss whether some topological properties are definable in subset space logic, then we have to require that the underlying space should have topology, not just a collection of subsets.

Therefore we need to revise the Definition 3.2.1 in such a way that obtained disjoint union will be a topological space.

Definition 3.2.2 (Topologic Disjoint Unions)**.** *For disjoint topologic subset space models $\mathcal{S}_i = \langle S_i, \sigma_i, v_i \rangle$, for $, \in I$ their disjoint union is the structure $\mathcal{S} = \biguplus_{i \in I} \mathcal{S}_i = \langle S, \sigma, v \rangle$ where $S = \bigcup_{i \in I} S_i$, $\sigma = \{U \in S : \forall i \in I, U \cap S_i \in \sigma_i\}$ and $v(p) = \bigcup_{i \in I} v_i(p)$.*

It is then easy to check topological disjoint unions are truth preserving.

Then, borrowing the ideas from topological interpretation, it is easy to see the following.

Proposition 3.2.1. *Compactness and connectedness are not definable in topologic subset space logic.*

Proof. We refer the interested reader to (Cate *et al.* , 2006) as there is no need to reproduce the same proof here. ■

Generated Subset Spaces Disjoint unions enable us to obtain larger spaces from smaller ones. However, we can also do the reverse. In other words, we can throw away the points in such a way that the satisfiability relation will not be affected after this operation. But, what kind of points we can throw away? Recall that, in subset spaces, we interpret the formulae at the neighborhood situations (s, U) where $s \in U \in \sigma$. Therefore, we claim that we can throw away the points which are not in any of the observation sets in σ. The reason for that is the fact that we cannot evaluate the formulae at these points as there are no observation sets attached to them. Intuitively, they are the points about which we did not make any observation, hence there is no harm to throw them away.

Proposition 3.2.2. *Let $\mathcal{S} = \langle S, \sigma, v \rangle$ be a subset space model. Let $S' = S - \{s : s \notin \cup\sigma\}$ and $v'(p) = v(p) \cap S'$. Then $\mathcal{S}' = \langle S', \sigma, v' \rangle$ and $\mathcal{S} = \langle S, \sigma, v \rangle$ satisfy the same formulae.*

Proof. It is easy to show that $s, U \models_{\mathcal{S}'} \varphi$ implies $s, U \models_{\mathcal{S}} \varphi$ as $S' \subseteq S$.

For the converse direction, assume $s, U \models_{\mathcal{S}} \varphi$. As $s \in U \in \sigma$, we observe $s \in \cup\sigma$. Hence $s \in S'$. The proof goes by induction on the complexity of φ. The propositional and Boolean cases are easy and hence skipped. However, the idea for $\varphi \equiv \mathsf{K}$ and $\varphi \equiv \Box\psi$ is very similar. We will only show the case for $\varphi \equiv \mathsf{K}$, and leave the other case to the reader.

Assume, $s, U \models_{\mathcal{S}} \mathsf{K}\psi$. Then for each $t \in U$, we have $t, U \models_{\mathcal{S}} \psi$. As $t \in U \in \sigma$, we observe $t \in S'$ for each $t \in U$. Then by the induction hypothesis, $t, U \models_{\mathcal{S}'} \psi$. As $s \in S'$ as well, we conclude $s, U \models_{\mathcal{S}} \mathsf{K}\psi$. ■

We can go further. We can consider a situation in which we are only interested in the observations and knowledge around one specific neighborhood situation (s, U). We can then throw away the observation sets which have an empty intersection with the designated observation set U. We will call this construction *generated subset spaces*.

Definition 3.2.3 (Generated Subset Spaces). *Let $\mathcal{S} = \langle S, \sigma, v \rangle$ be a subset space model. Let (s, U) be the designated neighborhood situation. Then we obtain the generated subset space $\mathcal{S}' = \langle S', \sigma', v' \rangle$ of $\mathcal{S}$ as follows.*

- $\sigma' := \sigma - \{V \in \sigma : V \not\subseteq U\}$
- $S' := S - \cup\sigma'$
- $v'(p) := v(p) \cap S'$ *for each propositional letter p.*

Proposition 3.2.3 (Invariance Under Generated Subset Spaces). *For each $s \in S'$, we have $s, U \models_{\mathcal{S}} \varphi$ if and only if $s, U \models_{\mathcal{S}'} \varphi$.*

Proof. We leave the easy proof to the reader. ■

Bounded Morphism The third validity preserving operation we will import from the basic modal language is bounded morphism. However, in the context of subset spaces, we will diverge a bit from the familiar Kripkean notion of morphisms. We will here call a function *topologic-continuous function* if the inverse image of an observation set is again an observation set. Likewise we will call a function *topologic-open function* if the image of an observation set is still an observation set. Now, we will adopt the definition of bounded morphisms from (Cate *et al.* , 2006).

Definition 3.2.4 (Bounded Morphism for Subset Space Logic). *Let $\mathcal{S} = \langle S, \sigma, v \rangle$ and $\mathcal{S}' = \langle S', \sigma', v' \rangle$ be two subset spaces. Let f be an topologic-open and topologic-continuous map from $\mathcal{S}$ to $\mathcal{S}'$. Then $s, U \models_{\mathcal{S}} \varphi$ if and only if $s', U' \models_{\mathcal{S}'} \varphi$ for each formula φ. We define $v(p) := f^{-1}(v'(p))$.*

Again, some basic facts and results about the bounded morphisms for the basic modal language can be found in (Blackburn *et al.* , 2001). Some further observations about bounded morphism in the context of topological interpretation of modal logic can be found in (Cate *et al.* , 2006).

Henceforth, we can claim the following invariance result.

Theorem 3.2.2 (Invariance Under Bounded Morphism). *Let f be a bounded morphism from $\mathcal{S} = \langle S, \sigma, v \rangle$ onto $\mathcal{S}' = \langle S', \sigma', v' \rangle$. Then $s, U \models_{\mathcal{S}} \varphi$ if and only if $fs, fU \models_{\mathcal{S}'} \varphi$ for each formula φ in the language of subset space logic $\mathcal{L}_S$.*

Proof. The proof is by induction on the complexity of φ. The propositional case and Boolean cases are easy and hence skipped.

Let $\varphi \equiv \mathsf{K}\psi$. Assume $s, U \models_{\mathcal{S}} \mathsf{K}\psi$. Then for each $t \in U$, we have $t, U \models_{\mathcal{S}} \psi$. By induction hypothesis, we have $ft, fU \models_{\mathcal{S}'} \psi$. As f is open $fU \in \sigma'$. As $fs \in fU$, we conclude $fs, fU \models_{\mathcal{S}'} \mathsf{K}\psi$. The converse direction is very similar. We only need to replace f by f^{-1} and use the continuity of f.

Let $\varphi \equiv \Box\psi$. Assume $s, U \models_{\mathcal{S}} \Box\psi$. Then there exists V with $s \in V \subseteq U$ such that $s, V \models_{\mathcal{S}} \psi$. Then by induction hypothesis we have $fs, fV \models_{\mathcal{S}'} \psi$. As f is open fV is in σ' and since $V \subseteq U$, we then trivially have $fV \subseteq fU$. Hence, $fs, fU \models_{\mathcal{S}'} \Box\psi$. The converse direction is also very similar to that of the previous case.

Hence the induction is complete. ∎

3.3 Bisimulation

Bisimulation is an equivalence relation which was defined, in the most general setting, between state transition systems. As Kripke models can also be seen as labelled state transition systems, bisimulations can also be defined in the context of Kripke semantics.

In this part, we will simply adopt the ideas from basic modal logic to define bisimulations in topologic. In the context of subset space logic, the introduction of the notion of bisimulations does not provide anything new, but instead points out the similarities of subset space logics with labelled transition systems of Kripke models.

Definition 3.3.1 (Bisimulation). *Let $\mathcal{S} = \langle S, \sigma, u \rangle$ and $\mathcal{T} = \langle T, \tau, v \rangle$ be two subset space. A topologic bisimulation is a non-empty relation $\rightleftarrows$ for neighborhood situations in $(S \times \sigma) \times (T \times \tau)$ such that if $(s, U) \rightleftarrows (t, V)$, then we have:*

1. ***Base Condition***

   $s \in u(p)$ *if and only if* $t \in v(p)$ *for each propositional variable* p

2. ***Back Conditions***

   (a) $\forall t' \in V$ *there exists* $s' \in U$ *with* $(s', U) \rightleftarrows (t', V)$.

   (b) $\forall V' \subseteq V$ *such that* $t \in V'$, *there is* $U' \subseteq U$ *with* $s \in U'$ *such that* $(s, U') \rightleftarrows (t, V')$

3. ***Forth Conditions***

(a) $\forall s' \in U$ *there exists* $t' \in V$ *with* $(s', U) \rightleftarrows (t', V)$.

(b) $\forall U' \subseteq U$ *such that* $s \in U'$*, there is* $V' \subseteq V$ *with* $t \in V'$ *such that* $(s, U') \rightleftarrows (t, V')$.

The immediate result of this definition is the Bisimulation Invariance Theorem for Subset Spaces.

Theorem 3.3.1 (Bisimulation Invariance for Subset Spaces)**.** *If* $(s, U) \rightleftarrows (t, V)$ *then they satisfy the same formulae.*

Proof. The proof is by the induction on the length of formulae. The Boolean cases are obvious. Let us start with the $\Diamond$ operator. Assume $(s, U) \rightleftarrows (t, V)$ and further $s, U \models \Diamond\varphi$. So for some $U' \subseteq U$ with $s \in U'$, we have $s, U' \models \varphi$. Then, by the back condition (b), we have $V' \subseteq V$ with $(s, U') \rightleftarrows (t, V')$. By the induction step, we conclude $t, V' \models \varphi$. As $V' \subseteq V$, the last statement means $t, V \models \Diamond\varphi$. This concludes the one direction of the proof. The other direction is similar.

For $\mathsf{L}\varphi$ case, assume that $s, U \models \mathsf{L}\varphi$ and $(s, U) \rightleftarrows (t, V)$. Then by definition, there is some $s' \in U$ such that $s', U \models \varphi$. As (s, U) and (t, V) are bisimular, by back condition (a), there exists $t' \in V$ such that $(s', U) \rightleftarrows (t', V)$. By the induction step, we conclude $t', V \models \varphi$. Hence, as $t' \in V$ we see $t, V \models \mathsf{L}\varphi$. This concludes the one direction of the proof. The other direction is similar.

Hence, bisimular neighborhood situations satisfy the same formulae. ■

The very first difference between the topologic bisimulations and the topo -bisimulations (See Definition 1.2.3) is the fact that the topological bisimulation is defined for the neighborhood situations whereas the topo-bisimulations are defined for the states. Second, the topologic bisimulations are more general than the topo-bisimulations for obvious reasons.

However, the converse of this statement is not necessarily true. If some restrictions are applied to the topologic spaces, converse of the Theorem 3.3.1 can be obtained.

Before presenting the converse of the Theorem 3.3.1, let us introduce the following notation. $(s, U) \leftrightsquigarrow (t, V)$ will denote that (s, U) and (t, V) satisfy the same formulae.

Theorem 3.3.2. *Let* $\mathcal{S} = \langle S, \sigma, u\rangle$ *and* $\mathcal{T} = \langle T, \tau, v\rangle$ *be two finite subset space. Then for each neighborhood situations* (s, U) *in* $S \times \sigma$ *and* (t, V) *in* $T \times \tau$*; we have* $(s, U) \rightleftarrows (t, V)$ *if and only if* $(s, U) \leftrightsquigarrow (t, V)$.

Proof. Sufficiency condition has been proven in Theorem 3.3.1. For the necessity condition, the idea is to show that $\leftrightsquigarrow$ is itself a bisimulation. So, assume $(s, U) \leftrightsquigarrow (t, V)$. Then, the base condition of the bisimulation is satisfied immediately. For the forth condition, let us assume further that $t' \in V$. We will try to get a contradiction by assuming that there is no $s' \in U$ such that $(s', U) \leftrightsquigarrow (t', V)$. Now, for each t'_i for ($i \leq n$, for some n as V is finite) there exists some formula ψ_i such that $s', U \models \psi_i$ wheras $t'_i, V \not\models \psi_i$. Hence, $s, U \models \bigwedge_{i \leq n} \mathsf{L}\psi_i$ whereas $t, V \not\models \bigwedge_{i \leq n} \mathsf{L}\psi_i$. However, we assumed that $(s, U) \leftrightsquigarrow (t, V)$, so we get a contradiction. Then we see that $\leftrightsquigarrow$ satisfies the forth condition (a). The back condition (a) is very similar.

