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Geometric Public Announcement Logics

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Introduction

In this paper, we consider public announcement logic (PAL, henceforth) in several different geometric models, and prove its completeness of those models. Moreover, we also consider some applications of our ideas in different fields varying from game theory to epistemic logic. What makes our work novel is the fact that PAL has never been investigated in geometric and topological models with further applications.

Topological Semantics for Modal Logic

Recall that *a topological space* $\mathcal{S} = \langle S, \sigma \rangle$ is a structure with a set S and a collection σ of subsets of S satisfying the following axioms:

1. The empty set and S are in σ,
2. The union of any collection of sets in σ is also in σ
3. The intersection of a finite collection of sets in σ is also in σ

Our main operator is *interior* operator $\mathbb{I}$ which returns the interior of a given set. A topological model $\mathcal{M}$ is a triple $\langle S, \sigma, v \rangle$ where $\mathcal{S} = \langle S, \sigma \rangle$ is a topological space, and v is a valuation function. The basic modal language $\mathcal{L}$ has a countable set of proposition letters P, a truth constant $\top$, the usual Boolean operators $\neg$ and $\wedge$, and a modal operator $\Box$. When we are in topological models, we will use the symbol I for $\Box$, and C for $\Diamond$ after the *interior* and *closure* operators respectively. We will call the set of points that satisfy φ in $\mathcal{M}$ *the extension* of φ, and denote as $(\varphi)^{\mathcal{M}}$. The extension of a modal formula in model $\mathcal{M}$, then, is given as follows $(\mathsf{I}\varphi)^{\mathcal{M}} = \mathbb{I}((\varphi)^{\mathcal{M}})$. Now, the model semantics of modal operators in topological models is given as follows.

$$\begin{array}{lll} \mathcal{M}, s \models \mathsf{I}\varphi & \text{iff} & \exists U \in \sigma.(s \in U \wedge \forall t \in U, \\ & & \mathcal{M}, t \models \varphi) \\ \mathcal{M}, s \models \mathsf{C}\varphi & \text{iff} & \forall U \in \sigma.(s \in U \rightarrow \exists t \in U, \\ & & \mathcal{M}, t \models \varphi) \end{array}$$

It is well known that the basic modal logic is complete with respect to the given semantics.

Subset Space Logic

Subset space logic (SSL, henceforth) was presented in early 90s as a bimodal logic to formalize reasoning about sets and points (Moss and Parikh 1992). The language of SSL has two modal operators K and $\Box$. A subset space model is a triple $\mathcal{S} = \langle S, \sigma, v \rangle$ where S is a non-empty set, σ is a collection of subsets (*not* necessarily a topology), v is a valuation function. Semantics of SSL for modal operators is given inductively as follows.

$$\begin{array}{llll} s, U \models \mathsf{K}\varphi & \text{iff} & t, U \models \varphi & \text{for all } t \in U \\ s, U \models \Box\varphi & \text{iff} & s, V \models \varphi & \text{for all } V \in \sigma \text{ such} \\ & & & \text{that } s \in V \subseteq U \end{array}$$

The axioms of SSL are as follows: K operator is S5, the $\Box$ operator is S4, and $\mathsf{K}\Box\varphi \rightarrow \Box\mathsf{K}\varphi$. Moreover, SSL is complete and decidable (Moss and Parikh 1992).

Public Announcement Logic

Public announcement logic is a way to represent changes in knowledge. A truthful announcement φ is made, and consequently, the agents updates their epistemic states by eliminating the possible states where φ is false (Plaza 1989). Notationwise, the formula $[\varphi]\psi$ is intended to mean that *after the public announcement of* φ, ψ *holds*. The language of PAL will be that of multi-agent (multi-modal) epistemic logic with an additional public announcement operator $[*]$. Let $\mathcal{M} = \langle W, \{R\}_{i \in I}, V \rangle$ be a Kripke model. For modal operators, we have the following semantics.

$$\begin{array}{ll} \mathcal{M}, w \models \mathsf{K}_i\varphi & \text{iff } \mathcal{M}, v \models \varphi \text{ for each } v \\ & \quad \text{such that } (w, v) \in R_i \\ \mathcal{M}, w \models [\varphi]\psi & \text{iff } \mathcal{M}, w \models \varphi \text{ implies } \mathcal{M}|\varphi, w \models \psi \end{array}$$

Here, $\mathcal{M}|\varphi$ is defined as $\langle W', \{R'_i\}_{i \in I}, V' \rangle$ where $W' = W \cap (\varphi)^{\mathcal{M}}$; $R'_i = R_i \cap (W' \times W')$, and $V'(p) = V(p) \cap W'$. Moreover, PAL is complete and decidable (Plaza 1989).

