Some Non-Classical Methods in (Epistemic) Modal Logic and Games: A Proposal

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Abstract

In this proposal, we discuss several non-classical frameworks, and their applications in epistemic modal logic. We largely consider topological semantics, as opposed to widely used Kripke semantics, paraconsistent systems, as opposed to consistent systems, and non-well-founded sets, as opposed to ZF(C) set theory. We discuss topological public announcement logics, introduce homotopies to modal logic, reason about topological semantics for paraconsistent logics, and introduce non-well-founded type spaces for games. Finally, we discuss several popular game theoretical issues: the Brandenburger - Keisler paradox, and strategy logic whose primitives are strategies. Then, we conclude with future research directions.
1 Dynamic Epistemic Logic and Topologies

1.1 Introduction

Public announcement logic is a well-known example of dynamic epistemic logics (Plaza, 1989; van Ditmarsch et al., 2007). Dynamic epistemic logics set out to formalize knowledge and knowledge changes in (usually) multi-agent setting by defining different ways of model updates and interaction among the agents. The contribution of public announcement logic (PAL, henceforth) to the field of knowledge representation is mostly due to its succinctness and clarity in reflecting the simple intuition as to how epistemic updates work in some situation. Moreover, as it will be clear in the next section, PAL is not more expressiveness than the basic epistemic logic. PAL updates the model by the announcements made by a truthful agent who can be one of the agents/knowers or “God”. After the truthful announcement, the model is updated by eliminating the states that do not satisfy the announcement. Public announcement logic has many applications in the fields of formal approaches to social interaction, dynamic logics, knowledge representation and updates (Balbiani et al., 2008; Baltag & Moss, 2004; van Benthem, 2006; van Benthem et al., 2005). Extensive application of PAL to different fields and frameworks has made PAL a rather familiar framework to many researchers. Especially, analyzing games such as “muddy children” made PAL a rather popular framework.

Virtually almost all applications of PAL make use of Kripke models for knowledge representation. However, as it is very well known, Kripke models are not the only representational tool for modal and epistemic logics.

In this section, we will consider PAL in two different geometrical frameworks: topological modal logic and subset space logic. Topological models are not new to modal logics, indeed they are the first models for modalities (McKinsey & Tarski, 1944; McKinsey & Tarski, 1946; Goldblatt, 2006). The past decades have witnessed a revival of academic interest towards the topological models for modal logics in many different frameworks (Aiello et al., 2003; van Benthem & Bezhanishvili, 2007; van Benthem et al., 2006; Bezhanishvili & Gehrke, 2005). However, to the best of our knowledge, topological models have not been applied to dynamic epistemic logics. But, there has been some influential work on common knowledge in topological models which motivated the current work (van Benthem & Sarenac, 2004). In the aforementioned reference, it was shown that the different definitions of common knowledge diverge in topological models even though these definitions are equivalent in Kripke structures, based on Barwise’s earlier investigation (Barwise, 1988). Nevertheless, the authors did not seem to take the next immediate step to discuss dynamic epistemologies in that framework. This is one of our goals in this paper: to apply topological reasoning to dynamic epistemological cases and present the immediate completeness results. The second framework that we will discuss, subset space logic, is a rather weak yet expressive geometrical structure dispensing with the topological structure (Moss & Parikh, 1992; Parikh et al., 2007). Subset space logic has been introduced to reason about the notion of
closeness and effort in epistemic situations. In this paper, we will also define PAL in subset space logic with its axiomatization and present the completeness of PAL in subset space logics improving the results based on an earlier work (Başkent, 2007).

The present section is organized as follows. First, we will introduce the geometrical frameworks that we will need: topological spaces and subset spaces. Then, after a brief interlude on PAL, we will give the axiomatizations of PAL in such spaces, and their completeness. The completeness proofs are rather immediate - which is usually the case in PAL systems. Then, we will make some observations on PAL in geometric models. Our observations will be about the stabilization of updated models, backward induction in games and persistency.

1.2 Geometric Models

In this section, we will briefly recall the geometric models for some modal logics. What we mean by geometric models is topological models and subset space logic models as they are inherently geometrical structures. We will first start with topological models and their semantics, and then discuss subset space models.