Next, let us assume again $(s, U) \leftrightsquigarrow (t, V)$. To get a contradiction, now we will assume that for $V' \subseteq V$ with $t \in V'$, there is no corresponding $U' \subseteq U$ with $s \in U'$ such that $(s, U') \leftrightsquigarrow (t, V')$. Then for each $V'_i \subseteq V$ with $t \in V'$ and $U' \subseteq U$ with $s \in U'$, there exist some formula ψ_i (for $i \leq n$ for some finite n) such that $s, U' \models \psi_i$ whereas $t, V'_i \not\models \psi_i$. But, as V is finite, then so is the number of V's. Then in a similar fashion, we see $s, U \models \bigwedge_{i \leq n} \Diamond\psi_i$ whereas $t, V \not\models \bigwedge_{i \leq n} \Diamond\psi_i$. However, we assumed that $(s, U) \leftrightsquigarrow (t, V)$, so we get a contradiction. Then we see that $\leftrightsquigarrow$ satisfies the forth condition (b). The back condition (b) is very similar.

Hence the proof is complete. ∎

In this proof, we simply incorporated the observations from basic modal logic. The proof of Hennesy - Milner Theorem in (Blackburn *et al.* , 2001) establishes the main argument of the proof.

In a similar manner, it is not difficult to define bisimulations for cross axiom models. The definition is as follows.

Definition 3.3.2 (Bisimulation on Cross Axiom Models). *Let* $\mathcal{M}_1 = \langle M_1, \xrightarrow{\mathsf{L}_1}, \xrightarrow{\Diamond_1}, V_1 \rangle$ *and* $\mathcal{M}_2 = \langle M_2, \xrightarrow{\mathsf{L}_2}, \xrightarrow{\Diamond_2}, V_2 \rangle$ *be two cross axiom models. A* cross axiom bisimulation *is a non empty relation* $\rightleftarrows \subseteq M_1 \times M_2$ *such that if* $\mathcal{M}_1, w_1 \rightleftarrows \mathcal{M}_2, w_2$ *then we have:*

1. ***Base Condition***

   w_1 *and* w_2 *satisfy the same propositional variables.*

2. ***Back Conditions***

   (a) *If* $w_2 \xrightarrow{\mathsf{L}_2} v_2$*, then there exists* $v_1 \in M_1$ *with* $v_1 \rightleftarrows v_2$ *and* $w_1 \xrightarrow{\mathsf{L}_1} v_1$

   (b) *If* $w_2 \xrightarrow{\Diamond_2} v_2$*, then there exists* $v_1 \in M_1$ *with* $v_1 \rightleftarrows v_2$ *and* $w_1 \xrightarrow{\Diamond_1} v_1$

3. ***Forth Conditions***

(a) *If* $w_1 \xrightarrow{\mathsf{L}_1} v_1$*, then there exists* $v_2 \in M_2$ *with* $v_1 \rightleftarrows v_2$ *and* $w_2 \xrightarrow{\mathsf{L}_2} v_2$

(b) *If* $w_1 \xrightarrow{\Diamond_1} v_1$*, then there exists* $v_2 \in M_2$ *with* $v_1 \rightleftarrows v_2$ *and* $w_2 \xrightarrow{\Diamond_2} v_2$

The usual consequence of the definition is the cross axiom bisimulation invariance between cross axiom models.

Theorem 3.3.3 (Cross Bisimulation Invariance). *If* $\mathcal{M}, w \rightleftarrows \mathcal{M}', w'$ *then* w *and* w' *satisfy the same formulae.*

Proof. The proof is a straight forward induction on the length of the formulae. We then leave it to the reader. ∎

3.4 Topologic Games

We can play games in subset spaces. We will here consider two kinds of games. *Topologic Evaluation Games* will evaluate whether a given formula φ holds in a given neighborhood (s, U). The *Topological Bisimulation Games*, on the other hand, will evaluate whether two topologic spaces are bisimular or not.

The players will be Eloise (denoted by $\exists$) and Abelard (denoted by $\forall$).

3.4.1 Topologic Evaluation Games

We will calculate whether a given formula φ holds in a given neighborhood (s, U) by topologic evaluation games. The positions in the game will be of the form $(\varphi, (s, U))$ where φ is a well formed formula in the language of basic topologic and (s, U) is a neighborhood situation. We will work with formulae in the positive normal form. Recall that a formula φ is in positive normal form if φ has no negation symbol, or $\varphi \equiv \neg\psi$ where ψ has no negation symbol. Observe that, $\neg$ symbol changes the roles of $\forall$ and $\exists$.

Hence the topologic evaluation game $\mathcal{E}(\varphi, \mathcal{S})$ for topologic space $\mathcal{S}$ is a board game with players $\exists$ and $\forall$ moving a token around the positions of the form $(\psi, (s, U))$ where ψ is a subformula of φ and (s, U) is a given neighborhood situation. The rules of the game is given below.

Position	Player	Admissible Moves
$(\bot, (s, U))$	$\exists$	$\emptyset$
$(\top, (s, U))$	$\forall$	$\emptyset$
$(p, (s, U))$ with $s \in v(p)$	$\forall$	$\emptyset$
$(p, (s, U))$ with $s \notin v(p)$	$\exists$	$\emptyset$
$(\psi_1 \wedge \psi_2, (s, U))$	$\forall$	$\{(\psi_1, (s, U)), (\psi_2, (s, U))\}$
$(\psi_1 \vee \psi_2, (s, U))$	$\exists$	$\{(\psi_1, (s, U)), (\psi_2, (s, U))\}$
$(\mathsf{L}\psi, (s, U))$	$\exists$	$\{(\psi, (t, U)) : t \in U\}$
$(\mathsf{K}\psi, (s, U))$	$\forall$	$\{(\psi, (t, U)) : t \in U\}$
$(\Diamond\psi, (s, U))$	$\exists$	$\{(\psi, (s, V)) : s \in V \subseteq U\}$
$(\Box\psi, (s, U))$	$\forall$	$\{(\psi, (s, V)) : s \in V \subseteq U\}$

Negations change the roles of $\forall$ and $\exists$ in the game. Observe that, since the language of subset spaces is an extension of first order logic, evaluation games will be finite.

Winning conditions can be formulated as follows: $\exists$ wins if $\forall$ gets stuck, and dually, $\forall$ wins if $\exists$ gets stuck. As a matter of notations, we will denote the winning positions for $\exists$ by $Win_\exists(\mathcal{E}(\varphi, (s, U)))$. However, one can feel the basic intuition behind the evaluation games by considering the winning positions. We will observe in the next theorem that, if a position lies $(\varphi, (s, U))$ in the winning positions for $\exists$, then it is equivalent to say $s, U \models \varphi$.

Theorem 3.4.1 (Adequacy Theorem for Topologic Evaluation Games)**.**

$$(\varphi, (s, U)) \in Win_\exists(\mathcal{E}(\varphi, (s, U))) \textit{ if and only if } s, U \models \varphi.$$

Proof. Proof goes by induction on the length of the formula. The case for propositional variables and Boolean operators are easy and hence skipped. The case for negation is also easy. As we work with positive normal formula, if the given formula is a negation, then the roles are swapped.

Let us consider the case for $\varphi \equiv \mathsf{L}\psi$. Assume we are in the position $(\mathsf{L}\psi, (s, U))$. Then it is $\exists$'s turn. She will pick a point t in U and move to $(\psi, (t, U))$. As $(\mathsf{L}\psi, (s, U))$ is a winning position for $\exists$, then so is $(\psi, (t, U))$. Then by induction hypothesis we see $t, U \models \psi$. But then, as $s \in U$ as well, we obtain $s, U \models \mathsf{L}\psi$. For the converse direction, assume we have $s, U \models \mathsf{L}\psi$. To get a contradiction assume further that $(\mathsf{L}\psi, (s, U))$ is not a winning situation for $\exists$ which means that after each move out of (s, U) under the formula $\mathsf{L}\psi$, $\exists$ will lose eventually. Now, as $s, U \models \mathsf{L}\psi$, for some $t \in U$ we then have $t, U \models \psi$. By induction hypothesis, we conclude, $(\psi, (t, U)) \in Win_\exists(\mathcal{E}(\psi, (t, U)))$. However, we observed that $(\mathsf{L}\psi, (s, U))$ is not a winning situation for $\exists$, then, $(\psi, (t, U))$ cannot be a winning situation for $\exists$ neither. Contradiction shows that, $(\mathsf{L}\psi, (s, U))$ is a winning position for $\exists$.

Let us now consider the case $\varphi \equiv \mathsf{K}\psi$. Assume we are in the position $(\mathsf{K}\psi, (s, U))$ which is a winning condition for $\exists$. However, now it is $\forall$'s turn. He will pick any point t in U and move to $(\psi, (t, U))$. But, for each $t \in U$, the position $(\psi, (t, U))$ is still a winning position for $\exists$. Hence, $t, U \models \psi$. But, this statement holds for each $t \in U$, and hence $s, U \models \mathsf{K}\psi$. For the converse direction assume $s, U \models \mathsf{K}\psi$. To get a contradiction assume further that $(\mathsf{K}\psi, (s, U))$ is not a winning situation for $\exists$ which means that after each move of $\forall$ out of (s, U) with the formula $\mathsf{K}\psi$, $\exists$ will lose eventually. Now, as $s, U \models \mathsf{K}\psi$, for each $t \in U$ we then have $t, U \models \psi$. By induction hypothesis, we conclude, $(\psi, (t, U)) \in Win_\exists(\mathcal{E}(\psi, (t, U)))$. However, we observed that $(\mathsf{K}\psi, (s, U))$ is not a winning situation for $\exists$, then, $(\psi, (t, U))$ cannot be a winning situation for $\exists$ neither. Contradiction shows that, $(\mathsf{K}\psi, (s, U))$ is a winning position for $\exists$.

Let us now consider the case for $\varphi \equiv \Diamond\psi$. Assume $\Diamond\psi, (s, U)$ is a winning position for $\exists$. So, when we are in the position $\Diamond\psi, (s, U)$, $\exists$ will pick a subset V of U with $s \in V$ and move to the position $\psi, (s, V)$. By induction hypothesis, we conclude $s, V \models \psi$. But now, it is easy to see that $s, U \models \Diamond\psi$. For the other direction, let us assume $s, U \models \Diamond\psi$ holds. To get a contradiction, assume $(\Diamond\psi, (s, U))$ is not a winning position for $\exists$. From, $s, U \models \Diamond\psi$ we conclude $s, V \models \psi$ for some $s, \in V \subseteq U$. By induction hypothesis we see$(\psi, (s, V))$ is a winning position for $\exists$ in the game $\mathcal{E}(\psi, (s, V))$. However, we assumed that $(\Diamond\psi, (s, U))$ was not a winning position, so $(\psi, (s, V))$ cannot be a winning position in the game $\mathcal{E}(\psi, (s, V))$. The contradiction shows that, $(\Diamond\psi, (s, U))$ is a winning position for $\exists$ in the game $\mathcal{E}(\Diamond\psi, (s, U))$.

The case for $\varphi \equiv \Box\psi$ is similar.

Hence the theorem is proved. ■

3.4.2 Topologic Bisimulation Games

Topologic bisimulation games will provide an alternative semantics to approach to the topologic bisimulations. In this game, $\forall$ and $\exists$ will compare neighborhood situations across the respective topologic spaces. $\exists$ wins if the given two neighborhood situations are bisimular, $\forall$ wins otherwise.

Assume we are given (s, U) and (t, V). $\forall$ starts. He can either pick another point t' in V (or s' in U) or pick a subset $V' \subseteq V$ such that $t \in V'$ (or a subset $U' \subseteq U$ such that $s \in U'$). If he picked another point t' in V, then $\exists$ must find a corresponding s' in U such that $(s', U) \rightleftarrows (t', V)$. She loses immediately, if she cannot find such a point. If $\forall$ picked a subset

$V' \subseteq V$ with $t \in V'$, then $\exists$ must find a corresponding subset $U' \subseteq U$ with $s \in U'$ such that $(s, U') \rightleftarrows (t, V')$. She loses immediately if she cannot find such a subset.