Subset Space PAL

Let $\mathcal{S} = \langle S, \sigma \rangle$. Announcement of φ gives $\mathcal{S}_\varphi = \langle S|\varphi, \sigma_\varphi \rangle$ where $S|\varphi = (\varphi)$ and $\sigma_\varphi = \{U_\varphi : U_\varphi = U \cap (\varphi) \neq \emptyset$, for each $U \in \sigma\}$. Then the corresponding semantics can be suggested as follows:

$$s, U \models [\varphi]\psi \text{ iff } s, U \models \varphi \text{ implies } s, U_\varphi \models \psi$$

The language of the topologic public announcement logic interpreted in subset spaces is given as follows:

$$p \mid \bot \mid \neg\varphi \mid \varphi \wedge \psi \mid \Box\varphi \mid \mathsf{K}\varphi \mid [\varphi]\psi$$

Theorem 1. *Axioms of the basic PAL are sound in subset space logic.*

The following axiomatize the topologic-PAL along with the axiomatization of SSL:

1. $[\varphi]p \leftrightarrow (\varphi \rightarrow p)$
2. $[\varphi]\neg\psi \leftrightarrow (\varphi \rightarrow \neg[\varphi]\psi)$
3. $[\varphi](\psi \wedge \chi) \leftrightarrow ([\varphi]\psi \wedge [\varphi]\chi)$
4. $[\varphi]\mathsf{K}\psi \leftrightarrow (\varphi \rightarrow \mathsf{K}[\varphi]\psi)$
5. $[\varphi]\Box\psi \leftrightarrow (\varphi \rightarrow \Box[\varphi]\psi)$

Theorem 2. *Topologic PAL is complete and decidable.*

Topological PAL

Let $\mathcal{T} = \langle T, \tau, v\rangle$ be a topological model and φ be an announcement. Define $\mathcal{T}_\varphi = \langle T_\varphi, \tau_\varphi, v_\varphi\rangle$ where $T_\varphi = T \cap (\varphi)$, $\tau_\varphi = \{O \cap T_\varphi : O \in \tau\}$ and $v_\varphi = v \cap T_\varphi$. Observe that τ here is a topology. The semantics for the public announcements in topological models is then given as follows.

$$\mathcal{T}, s \models [\varphi]\psi \text{ iff } \mathcal{T}, s \models \varphi \text{ implies } \mathcal{T}_\varphi, s \models \psi$$

The reduction axioms for PAL in topological spaces are given as follows.

1. $[\varphi]p \leftrightarrow (\varphi \rightarrow p)$
2. $[\varphi]\neg\psi \leftrightarrow (\varphi \rightarrow \neg[\varphi]\psi)$
3. $[\varphi](\psi \wedge \chi) \leftrightarrow ([\varphi]\psi \wedge [\varphi]\chi)$
4. $[\varphi]\mathsf{I}\psi \leftrightarrow (\varphi \rightarrow \mathsf{I}[\varphi]\psi)$

Theorem 3. *Topological PAL is complete and decidable.*

Applications

Announcement Stabilization For a model M and a formula φ, we define the announcement limit $\lim_\varphi M$ as the first model which is reached by successive announcements of φ that no longer changes after the last announcement is made. Announcement limits exist in both finite and infinite models (van Benthem and Gheerbrant 2010). In topological models, we observe the following.

Theorem 4. *For some formula φ and some topological model M, it may take more than ω stage to reach the limit model $\lim_\varphi M$.*

Theorem 5. *Limit models exist in topological models.*

Therefore, Kripke models and topological models differ in announcement stabilization.

Backward Induction (BI) Consider the backward induction solution for games where the player traces back his moves to develop a winning strategy. Notice that the Aumann's BI solution requires common knowledge of rationality (Aumann 1995). Recently, it has been shown that in any game tree model M taken as a PAL model, $\lim_{\mathsf{rational}} M$ is the actual subtree computed by the BI procedure where the proposition $\mathsf{rational}$ means that "at the current node, no player has chosen a strictly dominated move in the past coming here" (van Benthem and Gheerbrant 2010). Therefore, the announcement of node-rationality produces the same result as the backward induction procedure. Each backward step in the BI procedure can then be obtained by the public announcement of node rationality. However, there seems to be a problem. The admissibility of limit models can take more than ω steps in topological models. Therefore, based on the theorem we just stated, the BI procedure can take more than ω steps.

Theorem 6. *In topological models of games, under the assumption of rationality, the backward induction procedure can take more than ω steps.*

This is indeed a problem about the attainability in infinite games: how can a player continue playing the game when she hit the limit ordinal ω-th step in the backward induction procedure?

Persistence Let us now discuss stabilization in SSL framework. We already have a similar notion within the SSL context. A *persistent* formula in a model M is the formula φ whose truth is independent from the subsets in M. In other words, φ is persistent if for all states s and subsets $V \subseteq U$, we have $s, U \models \varphi$ impies $s, V \models \varphi$. Clearly, Boolean formulas are persistent in every model.

Theorem 7. *Let M be a model and φ be persistent in M. Then, for any formula χ and neighborhood situation (s, U), if $s, U \models$, then $s, U \models [\chi]\varphi$. In other words, true persistent formulas are immune to the public announcements.*

In other words, we can have some formulas in SSL framework that are immune to the announcements.

References

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