A topologic bisimulation game of length n, then can be defined as a game which can distinguish formulas of depth at most n. It is then easy to observe that, $\exists$ has a winning strategy in the bisimulation game of length n for (s, U) and (t, V) if and only if these two neighborhood situations are actually bisimular for formulas of depth at most n.

It is easy to see that when $\exists$ has a winning strategy in the bisimulation game, then the neighborhood situations are bisimular. The proof goes by induction on the depth of the formulae and the application of straight forward ideas.

To see the converse, assume we have a bisimular neighborhood situations (s, U) and (t, V). Then by following the definition of bisimulation, we will form a winning strategy for $\exists$. $\forall$ starts the game. He can pick a point t' in V. But as $(s, U) \rightleftarrows (t, V)$, we can find a corresponding point $s' \in U$ such that $(s', U) \rightleftarrows (t', V)$. If he picked a point $s'' \in U$, then by similar arguments, we can find $t'' \in V$ such that $(s'', U) \rightleftarrows (t'', V)$. Therefore, we can add these points to the winning strategy of $\exists$. On the other hand, $\forall$ can pick the subsets $V' \subset V$ where $t \in V'$ or $U' \subset U$ where $s \in U'$. But in any case, by following the above argumentation, we again come up with a corresponding subsets which maintain the bisimulation as $(s, U) \rightleftarrows (t, V)$. Hence, we established a winning strategy for $\exists$ by just following the bisimulation relation $\rightleftarrows$.

Hence we proved the Adequacy Theorem for Bisimulation Games.

Theorem 3.4.2 (Adequacy Theorem for Topologic Bisimulation Games). *$(s, U) \rightleftarrows_n (t, V)$ if and only if $\exists$ has a winning strategy in the topologic bisimulation game of length n.*

Proof. Given above the theorem. ■

3.4.3 Topologic Games in Extended Languages

Topologic games can easily be defined in extended languages. Therefore, in order to see that, we will show on example - subset space logic with overlap operator.

Heinamann's overlap operator O enables us to change the current neighborhood. Therefore, we can have much more freedom to move around the subset space. Then, the rules for this game is as follows. The rules of the game is given below.

Position	Player	Admissible Moves
$(\bot, (s,U))$	$\exists$	$\emptyset$
$(\top, (s,U))$	$\forall$	$\emptyset$
$(p, (s,U))$ with $s \in v(p)$	$\forall$	$\emptyset$
$(p, (s,U))$ with $s \notin v(p)$	$\exists$	$\emptyset$
$(\psi_1 \wedge \psi_2, (s,U))$	$\forall$	$\{(\psi_1, (s,U)), (\psi_2, (s,U))\}$
$(\psi_1 \vee \psi_2, (s,U))$	$\exists$	$\{(\psi_1, (s,U)), (\psi_2, (s,U))\}$
$(\mathsf{L}\psi, (s,U))$	$\exists$	$\{(\psi, (t,U)) : t \in U\}$
$(\mathsf{K}\psi, (s,U))$	$\forall$	$\{(\psi, (t,U)) : t \in U\}$
$(\Diamond\psi, (s,U))$	$\exists$	$\{(\psi, (s,V)) : s \in V \subseteq U\}$
$(\Box\psi, (s,U))$	$\forall$	$\{(\psi, (s,V)) : s \in V \subseteq U\}$
$(\mathsf{P}\psi, (s,U))$	$\exists$	$\{(\psi, (s,U')) : s \in U'\}$
$(\mathsf{O}\psi, (s,U))$	$\forall$	$\{(\psi, (s,U')) : s \in U'\}$

The adequacy theorem for extended topologic games is then straightforward.

Theorem 3.4.3 (Adequacy Theorem for Extended Topologic Evaluation Games)**.**

$$(\varphi, (s,U)) \in Win_{\exists}(\mathcal{E}(\varphi, (s,U))) \textit{ if and only if } s, U \models \varphi.$$

Proof. A simple extension of the Adequacy Theorem for Topologic Bisimulation Games - Theorem 3.4.2. Left as an exercise for the reader. ■

Chapter 4

Public Announcement Logic for Subset Space Logic

4.1 Introduction

Our main goal in this chapter is to give a subset space semantics for *Public Announcement Logic* (PAL, for short). The very first intuition for considering subset space semantics for PAL simply stems from the idea that the two modal operator of subset space logic can very well formalize the change in the knowledge after a public announcement. As we already noted, the shrinking operator has a dynamic nature.

In this chapter, starting from a familiar example of PAL, we will indicate some important motivations to formalize PAL in a geometrical (almost topological) setting. Then we will present the formal tools and develop them a bit further to get some more insights. Finally, the completeness of PAL for subset space semantics should not be a surprise. Our main reference for PAL is (Benthem *et al.* , 2005).

This section, on the other hand, can be read as a formal introduction for the next chapter in which we will extend the discussion with further examples.

4.2 Card Showing Game

Card showing game is one of the very simple examples to illustrate the main idea of public announcements. Suppose that we are playing a simple card game with three people A, B and C and three cards p, q and r. The aim of the game is to guess the card of each player. Let the actual distribution of the cards as follows: A has p, B has q and finally C has r. Suppose further that the players have looked at their cards but kept them hidden from the other players. For the sake of notation let the ordered tuple $\langle \alpha, \beta, \gamma \rangle$ denote the situation that the player A has the card α whereas the players B and C have the cards β and γ respectively. As the order matters $\langle \alpha, \beta, \gamma \rangle \neq \langle \gamma, \beta, \alpha \rangle$.

Therefore, in this situation, let us consider the set of observations for the agent A. As she already knows which card she has, and does not know the cards of the other people, her set of observations would be the following set $o = \{\langle p, q, r \rangle, \langle p, r, q \rangle\}$. Hence, by spending some effort such as cheating, bribing another player, agent A can increase her knowledge. But, the increase in the knowledge of A could also come in the form of *public announcement* (However, note that, in this simple game, when a public announcement of the form "Player X has the card x" is made, players other than X win the game.). Hence, if it is announced that the player B has card q, then the set o will shrink to, say $v = \{\langle p, q, r \rangle\}$. Similarly, the public announcement that B has r would lead to a shrinking of o to $w = \{\langle p, r, q \rangle\}$. Likewise for the public announcement for player C.

Hence, we can consider the public announcements as the (external) efforts spent. What motivates us from this simple example is the fact that, we can make the shrinking procedure explicit by a public announcement. Then why not identify this procedure with a mapping? Recall that we start with the set o and by throwing the refutative elements away from the set, we obtained the subsets $v \subseteq o$ or $w \subseteq o$ of u.

4.3 Formal Tools

Public announcement logic is typically interpreted on Kripke structures. So, before presenting the subset space semantics for PAL, let us review the Kripkean interpretation of PAL. Notationwise, the formula $[\varphi]\psi$ is intended to mean that *after the public announcement of φ, ψ holds*. As usual, K_i is the epistemic modality for the agent i. Likewise, R_i is the epistemic accessibility relation for the agent i and R stands for the set of accessibility relations. The language of PAL will be that of epistemic logic with an additional public announcement operator $[*]$ where $*$ can be replaced with any well formed formula.

Definition 4.3.1 (Semantics of PAL). *Let $\mathcal{M} = \langle W, R, V \rangle$ be a model and i be an agent. For atomic propositions, negations and conjunction the definition is as usual. For modal operators, we have the following semantics:*

$$\begin{array}{lll} \mathcal{M}, w \models \mathsf{K}_i\varphi & \textit{iff} & \mathcal{M}, v \models \varphi \textit{ for each } v \textit{ such that } (w, v) \in R_i \\ \mathcal{M}, w \models [\varphi]\psi & \textit{iff} & \mathcal{M}, w \models \varphi \textit{ implies } \mathcal{M}|\varphi, w \models \psi \end{array}$$

Here the updated model $\mathcal{M}|\varphi = \langle W', R', V' \rangle$ is defined by restricting $\mathcal{M}$ to those states where φ holds. Define $(\varphi)^{\mathcal{M}} = \{v \in W : \mathcal{M}, v \models \varphi\}$. Hence, $W' = \{w \in W : w \models \varphi\}$, i.e. $W' = W \cap (\varphi)^{\mathcal{M}}$; $R'_i = R_i \cap (W' \times W')$ and finally $V'(p) = V(p) \cap W'$.

The proof system of public announcement logic is the proof system of multi-modal S5 epistemic logic with the following additional axioms.

Atoms	$[\varphi]p \leftrightarrow (\varphi \rightarrow p)$
Partial Functionality	$[\varphi]\neg\psi \leftrightarrow (\varphi \rightarrow \neg[\varphi]\psi)$
Distribution	$[\varphi](\psi \wedge \chi) \leftrightarrow ([\varphi]\psi \wedge [\varphi]\chi)$
Knowledge Announcement	$[\varphi]\mathsf{K}_i\psi \leftrightarrow (\varphi \rightarrow \mathsf{K}_i[\varphi]\psi)$

The rule of inference is called the *announcement generalization* and is described as follows.

From $\vdash \psi$, derive $\vdash [\varphi]\psi$.

For the sake of simplicity, we leave the common knowledge out in the definitions. The interested reader is referred to (Benthem *et al.* , 2005).

The key idea is that, after the public announcement φ, the states which are incompatible with φ are *discarded*. In other words, the effort spent by the public announcement, makes our accessible states smaller by getting rid of refutative observations (recall that $R'_i = R_i \cap (W \times W')$ in the new model $\mathcal{M}|\varphi$). However, throughout the process, we do not change our point of view, i.e. our state. Therefore, the public announcement φ makes sense if the current state realizes φ. Otherwise the state of the observer itself will be discarded, too. This observation manifests itself in the axioms.

4.4 Subset Space PAL

Following the aforementioned motivations and considerations, let us now formalize the public announcement logic in the language of the basic subset space logic. As we already pointed out, in order to eliminate some of the refutative observations, we will use the public announcements. Let us make this observation more explicit. An illustrative example can be given by following the speed measurement of the policeman case. Let us recall that the speed limit is 50 mph and policeman made an measurement that a particular car has the velocity of 51 mph. As the error range is 2 mph, we have the interval (49, 53) as our set of observation. However, a public announcement can shrink this set as well. Assume, that we have the following public announcement: "The velocity of the car is not greater than 52 mph". Hence, the updated observation set will be (49, 52). So, we have more knowledge now as we reduced the set of possibilities significantly. Examples can be multiplied easily.

In the subset space language, as we underlined before, instead of accessibility relations, we depend on neighborhood situations. Therefore, if we want to adopt public announcement logic to the context of subset space logic, we first need to focus on the fact that the public announcements shrink the observation sets for each agent. Hence, assume we are in a subset space frame $\mathcal{S} = \langle S, \sigma \rangle$. Then, after public announcement logic φ, we will move to another subset space frame, say $\mathcal{S}_\varphi = \langle S|\varphi, \sigma_\varphi \rangle$ where $S|\varphi = (\varphi)_1$ and σ_φ is the reduced collection of subsets after the public announcement φ. The saddle point is to construct σ_φ. As we need to get rid of the refutative states, for each observation set U in σ, we eliminate the points which do not satisfy φ. We will disregard the empty set as no neighborhood situations can be formed with empty set. Hence $\sigma_\varphi = \{U_\varphi : U_\varphi = U \cap (\varphi)_2 \neq \emptyset, \text{ for each } U\}$. Note that $(\varphi)_i$ is the set of projection onto the i-th coordinate of the extension of φ, i.e. $(\varphi)_1 = \{s : (s, U) \in (\varphi) \text{ for some } U\}$ and $(\varphi)_2 = \{U : (s, U) \in (\varphi) \text{ for some } s\}$[1].

But then, how would the neighborhood situations be effected from the public announcement? Consider the neighborhood situation (s, U) and the public announcement φ. Then the statement $s, U \models [\varphi]\psi$ will mean that after the public announcement of φ, ψ will hold in the neighborhood situation (s, U_φ). So, first we will remove the points in U who refutes φ, and then ψ will hold in the updated set U_φ which was obtained from the original set U. Then the corresponding semantics can be suggested as follows:

$s, U \models [\varphi]\psi$ if and only if $s, U \models \varphi$ implies $s, U_\varphi \models \psi$

Before checking whether this semantics satisfies the axioms of public announcement logic, let us give the language and semantics of the topologic PAL.

The language of the topologic public announcement logic interpreted in subset spaces is given as follows:

$$p \mid \bot \mid \neg\varphi \mid \varphi \wedge \psi \mid \Box\varphi \mid \mathsf{K}\varphi \mid [\varphi]\psi$$

The semantics for topologic PAL differs only on public announcement operator whose semantics is given as follows:

[1] Observe that the projection on the first coordinate will give a set of points, on the second coordinate will give a set of sets.

$$s, U \models [\varphi]\psi \quad \text{if and only if} \quad s, U \models \varphi \text{ implies } s, U_\varphi \models \psi$$

Now, let us consider the soundness of the following axioms of basic PAL.

Atoms	$[\varphi]p \leftrightarrow (\varphi \rightarrow p)$
Partial Functionality	$[\varphi]\neg\psi \leftrightarrow (\varphi \rightarrow \neg[\varphi]\psi)$
Distribution	$[\varphi](\psi \wedge \chi) \leftrightarrow ([\varphi]\psi \wedge [\varphi]\chi)$
Knowledge Announcement	$[\varphi]\mathsf{K}\psi \leftrightarrow (\varphi \rightarrow \mathsf{K}[\varphi]\psi)$

The following theorem establishes the soundness of the topologic PAL.

Theorem 4.4.1 (Soundness of Topologic PAL). *Above axioms are sound in topologic PAL.*

Proof. As the atomic propositions do not depend on the neighborhood, the *Atoms* axiom is satisfied by the subset space semantics of public announcement modality. To see this, assume $s, U \models [\varphi]p$. So, by the semantics $s, U \models \varphi$ implies $s, U_\varphi \models p$. So, $s \in v(p)$. So for any set V where $s \in V$, we have $s, V \models p$. Hence, $s, U \models \varphi$ implies $s, U \models p$, that is $s, U \models \varphi \rightarrow p$. Conversely, assume $s, U \models \varphi \rightarrow p$. So, $s, U \models \varphi$ implies $s \in v(p)$. As $s, U \models \varphi$, s will lie in U_φ, thus (s, U_φ) will be a neighborhood situation. Thus, $s, U_\varphi \models p$. Then, we conclude $s, U \models [\varphi]p$.

Partial functionality and distribution axioms are also straight forward formula manipulations and hence skipped.

The important reduction axiom is the knowledge announcement axiom. Assume, $s, U \models [\varphi]\mathsf{K}\psi$. Suppose further that $s, U \models \varphi$. Then we have

$$\begin{array}{lll} s, U \models [\varphi]\mathsf{K}\psi & \text{iff} & s, U_\varphi \models \mathsf{K}\psi \\ & \text{iff} & \text{for each } t_\varphi \in U_\varphi \text{ we have } t_\varphi, U_\varphi \models \psi \\ & \text{iff} & \text{for each } t \in U,\ t, U \models \varphi \text{ implies } t, U \models [\varphi]\psi \\ & \text{iff} & s, U \models \mathsf{K}(\varphi \rightarrow [\varphi]\psi) \\ & \text{iff} & s, U \models \mathsf{K}[\varphi]\psi \end{array}$$

Thence, the above axioms are sound for the subset space semantics of public announcement logic. ■

Let us explain the proof further. The first step is the definition of the public announcement in the subset space semantics. Then, in the second step, we unravel the knowledge modality. However, the second step says that the neighborhood situations that satisfy φ will satisfy ψ after an update with φ. Hence, for each point t in U, if φ is true at (t, U) then after an update with φ, ψ will be true at (t, U). However, this statement is true for each t in U, so we can go back to our starting point s by knowledge modality. As $[\varphi]$ is a partial operation, that is only applicable to the neighborhood situations which satisfy φ, we can simplify the statement in the forth step to the one in the last step. Hence, the result follows.

However, subset space logic has an indispensable modal operator, namely the effort modality. One can wonder whether we have a reduction axiom for it as well. We start with considering the statement $[\varphi]\Box\psi \leftrightarrow (\varphi \to \Box[\varphi]\psi)$. We will call it the *reduction axiom for shrinking operator*. Assume, $s, U \models [\varphi]\Box\psi$. Suppose further that $s, U \models \varphi$. Then we have

$$\begin{array}{lll} s, U \models [\varphi]\Box\psi & \text{iff} & s, U_\varphi \models \Box\psi \\ & \text{iff} & \text{for each } V_\varphi \subseteq U_\varphi \text{ we have } s, V_\varphi \models \psi \\ & \text{iff} & \text{for each } V \subseteq U,\ s, V \models \varphi \text{ implies } s, V \models [\varphi]\psi \\ & \text{iff} & s, U \models \Box(\varphi \to [\varphi]\psi) \\ & \text{iff} & s, U \models \Box[\varphi]\psi \end{array}$$

The first step here is again the definition of $[\varphi]$ operator. In the second step, we unravel the effort modality. We go back to the initial model in the third step by considering the each subset of the given neighborhood. We move to the fourth step by quantifying over the subsets of the given neighborhood by $\Box$ modality. Hence the result follows.

In conclusion, we observed that both for K and $\Box$, we have a reduction axiom and hence reduced the complexity of the formulae in the language of topologic PAL step by step. Starting with a formula in the language of topologic PAL, by following the reduction axioms we discussed, eventually we will end up with a formula in the language $\mathcal{L}_S$ of subset space logic. This is the key idea for the completeness of topologic PAL.

Therefore, it is easy to see that the following axiomatize the topologic-PAL:

$$\begin{array}{rl} \textit{Atoms} & [\varphi]p \leftrightarrow (\varphi \to p) \\ \textit{Partial Functionality} & [\varphi]\neg\psi \leftrightarrow (\varphi \to \neg[\varphi]\psi) \\ \textit{Distribution} & [\varphi](\psi \wedge \chi) \leftrightarrow ([\varphi]\psi \wedge [\varphi]\chi) \\ \textit{Knowledge Announcement} & [\varphi]\mathsf{K}\psi \leftrightarrow (\varphi \to \mathsf{K}[\varphi]\psi) \\ \textit{Shrinking Reduction} & [\varphi]\Box\psi \leftrightarrow (\varphi \to \Box[\varphi]\psi) \end{array}$$

Referring to the above discussions, the completeness result for topologic PAL follows easily.

Theorem 4.4.2 (Completeness of Topologic PAL). *Topologic PAL is complete with respect to the axiom system given above.*

Proof. By reduction axioms we can reduce each formula in the language of topologic PAL to a formula in the language of topologic. As topologic is strongly complete, so is topologic PAL. ∎

4.4.1 PAL with Overlap Operator

In this section, we will enrich the language of subset space PAL with overlap operator. Our new language will be as follows.

$$p \mid \bot \mid \neg\varphi \mid \varphi \wedge \psi \mid \Box\varphi \mid \mathsf{K}\varphi \mid [\varphi]\psi \mid \mathsf{O}\varphi$$

Now we observe that we can come up with a reduction axiom for overlap operator as well.

Theorem 4.4.3 (Reduction Axiom for Overlap Operator)**.** $[\varphi]\mathsf{O}\varphi \leftrightarrow (\varphi \rightarrow \mathsf{O}[\varphi]\psi)$ *is sound.*

Proof. Assume $s, U \models [\varphi]\mathsf{O}\psi$. Suppose further that $s, U \models \varphi$. Then we have,

$$\begin{array}{lll} s, U \models [\varphi]\mathsf{O}\psi & \text{iff} & s, U_\varphi \models \mathsf{O}\psi \\ & \text{iff} & \text{for each } V_\varphi \in \sigma_\varphi, \text{ we have} \\ & & s \in V_\varphi \text{ implies } s, V_\varphi \models \psi \\ & \text{iff} & \text{for } s \in V \; s, V \models \varphi \text{ implies } s, V \models [\varphi]\psi \\ & \text{iff} & s, U \models \mathsf{O}(\varphi \rightarrow [\varphi]\psi) \\ & \text{iff} & s, U \models \mathsf{O}[\varphi]\psi \end{array}$$

■

We can celebrate the above theorem with a completeness result.

Theorem 4.4.4 (Completeness of Topologic PAL with Overlap)**.** *Topologic public announcement logic with overlap operator is complete.*

Proof. Recall that the subset space logic extended with overlap operator is complete. Then, the result follows. ■

4.5 Conclusion

We observed that the knowledge and shrinking operators behave nice in the context of public announcement logics. They both satisfy the appropriate reduction axioms. Therefore, using these reduction axioms, we can easily get the completeness of this logic. Furthermore, we used the very same idea to obtain a completeness result for PAL with overlap operator.

Chapter 5

Controlled Shrinking for Subset Space Logic

5.1 Introduction

There are many ways to construct the observation sets. Let us start with reviewing the example given in (Dabrowski *et al.* , 1996).

Moss, Parikh & Dabrowski gives the example of a policeman measuring the speed of passing vehicles (Dabrowski *et al.* , 1996), (Moss *et al.* , 2007). Assume that the speed limit is 50 mph and the measurement device of the policeman has an error range of 2 mph. Assume further that at a specific time the policeman made a measurement that the velocity of a car is 51 mph. Considering the error range, we interpret it as an open set (49, 53). Therefore, the policeman is in no position of deciding whether the car exceeded the speed limit or not based on his measurement. However, he can increase his knowledge by using more accurate or hi-tech measurement device. Suppose now that the policeman has a more sophisticated measurement device with an error range of 0.8 mph. In this case, when the policeman made an observation that a car had a velocity of 51 mph, then this observation will correspond to the open set (50.2, 51.8) which entirely lies in above the permitted speed limit. To sum up, quoting from the paper (Dabrowski *et al.* , 1996), it can be concluded that,

> ... [policeman's] knowledge of the speeds of the cars that he can *see* will depend on the accuracy of his measuring instrument. He can increase his knowledge *without* changing his view, by just using a more accurate measuring instrument. Nonetheless, his knowledge will generally be such that he can always improve it.

Hence, in the previous example, by spending some effort (which corresponds to using more sophisticated measuring device), the policeman increased his knowledge by making his set of observations smaller. But, in this context, Moss, Parikh & Steinsvold (Moss *et al.* , 2007) remark that,

> Our treatment of effort is in a sense fairly crude: we take the subsets of a space as the possible observations, and then more effort corresponds to a smaller set, a better observation to being closer to the "real" point. We say that this is crude because it does not *measure* [emphasize is mine. C.B.] the amount of effort in any real sense.

Therefore, starting from the all possible sets of observations, we can come up with some reduced sets of observations. What are the reduced sets of observations then? In the context of measuring the speed of the cars, the reduced sets of observation corresponds to a speeding device with a smaller error range. By using a more sophisticated measuring method, then, some observations are eliminated as they are now redundant.

In this chapter, we will provide some motivations for our discussion. We picked these examples from different areas of formal sciences: philosophy of science, belief revision, topology and logic.

5.2 Motivations

5.2.1 An Example from Philosophy of Science

The approach we presented above is rather visible in the philosophy of science. Starting from some specific set of observations and, for instance, by applying some methods or by carrying out some relevant experiments, we obtain a restricted set of observations which eventually (and hopefully) lead to an increase in knowledge.

In this aspect, Lakatos's seminal work *Proofs and Refutations* illustrates our approach (Lakatos, 2005),(Başkent & Bağçe, 2007). *Proofs and Refutations* gives a rationally reconstructed account of the methodological evaluation of Euler's formula for polyhedra: $V - E + F = 2$, where V is the number of vertices, E is the number of edges and F is the number of faces of the polyhedron in question. Starting from a collection of observations (or assertions) about some peculiar properties of polyhedron, the arguments proceed by reducing these observations (or assertions) by some mathematical *thought experiments* as Lakatos himself called. The effort in this context corresponds to some mathematical calculations and analysis or suggesting a counter example or even refuting a counterexample. However, there might be several (correct or incorrect; plausible or implausible) ways of reducing the collection of observations which are simple and usually naive thought experiments. Hence, we can identify and distinguish these procedures.

For example, if we establish that the Euler formula holds for simply connected polyhedra, then, we will discard some observations about the polyhedra which are not simply connected - such as torus. Hence, without changing our point of view, we changed our neighborhood situation by considering some smaller set around the reference point we are occupying. Therefore, in this context, the neighborhood of the connected polyhedra will be the polyhedra which are suggested to be connected polyhedra.

However, Lakatos's constructions are not in one-to-one correspondence with the subset spaces enriched with functional mappings. The reason for this is the occurrence of the fallacies throughout the course of mathematical development. As we have no modality for a decrease in knowledge caused by these fallacies since we cannot enlarge the neighborhood we are occupying as we please, functional subset spaces are far from formalizing the entire course of the methodology of mathematics. But, as we underlined, several motivations which we hitherto pointed out can be derived from some specific instances in the history of mathematics.

5.2.2 An Example from Belief Revision

The notion of belief revision is raised from the necessity to revise ones own beliefs which was usually caused by a piece of information leading to a contradiction. Gärdenfors gave an example of a situation, when you are about the pay your bill in a restaurant, and suddenly realize that, you do not have sufficient money in your wallet, although you were sure that you had the money (Gärdenfors, 1988). In this example, the contradiction, which arose from the idea of having money and the fact of not having money, makes us revise our belief that, we had the money.

The important aspect of belief revision lies in its consistency. It is not only adding new pieces of information to the initial set of information or belief, but instead, learning a new information that caused a contradiction. This new enlarged set has to be revised in order to maintain the consistency. Thus, some criteria for rationality of the belief revisions needs to be given. AGM postulates are one of this aforementioned rationality criteria. They state the very basic properties of the operations on belief sets. So, AGM postulates give an account of belief revision (Gärdenfors, 1988). In order to get acquainted with the AGM postulates, the interested reader is referred to the aforementioned reference.

However, the easiest way to get acquainted with the underlying motivations for belief revision might be to review the following example from Gärdenfors (Gärdenfors, 1988). Suppose, among others, we have the following informations and observations:

α All European swans are white.

β The bird caught in the trap is a swan.

γ The bird caught in the trap comes from Sweden.

δ Sweden is a part of Europe.

Therefore, one can easily derive that:

ϵ The bird caught in the trap is white.

However, it might be the case that, the bird caught in the trap turned out to be black. Then we have basically two options: either we can add the negation of ϵ to our set of propositions and get an inconsistent set of propositions (which is not desirable for obvious reasons), or revise our set of knowledge. In order to achieve this, we may need to retract some of the observations which we had before. However, the formal rules themselves cannot tell us which propositions we should revise: α, β, γ or δ or even some combination of them? For example, if we need to revise δ, we could come up with the following

δ^* Some parts of Sweden does not belong to Europe.

Similarly, we can revise δ as

ϵ^* The bird caught in the trap comes from either Sweden or Russia.

The other observations can be revised as well in a similar fashion.

The saddle point of belief revision which relates to our focus is the necessity of shrinking the set of observations by observing the maintenance the consistency. For instance, in the case of contracting or shrinking some specific set of observations, we remove some sentences from the observation set O maintaining the consistency. For instance if we want to retract a sentence φ from O, some more sentences might be needed to removed from O as well in order to preserve the consistency. Hence, there appears to be a method (one by one checking the sentences to spot the ones who might lead to inconsistency) for shrinking.

On the other hand, the language of subset spaces is far from being adequate for formalization of AGM belief revision postulates. The simplest reason for this is the fact that we cannot talk about expanding sets in the language of subset space logics. However, in the context of belief revision, by adding some extra information or observation with all its logical consequences to our set of observation, we can expand the set. This is called expansion in belief revision.

Our motivation for mentioning belief revision for our goals was simply to pay attention to the existence of some methods for shrinking the set of observations. Contracting the belief sets or observation sets cannot hence be performed randomly. At least, we have to preserve consistency and therefore there is a rational for leaving some observations out.

5.2.3 An Example from Basic Topology

One of the examples given in (Dabrowski *et al.* , 1996) considers the standard topology τ of reals. As we underlined in the Introduction chapter, subset space logic was aimed to be strong enough to formalize topological reasoning. We will here reproduce this example to give insights about the constructions we are about to present. Let us start with recalling the original example from the article.

Assume, among others, we have an atomic predicate p where $v(p) = [0, 2]$. Then, it is easy to see $1, (0, 4) \models p$ as $v(p) \subseteq (0, 4)$. As $3 \in (0, 4)$, then there is a point (namely 3) in the subset $(0, 4)$ in which p fails. Hence, we write $1, (0, 4) \models \mathsf{L}\neg p$. Furthermore, we can shrink the open interval $(0, 4)$ for instance, to $(0.7, 1.8)$ around 1 where $v(p)$ entirely lies in $(0.7, 1.8)$. Therefore, we conclude $1, (0, 4) \models \Diamond \mathsf{K} p$. Examples can be multiplied in a similar fashion.

However, observe that, out of the continuum many open subintervals of $(0, 4)$, we picked one, namely $(0.7, 1.8)$ in this case, in which p holds. But, we can make this selection by observing some guidelines, methods etc. In other words, we can say, we only need the open intervals with a length smaller than 1, thus we can discard the open intervals longer than 1 and get a new collection of subsets, say τ', of $\mathbf{R}$. Therefore, the shrinking function in this context makes the intervals shorter than 1. So, in the reduced collection $(1, (0.7, 1.8))$ would not constitute a neighborhood situation.

But, observe τ' is not a topology. For instance consider the intervals $(4, 5)$ and $(4.5, 5.5)$. Then the union of these two $4, 5.5$ does not lie in τ' as its length is longer than 1. Hence, we can lose some nice properties while reducing our sets.

There are several methods to reduce the intervals to an interval of length smaller than 1. Let us present one for the sake of the argument. For a given interval (a, b) in τ, we will leave it intact if it is shorter than 1. If it is length is greater than 1, we will shrink it around its middle point $(b - a)/2$ to an interval of length 1. So, we will obtain the interval $((b - a - 1)/2, (b - a + 1)/2)$. Obviously the mapping from (a, b) to $((b-a-1)/2, (b-a+1)/2)$ is not injective, as the interval $(a-1, b-1)$ is also mapped to $((b-a-1)/2, (b-a+1)/2)$. But, for obvious reasons, this mapping is surjective. Plenty more methods for shrinking the opens can be given. For instance, from an interval (a, b)longer than 1, one can get the interval $(a, a+1)$ or $(b-1, b)$ etc.

5.2.4 An Example from Logic

Several notions from logic can be helpful to underline the intuition and motivation behind our system. The public announcement logic, in this respect is helpful. See the previous chapter for the construction of PAL and its subset space semantics.

What is significant about subset space PAL is the fact that it gives us a method to contract the set of observations we already have to get a restricted set of observations. The restriction in this context has been carried out by the public announcement. In other words, by the public announcement, we eliminated the observations that refutes the public announcement or contracted the observation sets by eliminating the points (in these sets) which refute the public announcement. We gave the technical details in the previous chapter.

5.3 Motivations: Reconsidered

The examples we presented might not be seem obvious at first glance. Now we will elaborate more on them and establish the connection with subset spaces.

Lakatosian approach to the methodology of mathematics actually underlines our first starting point. We want to keep the point of reference intact and change only the neighborhood of it. In Lakatos's examples, thought experiments about polyhedra represents the change of the neighborhoods while the mathematical assertions discussed by Lakatos are represented by the points in subset space logic. As we considered already, for instance, when simply- or multiply-connectedness property of polyhedra is considered, we can have several methods to eliminate some polyhedra. For instance, counting edges, vertices and faces and making the calculations according to the Euler formula is one method. As a second method, one can approach to the problem from a geometric topological point of view and consider the problem with some sophisticated mathematical tools. In any case, these different approaches for obtaining smaller observation sets will yield different observation sets.

Belief revision example can also be approached in a similar manner. There may be several ways to retract the belief sets. Therefore, the shrunk observation sets differ. So that, we can identify these different methods for retracting.

Henceforth, we leave it to the reader to convince herself that the remaining motivational cases we considered above can also be analyzed in a similar fashion.

In the next section, we will identify these different ways to shrink observation sets with a function. But a strong caution should be observed here. These functions do not impose the method for shrinking the observation sets. But rather they are imposed by the specific methods used in these specific cases.

5.4 Formalization

Recall that, in subset space logic, we have a collection of observations and a bimodal logic based on this collection of sets. As we emphasized earlier, our aim is to reduce the collection of opens to a smaller collection. To achieve this goal, we will use functions. Therefore, starting from the collection of all possible observations, we will then apply the function(s) - which corresponds to some specific methods for spending effort to increase knowledge - and get a new collection. However, as this new operation should be information increasing, the image sets under these functions must be smaller. In other words, for each such function f and each observation set U, we will have $fU \subseteq U$

5.4.1 Semantics for Controlled Subset Structures

We will now generalize the above observations and remarks. Let $\mathcal{F}$ be a collection of functions from S to S, and further let $F \subseteq \mathcal{F}$. Take two subset spaces $\mathcal{S} = \langle S, \sigma, v \rangle$ and $\mathcal{S}_F = \langle S, \sigma_F, v \rangle$. Here, σ_F is the image of each $U \in \sigma$ under each function $f \in F$. In other words, $\sigma_F := \{fU : f \in F, U \in \sigma\}$. We will call $\mathcal{S}_F$ *the image space of* $\mathcal{S}$ *under* F.

Note that, as we already underlined, each function $f \in F$ are contracting mappings intended to represent the increase in the information. Hence, $fU \subseteq U$ should hold for each function f and for each observation set U. On the other hand, observe that each $V \in \sigma_F$ is the image of an observation set $U \in \sigma$ under some function $f \in F$. Given a neighborhood situation (s, U) in $\mathcal{S}$, we will get another neighborhood situation (s, fU) in $\mathcal{S}_F$ for some $f \in \mathcal{F}$ by throwing away the points in U which are not in the image of U under f. But, is s in fU? Because otherwise (s, fU) would not be a neighborhood situation. Therefore, we have to *force* this condition. In other words, we can only evaluate the formulae if $s \in fU$. If this condition was not met, then we would not evaluate the formuale at (s, fU).

Definition 5.4.1 (Controlled Subset Space). *$\mathcal{S} = \langle S, \sigma, v, \mathcal{F} \rangle$ is a controlled subset space where S is a set, σ is any collection of subsets of S, $v : \mathsf{At} \to \wp(S)$ is a valuation function and $\mathcal{F}$ is a collection of functions $\mathcal{F} = \{f : S \to S\}$.*

The idea behind the controlled subset spaces is to control the shrinking of sets by (a collection of) functions. As we emphasized earlier these functions will be contracting functions so that we will guarantee the increase in the knowledge. Recall that, the more effort we spend the more closer we get to the knowledge.

We will now introduce an additional modality $[F]$ representing the controlled shrinking. The intended meaning of the statement $[F]\varphi$ is "after the application of each function $f \in F$, φ is true". Note that, after application of function, we evaluate the formula φ in $\mathcal{S}_F$. The dual of $[F]$ is $\langle F \rangle$ and defined as usual. Therefore, we will evaluate the knowledge and effort modality in the given subset space $\mathcal{S}$. However, while evaluating the controlled shrinking modality we will move into the image space $\mathcal{S}_F$ of $\mathcal{S}$.

We define the semantics for controlled subset spaces as follows.

Definition 5.4.2 (Semantics of Controlled Subset Spaces). *For a controlled subset space $\mathcal{S} = \langle S, \sigma, v, \mathcal{F} \rangle$ and for $F \subseteq \mathcal{F}$, the semantics is defined as follows.*

$s, U \models_{\mathcal{S}} p$	*iff*	$v(p) \in U$ *for each propositional variable* p
$s, U \models_{\mathcal{S}} \neg\varphi$	*iff*	$s, U \not\models_{\mathcal{S}} \varphi$
$s, U \models_{\mathcal{S}} \varphi \wedge \psi$	*iff*	$s, U \models_{\mathcal{S}} \varphi$ *and* $s, U \models_{\mathcal{S}} \psi$
$s, U \models_{\mathcal{S}} \mathsf{K}\varphi$	*iff*	$t, U \models_{\mathcal{S}} \varphi$ *for each* $t \in U$
$s, U \models_{\mathcal{S}} \Box\varphi$	*iff*	$s, V \models_{\mathcal{S}} \varphi$ *for each* $V \subseteq U$
$s, U \models_{\mathcal{S}} [F]\varphi$	*iff*	$s, fU \models_{\mathcal{S}_F} \varphi$ *for each* $f \in F$

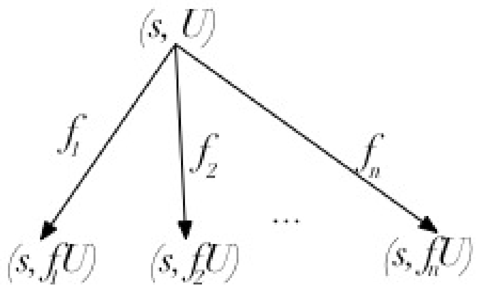
Let L be the dual of K and $\Diamond$ be the dual of $\Box$ defined in the usual way. The dual of $[F]$ will be defined as follows:

$$s, U \models_{\mathcal{S}} \langle F \rangle \varphi \quad \textit{iff} \quad s, fU \models_{\mathcal{S}_F} \varphi \textit{ for some } f \in F$$

In order to be able to evaluate formulae with $[F]$ modality, we will need a corresponding neighborhood situation in the image space $\mathcal{S}_F$. Therefore, the functions f in F will be chosen in such a way that the reference point s will surely be in the image of any of its neighborhoods. In other words, we will not alter our reference point after shrinking the neighborhood in a controlled manner. We furthermore *control* the shrinking in such a way that we will not lose our reference point s and so that s will still be in the image set fU for each $f \in F$. This is the basic motivation behind the controlled shrinking. We are increasing our knowledge by following some methods in such a way that after applying this method (or these methods), our reference point will still be *meaningful*. Because otherwise, we did not follow the aforementioned methodology as they would have make our reference point *meaningless* after their application.

For notational clarity, if F is singleton $F = \{f\}$, instead of $[F]$ we will write $[f]$.

For a picture of the concept of controlled shrinking see the picture.



5.4.2 Some Properties

We are far from offering an axiomatization for controlled subset spaces. Yet, we are still willing to make some observations regarding the modal properties of $[F]$.

1. $[F](\varphi \to \psi) \to ([F]\varphi \to [F]\psi)$

   It is easy to see that $[F]$ modality realizes the **K** axiom

2. $[F][F]\varphi \to [F]\varphi$

   This axiom is valid if F is closed under function decomposition i.e., for each $f \in F$, we have $g, h \in F$ such that $f = gh$.

3. $[F]\varphi \to [F][F]\varphi$

   This axiom is valid if F is closed under function composition i.e., for each $f, g \in F$, we also have $fg \in F$.

4. $[F]\varphi \to \varphi$

   This axiom is valid if the identity function id_F is in F.

5. $\Box\varphi \to [F]\varphi$

   This axiom states that there may be some subsets of U which are not the image of fU for each $f \in F$. This axiom is the essence of our approach. We discard some subsets of U by observing the functions in F.

6. $\mathsf{K}[F]\varphi \to [F]\mathsf{K}\varphi$

   This is the cross axiom for $[F]$ and K

We leave it to the reader to justify above observations with simple yet enjoyable proofs.

In this section, in conclusion, we pointed out that the semantics of subset spaces is strong and powerful enough to formalize several notions in many areas of formal sciences. The additional control operator $[F]$ is introduced in order to be able to distinguish the shrinking methods and henceforth we did some simple observations on the control modality $[F]$.

Chapter 6

Multi-Agent Subset Space Logic

6.1 Introduction

Moss & Parikh introduced the logic of subset spaces for the single agent case. However, when the observation of more than one agents matter, we need to extend the logic in a way that, it can talk about more than one agents.

One of the first application of topologic framework to the multi-agent case was carried out in (Pacuit & Parikh, 2007) where the authors developed "a multi-agent epistemic logic with a communication modality". However, in the aforementioned article, the communication of the agents was represented by a graph rather than a subset space. Apart from the knowledge modality, they had the $\Box$ modality interpreted as "$\Box\varphi$ is true iff φ is true after every sequence of communications that respect the communication graph". Therefore, the relation between the effort modality in subset space logic and the communication modality in communication graphs is clear - that is the reason why we considered the logic of communication graphs is an application of topologic to the multi agent cases. However, as we already pointed out, the communication graphs are Kripke structures - not subset spaces. We thus, refer the interested reader to the original article for further information.

In this section, we will consider several different ways of combining knowledge in subset space logic. Before going into the details let us now fix some notation.

$\mathrm{COM}_{\mathsf{K}}$ will denote the commutativity property for the knowledge operator, that is $\mathsf{K}_1\mathsf{K}_2\varphi \leftrightarrow \mathsf{K}_2\mathsf{K}_1\varphi$ and likewise $\mathrm{COM}_{\Box}$ will denote the commutativity property for the shrinking operator that is $\Box_2\Box_1\varphi \leftrightarrow \Box_1\Box_2\varphi$. In a similar manner, $\mathrm{CHR}_{\mathsf{K}}$ will denote the Church-Russer property for the knowledge operator, that is $\mathsf{K}_1\mathsf{L}_2\varphi \leftrightarrow \mathsf{L}_2\mathsf{K}_1\varphi$ and $\mathrm{CHR}_\Box$ will denote the Church-Russer property for the shrinking operator $\Box_1\Diamond_2\varphi \leftrightarrow \Diamond_2\Box_1\varphi$.

6.2 Intersection of Observations

Consider a case in which we need to take into account that we need the observations which are common in all agents. For example, if we go back to the measuring the speed of cars example, we can very well consider a situation in which we need to take into account of the observations which are common in all officers.

The intersection subset frame $\mathcal{T} = \mathcal{S} \cap \mathcal{S}'$ is the frame $\mathcal{T} = \langle S, \tau \rangle$ where $\tau = \{U : U \in \sigma_1 \cap \sigma_2\}$. We assume that both $\mathcal{S}$ and $\mathcal{S}'$ are in the same language, so is $\mathcal{T}$.

Intersection spaces reason over the observations that both (all) agents have. Then we have the following observation.

Lemma 6.2.1. *If $s, U \models_{\mathcal{T}} \varphi$, then $s, U \models_{\mathcal{S}} \varphi$ and $s, U \models_{\mathcal{S}'} \varphi$ where φ does not contain $\Box$ or $\Diamond$.*

Proof. Obvious. ■

It should be underlined that $\Box\psi$ does not satisfy the above lemma. The reason is simple. Not each subsets of U in σ or in σ' might be in τ.

However, if we can restrict our attention to some specific class of topologic classes, we can get a more general result.

Definition 6.2.1 (Downward Closure)**.** *The topologic space $S = \langle S, \sigma \rangle$ is downward closed if $U \in \sigma$ and $V \subseteq U$, then $V \in \sigma$ for each U, V as such.*

Therefore we get the desired result.

Lemma 6.2.2. *For downward closed set U, if $s, U \models_{\mathcal{T}} \varphi$, then $s, U \models_{\mathcal{S}} \varphi$ and $s, U \models_{\mathcal{S}'} \varphi$ for each formula φin the language of subset space logic.*

Proof. Obvious. ■

Another way to define an intersection of observations can be given as follows.

Definition 6.2.2 (Direct Intersections)**.** *Given two subset space frames $\mathcal{S}_1 = \langle S, \sigma_1 \rangle$ and $\mathcal{S}_2 = \langle S, \sigma_2 \rangle$, the direct intersection of $\mathcal{S}_1$ and $\mathcal{S}_2$ is $\mathcal{S}_1 \sqcap \mathcal{S}_2 = \langle S, \tau \rangle$ where $\tau = \{X : X = U \cap V$ where $U \in \sigma_1$ and $V \in \sigma_2\}$.*

An immediate observation is the fact that if a formula is true at the neighborhood situation (s, U) in $\mathcal{S}_1$ and (s, V) in $\mathcal{S}_2$, then it is also true at $s, U \cap V$ in $\mathcal{S}_1 \sqcap \mathcal{S}_2$.

Proposition 6.2.1. *If* $s, U \models_{\mathcal{S}_1} \varphi$ *and* $s, V \models_{\mathcal{S}_2}$*, then* $s, U \cap V \models_{\mathcal{S}_1 \sqcap \mathcal{S}_2} \varphi$ *for each* $\varphi \in \mathcal{L}_S$*.*

Proof. Obvious. ■

A closer look at the direct intersection reveals that this is essentially the distributed knowledge (Fagin *et al.* , 1995). Therefore, in the context of direct intersections, we are only interested in the knowledge which are only obtained when the agent combine their knowledge. We stated the definition for two agents, but the generalization to n-agents is straightforward.

6.3 Product

A *product multi-agent subset frame* is a tuple $\mathcal{S} = \langle S, \prod_i \sigma_i \rangle$ where $\langle S, \sigma_i \rangle$ is a subset space frame for each agent i in the set of agents I. In this context each $\langle S, \sigma_i$ is in the language $\mathcal{L}_{\mathsf{K}_i, \Box_i}$. Therefore the product S, Π_i is in the language $\mathcal{L}_{\mathsf{K}_1, \Box_1, \ldots, \mathsf{K}_n, \Box_n}$ where n is the number of agents.

The reason we define the collection of subsets as *vectors* is because, at each state s, the agents might have a different set of observations. Hence, at a state s, we will have $\vec{U} = \langle U_1, U_2, \ldots, U_n \rangle$ for n agents where each U_i lies in σ_i. In other words, the vector $\vec{U}$ is the ordered tuple of the observations of each agent where U_i is the observation of agent i. Furthermore, we will call $(s, \vec{U})$ a *neighborhood situation* if $s \in U_i$ for each i. The neighborhood situation $(s, \vec{U})$ then is the merge of the neighborhood situations $\{(s, U_1), (s, U_2), \ldots, (s, U_n)\}$. Observe that all these observations (i.e. the subsets U_i) are neighborhoods about the given point s in their respective frames.

Let us illustrate the notion of vectoral neighborhood situation with an example. Recall the example of a policeman who measures the speeds of the cars. Let us now have two policemen 1 and 2 measuring the speeds of cars simultaneously. Assume again that the speed limit is 50 mph. Let us further suppose that policeman 1 can makes the measurement with an error range of 2 mph whereas policeman 2 makes the measurement with an error range of 3 mph. Consider the situation that a car with a speed of 51 mph passed by the both policeman. Policeman 1 measured the car's speed as $(49, 53)$ while for policeman 2 the speed was in the interval $(48, 54)$. Take the point 52 in both intervals and consider the following statement:

$52, \langle (49, 53), (48, 54) \rangle \models \mathsf{K}_2\varphi$. Intuitively, this statement says that the policeman, based on his observation $(48, 54)$ knows that φ holds at the point 52. Similarly, we also have $52, \langle (49, 53), (48, 54) \rangle \models \mathsf{K}_1\varphi$. However, as for the policeman 2, the points in the interval $(48, 54)$ are indistinguishable, he can as well move to the point 53.4. However, this point does not lie in the set of observations of policeman 1. Hence, $53.4, \langle (49, 53), (48, 54) \rangle$ is not a neighborhood situation and we cannot interpret $\mathsf{K}_1\varphi$ at $53.4, \langle (49, 53), (48, 54) \rangle$

We can define the satisfaction relation $\models$ on by recursion for $s \in S$, $U \in \Pi_i \sigma_i$, $p \in P$, $i \in I$, φ and ψ in $\mathcal{L}$, we have the following semantics for n agents.

$s, \vec{U} \models p$	if and only if	$p \in v(A)$
$s, \vec{U} \models \varphi \wedge \psi$	if and only if	$s, \vec{U} \models \varphi$ and $s, \vec{U} \models \psi$.
$s, \vec{U} \models \neg\varphi$	if and only if	$s, \vec{U} \not\models \varphi$.
$s, \vec{U} \models \mathsf{K}_i\varphi$	if and only if	$t, \vec{U} \models \varphi$ for all $t \in \cap_i U_i \in \sigma_i$.
$s, \vec{U} \models \Box_i\varphi$	if and only if	$s, \vec{V} \models \varphi$ for all $\vec{V}$ where $U_j = V_j$ for $j \neq i$, and $V_i \subseteq U_i$.

The duals can be defined easily. Let $\mathsf{L}_i\varphi$ be defined as $\neg\mathsf{K}_i\neg\varphi$ and $\Diamond_i\varphi$ be defined as $\neg\Box_i\neg\varphi$. Therefore, the corresponding semantics is given as follows.

$s, \vec{U} \models \mathsf{L}_i\varphi$	if and only if	$t, \vec{U} \models \varphi$ for some $t \in \cap_i U_i \in \sigma_i$.
$s, \vec{U} \models \Diamond_i\varphi$	if and only if	$s, \vec{V} \models \varphi$ for some $\vec{V}$ where $U_j = V_j$ for $j \neq i$, and $V_i \subseteq U_i$.

As we emphasized earlier, if we have the neighborhood situation $s, \vec{U}$ for $\vec{U} = \langle U_1, \ldots, U_n \rangle$ in the product then we have $s \in U_i$ for each agent i.

COM and CHR It is easy to see that in product subset spaces $\mathrm{COM}_{\mathsf{K}}$ and $\mathrm{COM}_{\Box}$ together with $\mathrm{CHR}_{\mathsf{K}}$ and $\mathrm{CHR}_{\Box}$ are valid.

Proposition 6.3.1. *$\mathrm{COM}_{\mathsf{K}}$ and $\mathrm{COM}_{\Box}$ together with $\mathrm{CHR}_{\mathsf{K}}$ and $\mathrm{CHR}_{\Box}$ are valid in product subset spaces.*

Proof. Easy. ∎

6.4 Multi-Product

The standard way of capturing the multi-dimensionality in mathematics is the formation of Cartesian products. The construction of direct products are apparent, for instance in topology and algebra when the formation of multi-dimensional vector spaces or topological spaces etc. are considered.

We will follow the very same intuition to form the multi-product of topologic logics. Now, let $\mathbb{S}_1$ and $\mathbb{S}_2$ be two topologic logics.

The product of the frames $\mathcal{S}_1 = \langle S_1, \sigma_1 \rangle$ and $\mathcal{S}_2 = \langle S_2, \sigma_2 \rangle$ is the frame $\mathcal{F}_1 \times \mathcal{F}_2 = \langle S_1 \times S_2, \sigma_h, \sigma_v \rangle$ in which for each $u, u' \in U$ and $v, v' \in V$ and for each $U' \subseteq U$ and $V' \subseteq V$, we then have,

$$
\begin{array}{lll}
(u,v), U \times V \models \mathsf{K}_1\varphi & \text{iff} & \text{for all } u' \in U, \text{ we have } (u',v), U \times V \models \varphi \\
(u,v), U \times V \models \Box_1\varphi & \text{iff} & \text{for all } u \in U' \subseteq U, \text{ we have } (u,v), U' \times V \models \varphi \\
\\
(u,v), U \times V \models \mathsf{K}_2\varphi & \text{iff} & \text{for all } v' \in V, \text{ we have } (u,v'), U \times V \models \varphi \\
(u,v), U \times V \models \Box_2\varphi & \text{iff} & \text{for all } v \in V' \subseteq V, \text{ we have } (u,v), U \times V' \models \varphi
\end{array}
$$

Similar to the topological semantics of basic modal logic, hereby we will call K_1 and $\Box_1$ the horizontal topologic modalities and, K_2 and $\Box_2$ the vertical topologic modalities.

In this context $((u,v), U \times V)$ is a neighborhood situation in $\mathcal{S}_1 \times \mathcal{S}_2$ if (u, U) and (v, V) are neighborhood situations in $\mathcal{S}_1$ and $\mathcal{S}_2$ respectively.

Now let us consider the formulae COM and CHR whether they are valid in horizontal and vertical topologic product logics.

Proposition 6.4.1. *COM*$_{\mathsf{K}}$ *(*$\mathsf{K}_1\mathsf{K}_2\varphi \leftrightarrow \mathsf{K}_2\mathsf{K}_1\varphi$*) and COM*$_{\Box}$ *(*$\Box_2\Box_1\varphi \leftrightarrow \Box_1\Box_2\varphi$*) are valid in topologic products. Moreover, they can be generalized to* n*-agents case, i.e.,* $\mathsf{K}_1\mathsf{K}_2 \ldots \mathsf{K}_n\varphi \leftrightarrow \mathsf{K}_{\pi(1)}\mathsf{K}_{\pi(2)} \ldots \mathsf{K}_{\pi(n)}\varphi$ *and* $\Box_1\Box_2 \ldots \Box_n\varphi \leftrightarrow \Box_{\pi(1)}\Box_{\pi(2)} \ldots \Box_{\pi(n)}\varphi$ *hold in product topologic spaces where* π *is any permutation function, valid in topologic product frames.*

Proof. Assume $(u,v), U \times V \models \mathsf{K}_1\mathsf{K}_2\varphi$. For each u' in U, we have $(u',v), U \times V \models \mathsf{K}_2\varphi$, and consequently for each v' in V we then have $(u',v'), U \times V \models \varphi$. Now, $(u,v'), U \times V \models \mathsf{K}_1\varphi$ is obtained. Finally, $(u,v), U \times V \models \mathsf{K}_2\mathsf{K}_1\varphi$. Converse direction is very similar.

Now, assume $(u,v), U \times V \models \Box_1\Box_2\varphi$. For each U' in U, we have $(u,v), U' \times V \models \Box_2\varphi$, and consequently for each V' in V we then have $(u,v), U' \times V' \models \varphi$. Now, we can go backwards. Therefore, $(u,v), U \times V' \models \Box_1\varphi$ is obtained. Finally, $(u,v), U \times V \models \Box_2\Box_1\varphi$. Converse direction is very similar.

Generalization to n-agents case is a straight forward induction on n.

We also note that, we can permute the modalities $\Box$ and K as the universal quantifiers commute. ■

In a similar fashion, CHR is also valid in topologic products for both modalities.

Proposition 6.4.2. *CHR_{K} ($\mathsf{K}_1\mathsf{L}_2\varphi \leftrightarrow \mathsf{L}_2\mathsf{K}_1\varphi$) and $CHR_{\Box}$ ($\Box_1\Diamond_2\varphi \leftrightarrow \Diamond_2\Box_1\varphi$) are valid in topologic products.*

Proof. Assume $(u,v), U \times V \models \mathsf{K}_1\mathsf{L}_2\varphi$. Then, for each u' in U, we have $(u',v), U\times V \models \mathsf{K}_2\varphi$, and consequently there is a v' in V such that $(u',v'), U\times V \models \varphi$. Now, $(u',v), U\times V \models \mathsf{L}_2\varphi$ is obtained. Finally, $(u,v), U\times V \models \mathsf{K}_1\mathsf{L}_2\varphi$.

The converse direction is similar.

Now, assume $(u,v), U \times V \models \Box_1\Diamond_2\varphi$. Then, for each $U' \subseteq U$, we have $(u,v), U' \times V \models \Diamond_2\varphi$, and consequently there is a $V' \subseteq V$ such that $(u,v), U' \times V' \models \varphi$. Now, $(u,v), U \times V' \models \Box_1\varphi$ is obtained. Finally, $(u,v), U \times V \models \Diamond_2\Box_1\varphi$.

The converse direction is similar. ■

It is also obvious that the horizontal cross axiom and vertical cross axiom are valid in topologic products.

Proposition 6.4.3. *The horizontal cross axiom $\mathsf{K}_1\Box_1\varphi \to \Box_1\mathsf{K}_1\varphi$ and vertical cross axiom $\mathsf{K}_2\Box_2\varphi \to \Box_2\mathsf{K}_2\varphi$ are valid in topologic direct products.*

Proof. Let us focus on horizontal cross axiom as the vertical cross axiom is the symmetric of it.

Assume $(u,v), U \times V \models \mathsf{K}_1\Box_1\varphi$. Then, for each $u' \in U$, we have $(u',v), U\times V \models \Box_1\varphi$, and consequently for each $U' \subseteq U$ we have $(u',v), U'\times V \models \varphi$. As the first component of the product subset space is also a subset space, it validates the cross axiom. Thus, $\Box_1\mathsf{K}_1\varphi$ holds at $(u,v), U \times V \models \mathsf{K}_1\Box_1\varphi$. ■

Another observation is about the commutativity of horizontal and vertical modalities.

Proposition 6.4.4. *$\mathsf{K}_1\Box_2\varphi \leftrightarrow \Box_2\mathsf{K}_1\varphi$ and $\mathsf{K}_2\Box_1\varphi \leftrightarrow \Box_1\mathsf{K}_2\varphi$ are valid in topologic direct products.*

Proof. Straight forward by using the above ideas. ■

Previously, we lifted the individual modalities of the each component of the product to the product subset spaces. However, one can also define a direct product of subset spaces.

Definition 6.4.1 (Direct Multi-Product of Topologic Frames)**.** *Let $\mathcal{S} = \langle S, \sigma\rangle$ and $\mathcal{S}' = \langle S', \sigma'\rangle$ be given. The direct product of $\mathcal{S}$ and $\mathcal{S}'$ is $\mathcal{S} \otimes \mathcal{S}' = \langle S \times S', \sigma \times \sigma'\rangle$.*

Neighborhood situations in $\mathcal{S} \otimes \mathcal{S}'$ are defined similarly. Thus, $(u, v), U \times V$ is a neighborhood situation in $\mathcal{S} \otimes \mathcal{S}'$ if both (u, U) is a neighborhood situation in $\mathcal{S}$ and (v, V) is a neighborhood situation in $\mathcal{S}'$.

We define the knowledge operator K and effort operator $\Box$ in the direct multi-product subset space as follows:

$$\begin{array}{lll} (u, v), U \times V \models \mathsf{K}\varphi & \text{iff} & \forall (u', v') \in U \times V \text{ then } (u', v'), U \times V \models \varphi \\ (u, v), U \times V \models \Box\varphi & \text{iff} & \forall (U' \times V') \subseteq (U \times V) \text{ then } (u, v), U' \times V' \models \varphi \end{array}$$

Duals can also be defined easily.

Proposition 6.4.5. *The cross axiom* $\mathsf{K}\Box\varphi \to \Box\mathsf{K}\varphi$ *holds in topologic direct products.*

Proof. Obvious. ∎

The modalities in direct topologic multi-products can be defined in terms of horizontal and vertical modalities easily.

Proposition 6.4.6. $\Box\varphi \leftrightarrow \Box_1\Box_2\varphi$ *and* $\mathsf{K}\varphi \leftrightarrow \mathsf{K}_1\mathsf{K}_2\varphi$ *are valid in direct topologic products.*

Proof. Obvious ∎

6.5 Common Knowledge

As it has been underlined, "common knowledge is a phenomenon which underwrites much of social life" (Vanderschraaf & Sillari, 2007). In this section, we will consider the notion of common knowledge.

6.5.1 What is Common Knowledge

Being a rather intuitive but significantly important concept in logic, common knowledge has a history that can be traced back to Hume according to (Vanderschraaf & Sillari, 2007). They indicated that,

> David Hume (1740) was perhaps the first to make explicit reference to the role of mutual knowledge in coordination. In his account of convention in A Treatise of Human Nature, Hume argued that a necessary condition for coordinated activity was that agents all know what behavior to expect from one another. Without the requisite mutual knowledge, Hume maintained, mutually beneficial social conventions would disappear.
>
> (Vanderschraaf & Sillari, 2007)

However, without much surprise it was David Lewis who gave the explicit analysis of common knowledge. Several additional account for giving the definition of common knowledge can be found in the literature, e.g. Aumann, Barwise etc.

There are plenty daily life examples explaining the concept of common knowledge. We follow from (Vanderschraaf & Sillari, 2007):

> Informally, a proposition A is mutually known among a set of agents if each agent knows that A. Mutual knowledge by itself implies nothing about what, if any, knowledge anyone attributes to anyone else. Suppose each student arrives for a class meeting knowing that the instructor will be late. That the instructor will be late is mutual knowledge, but each student might think only she knows the instructor will be late. However, if one of the students says openly "Peter told me he will be late again," then the mutally known fact is now commonly known. Each student now knows that the instructor will be late, and so on, ad infinitum. The agents have common knowledge (...).
>
> (Vanderschraaf & Sillari, 2007)

There are plenty of applications of common knowledge in the literature. The one we feel like stressing is due to Aumann (Aumann, 1976). He proved that when agents' beliefs are formalized as probability distributions, then if these agents start with common prior beliefs, then they cannot "agree to disagree". For the formal details and the proof, the reader is referred to the aforementioned paper.

6.5.2 Formalizing Common Knowledge

Recall that in Kripkean setting, the common knowledge is usually defined as the reflexive and transitive closure of the union of the accessibility relations. For instance, consider the simple case for two agents 1 and 2, and their accessibility relation R_1 and R_2. Then in Kripke models, the common knowledge operator $\mathsf{C}_{1,2}\varphi$ is defined as follows:

$$\mathcal{M}, w \models \mathsf{C}_{1,2}\varphi \text{ iff } \mathcal{M}, v \models \varphi \text{ for each } (w, v) \in (R_1 \cup R_2)^*$$

where $\mathcal{M}$ is the model in question and w is a state in the model, and $(R_1 \cup R_2)^*$ is the reflexive and transitive closure of the union $R_1 \cup R_2$. Intuitively, reflexive and transitive closure means that if $(w, v) \in (R_1 \cup R_2)^*$, then there is a finite sequence of successive steps from w to v via either of the two accessibility relations.

However, this construction will not work in subset spaces. Therefore we need to reconstruct the notion of common knowledge from the scratch. In the literature, there are two approaches to this problem. One is due to (Moss *et al.* , 2007) and the other is due to (Heinemann, 2006b). In other to get to these results, we will proceed step by step. We will first consider the mutual knowledge and distributed and knowledge and then move into common knowledge.

Definition 6.5.1 (Mutual Knowledge)**.**
$s, \langle U_1, \dots, U_n \rangle \models \mathsf{E}\varphi$ *iff* $s, \langle U_1, \dots, U_n \rangle \models \mathsf{K}_i\varphi$ *for each agent* i.

Mutual knowledge operator, therefore, indicates the information which is known by each agent. Hence, it is not difficult to see that if φ is a mutual knowledge, then for each $t_i \in U_i$ we have $t_i, U_i \models \varphi$ where $i \in \{1, \dots, n\}$.

Definition 6.5.2 (Distributed Knowledge)**.**
$s, \langle U_1, \dots, U_n \rangle \models \mathsf{D}\varphi$ *iff* $s, \bigcap_{1 \leq i \leq n} U_i \models \varphi$ *for each agent* i

Distributed knowledge, then indicates the information known after each agent combined their knowledge.

An straight forward observation about distributed knowledge is the following.

Lemma 6.5.1. $s, \langle U_1, \dots, U_n \rangle \models \mathsf{D}\varphi$ *implies* $s, U_i \models \mathsf{L}\varphi$ *and* $s, U_i \models \Diamond\varphi$ *for each agent* i.

Proof. For an arbitrary agent i, observe that $\bigcap_{1 \leq i \leq n} U_i \subseteq U_i$. Then the result follows easily. ∎

As we already emphasized there are two suggestions to define common knowledge in subset spaces. The first approach which was put forward in (Moss *et al.* , 2007) uses the basic language whereas the one in (Heinemann, 2006b) uses the extended language of subset space logic with overlap operator.

Common Knowledge in Basic Subset Space Language In (Benthem & Sarenac, 2004), the common knowledge is discussed in the setting of topological interpretation of modal logic. They consider several approaches to common knowledge. One of these approaches is the infinite iteration definition of common knowledge which can be given as $\mathsf{C}\varphi$ is $\varphi \wedge \mathsf{I}\varphi \wedge \mathsf{II}\varphi \wedge \dots$.

Recall now the translation map $^{\mathsf{t}}$ from the language of basic modal language $\mathcal{L}$ to the language of subset space logic$\mathcal{L}_S$ we discussed in Section 3.2. Then, under this translation the common knowledge operator reduces to the following:

$$\mathsf{C}\varphi \equiv \varphi \wedge \Diamond\mathsf{K}\varphi \wedge \Diamond\mathsf{K}\Diamond\mathsf{K}\varphi \dots$$

Common Knowledge in Extended Subset Space Language We can use the idea of common knowledge in Kripke semantics in the context of extended subset space logic. For this purpose of ours, we need to be able to move between neighborhood situations. Hence, we will use the overlap operator O for this discussion.

The precise definition of common knowledge for the subset space $\langle S, \sigma, v\rangle$ is given as follows (Heinemann, 2006b).

$$\begin{aligned} s, U \models \mathsf{C}\varphi \quad := \quad & \forall n \in \mathbb{N} \text{ and } t \in S, \text{ we then have:} \\ & \text{if } U_0, U_1, \ldots, U_n \in \sigma \text{ satisfy } U_0 = U \text{ and } U_i \cap U_{i+1} \neq \emptyset \\ & \text{for } i = 0, \ldots, n-1 \text{ and, } t \in U_n, \text{ then } t, U_n \models \varphi \end{aligned}$$

This is also an iteration definition of common knowledge. This definition states that, step by step we will first change our current state in the same neighborhood and then change the neighborhood by keeping the new point, and then repeat the same process with the new point and its new neighborhood situation. Note that, we alter the current point by K modality and alter the current neighborhood by O modality. Therefore this definition simplifies to the following equation.

$$s, U \models \mathsf{C}\varphi \equiv s, U \models \underbrace{\mathsf{KO} \ldots \mathsf{KO}}_{n-\text{times}} \varphi$$

6.6 Common Knowledge on Topologic PAL

On the other hand, remark that PAL with common knowledge operator does not have a reduction axiom for the formulae of the form $[\varphi]\mathsf{C}_{1,2}\psi$ (Benthem *et al.* , 2005). But, what about topologic PAL?

In order to discuss the common knowledge operator in the context of public announcement logic in subset spaces, we will start with *relativized common knowledge* and then proceed into common knowledge step by step.

6.6.1 Relativized Common Knowledge

In Kripke semantics, relativized common knowledge $\mathsf{C}(\varphi, \psi)$ states that in the model, "each path which consists exclusively of φ-worlds ends in a ψ world" (Benthem *et al.* , 2005).

We can adopt the definition of common knowledge given in Section 6.5.2 to the context of relativized common knowledge.

$$
\begin{aligned}
s, U \models \mathsf{C}(\varphi, \psi) \quad := \quad & \forall n \in \mathbb{N} \text{ and } t \in S, \text{ we then have:} \\
& \text{if } U_{0_\varphi}, \ldots, U_{n_\varphi} \in \sigma \text{ satisfy } U_{0_\varphi} = U \text{ and } U_{i_\varphi} \cap U_{(i+1)_\varphi} \neq \emptyset \\
& \text{for } i = 1, \ldots, n-1, \text{ and } t \in U_{n_\varphi}, \text{ then } t, U_{n_\varphi} \models \psi
\end{aligned}
$$

where U_{i_φ} is $U_i \cap (\varphi)$.

In other words, we travel through the sets in which φ holds, and at each step we end up with a neighborhood situation in which ψ holds.

Observe that $\mathsf{C}(\varphi) \equiv \mathsf{C}(\top, \varphi)$.

Then we have the following more intuitive definition for the subset space $\mathcal{S} = \langle S, \sigma, v \rangle$.

$$s, U \models \mathsf{C}(\varphi, \psi) \equiv s, U_\varphi \models \underbrace{\mathsf{KO} \ldots \mathsf{KO}}_{n-\text{times}} \psi$$

Now, it appears that, we extended the language of topologic PAL with overlap operator. Thus, the language of topologic public announcement logic with additional relativized common knowledge and overlap operators interpreted in subset spaces is given as follows:

$$p \mid \bot \mid \neg\varphi \mid \varphi \wedge \psi \mid \Box\varphi \mid \mathsf{K}\varphi \mid [\varphi]\psi \mid \mathsf{C}(\varphi, \psi) \mid \mathsf{O}\varphi$$

Recall that we already observed that the PAL with the overlap operator is complete. See Theorem 4.4.4 for the details.

In (Benthem *et al.* , 2005), a reduction axiom for relativized common knowledge operator is suggested in Kripke semantics. The axiom is as follows

$$[\varphi]\mathsf{C}(\psi, \chi) \leftrightarrow (\varphi \to \mathsf{C}(\varphi \wedge [\varphi]\psi, [\varphi]\chi))$$

The statement $[\varphi]\mathsf{C}(\psi, \chi)$ simply expresses that after the public announcement of φ, every ψ path leads to a χ world. In other words $[\varphi]\mathsf{C}(\psi, \chi)$ holds in those worlds where each $\varphi \wedge [\varphi]\psi$ path ends in a worlds where $[\varphi]\chi$ holds.

We claim the very same reduction axiom also holds in topologic PAL. Hence the next theorem follows.

Theorem 6.6.1 (Reduction Axiom for Relativized Common Knowledge Operator)**.** *$[\varphi]\mathsf{C}(\psi, \chi) \leftrightarrow (\varphi \to \mathsf{C}(\varphi \wedge [\varphi]\psi, [\varphi]\chi))$ is sound.*

Proof. Assume $s, U \models [\varphi]\mathsf{O}\psi$. Suppose further that $s, U \models \varphi$. Then we have,

$$
\begin{aligned}
s, U \models [\varphi]\mathsf{C}(\psi, \chi)\psi \quad & \text{iff} \quad s, U_\varphi \models \mathsf{C}(\psi, \chi) \\
& \text{iff} \quad s, U_{\varphi_\psi} \models \mathsf{KO} \ldots \mathsf{KO}\chi \\
& \text{iff} \quad s, U \models \mathsf{C}([\varphi]\psi, [\varphi]\chi) \\
& \text{iff} \quad s, U \models \mathsf{C}(\varphi \wedge [\varphi]\psi, [\varphi]\chi)
\end{aligned}
$$

■

The expected completeness theorem then comes.

Theorem 6.6.2 (Completeness of topologic PAL with Overlap and Relativized Common Knowledge)**.** *Topologic public announcement logic with overlap and relativized common knowledge operators is complete.*

Proof. Follows from the completeness of subset space logic extended with overlap operator and the completeness of public announcement logic with relativized common knowledge. For the aforementioned proofs, see (Heinemann, 2006b) and (Benthem *et al.* , 2005). ■

In conclusion, in this chapter we observed that the basic language of subset space logic is strong enough to express the public announcement logic. Once equipped with the overlap operator, then it becomes stronger and can expresses the relativized common knowledge operator. We then concluded this chapter with a straightforward completeness theorem.

Chapter 7

Conclusion

7.1 Recap of the Results

The first part of this work was rather intended to fill in the gaps in subset space logic by importing some simple notions from basic modal logic. We also showed the validity of these operations and furthermore introduced a game theoretical semantics for the subset space logic.

In the second part, we applied the logic to a dynamic logic - namely public announcement logic. As emphasized, we proceeded further to obtain several more nice results. Furthermore, we did some intuitive observations about the *nature* of shrinking operator.

However, we cannot claim that our work is complete. There are still some open problems and possible research directions that can be carried out further.

7.2 Open Problems / Future Work

The multi-agent version of subset space is still not very well developed and maybe even not very promising[1].

Several reasons can be given to account for these obstacles. First of all, it is not very clear how the agents should merge their information in multi-agent case. The behavior of neighborhood situations in multi-agent case is not as straight forward as in the case of Kripke structures. Second, as we noted, subset spaces do not diverge much from the behavior of Kripke structures in multi-agent case.

[1]Bernard Heinemann: Personal communication, May 2007

As we indicated, subset spaces can be interpreted in two different structures: subset spaces and cross axioms. However, the relation between cross axiom frames and subset frames is not clear. It is especially vague how to obtain a subset frame from cross axiom frame.

Another vagueness we did not shed light on is the size of collection of the observation sets. In other words, it is not clear if we are supposed to consider the all possible observations or some selected or given collection of observations for a given set. This does not change anything as the technical results do not depend on this. However, from a semantical point of view, we believe, an agent cannot possibly consider the all possible observations in any case. As each agent has time, effort etc. limits, she cannot possibly consider all possible observations. In other words, she is limited to her abilities and options. This was our underlying starting point to formalize the concept of controlled shrinking.

As we briefly pointed out, the elimination of observation sets - based on the agent's limitations and preferences - should be investigated deeply. The easiest way we can think of is utilizing functions for this purpose.

As a future work, the complexity of various multi-agent subset spaces can be considered as well[2]. This was one of the aspects we did not even touch in this work.

In addition to that, it is also possible to extend the language with the universal modalities E and A in order to increase the expressivity. However, we did not consider this case as it was not entirely in our scope.

In conclusion, several directions for further research are, in our opinion, apparent. One is for expressive strength of the logic and the complexity issues of the extended languages of subset space logics. The second research areas is the dynamic aspects of subset space logics. We believe that in this some simple aspects of these two directions have been pointed out.

[2]Bernard Heinemann: Personal communication, May 2007

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