1.2.1 Topological Semantics for Modal Logic

Topological interpretations for modal logic historically precede the relational semantics (McKinsey & Tarski, 1944; Goldblatt, 2006). Moreover, as we will observe very soon, topological semantics is arithmetically more complex than relational semantics. Let us start by introducing the definitions.

Definition 1. A topological space \( S = \langle S, \sigma \rangle \) is a structure with a set \( S \) and a collection \( \sigma \) of subsets of \( S \) satisfying the following axioms:

1. The empty set and \( S \) are in \( \sigma \).
2. The union of any collection of sets in \( \sigma \) is also in \( \sigma \).
3. The intersection of a finite collection of sets in \( \sigma \) is also in \( \sigma \).

The collection \( \sigma \) is said to be a topology on \( S \). The elements of \( S \) are called points and the elements of \( \sigma \) are called opens. The complements of open sets are called closed sets. Our main operator in topological spaces is called interior operator \( \downarrow \) which returns the interior of a given set. The interior of a set is the largest open set contained in the given set. A topological model \( M \) is a triple \( \langle S, \sigma, v \rangle \) where \( S = \langle S, \sigma \rangle \) is a topological space, and \( v \) is a valuation function sending propositional letters to subsets of \( S \), i.e. \( v: P \to \wp(S) \) for a countable set of propositional letters \( P \).

The basic modal language \( \mathcal{L} \) has a countable set of proposition letters \( P \), a truth constant \( \top \), the usual Boolean operators \( \neg \) and \( \land \), and a modal operator \( \Box \). The dual of \( \Box \) is denoted by \( \Diamond \) and defined \( \Box \varphi \equiv \neg \Diamond \neg \varphi \). When we are in topological models, we will use the symbol \( I \) for \( \Box \) after the interior operator for intuitive reasons, and to prevent any future confusion. Likewise, we will use
the symbol $\diamond$ for $\square$. The notation $\mathcal{M}, s \models \varphi$ will read the point $s$ in the model $\mathcal{M}$ makes the formula $\varphi$ true. We will call the set of points that satisfy a given formula $\varphi$ in model $\mathcal{M}$ the extension of $\varphi$, and denote as $\langle \varphi \rangle^\mathcal{M}$.

In topological models, the extension of a Boolean formula is obtained in the familiar sense. The extension of a modal formula in model $\mathcal{M}$, then, is given as follows $\langle I \varphi \rangle^\mathcal{M} = I(\langle \varphi \rangle^\mathcal{M})$ - namely, the extension of $I \varphi$ is the interior of the extension of $\varphi$. Now, based on this framework, the model theoretical semantics of modal logic in topological spaces is given as follows.

\begin{align*}
\mathcal{M}, s \models p & \text{ iff } s \in v(p) \text{ for } p \in P \\
\mathcal{M}, s \models \neg \varphi & \text{ iff } \mathcal{M}, s \not\models \varphi \\
\mathcal{M}, s \models \varphi \land \psi & \text{ iff } \mathcal{M}, s \models \varphi \text{ and } \mathcal{M}, s \models \psi \\
\mathcal{M}, s \models I \varphi & \text{ iff } \exists U \in \sigma(s \in U \land \forall t \in U, \mathcal{M}, t \models \varphi) \\
\mathcal{M}, s \models C \varphi & \text{ iff } \forall U \in \sigma(s \in U \rightarrow \exists t \in U, \mathcal{M}, t \models \varphi)
\end{align*}

A few words on the semantics are in order here. The necessity modality $I \varphi$ says that there is an open set that contains the current state and the formula $\varphi$ is true everywhere in this open. Obviously, this is a rather complex statement, first, it requires us to determine an open set that would work, and then check whether each point in this open set satisfies the given formula or not. On the other hand, the possibility modality $C \varphi$ manifests the idea that for every open set that includes the current state, there is point in the same set that satisfies $\varphi$.

It is been shown by McKinsey and Tarski that the modal logic of topological spaces is S4 (McKinsey & Tarski, 1944). Moreover, the logic of many other topological spaces has also been investigated (Aiello et al., 2003; Cate et al., 2009; Bezhanishvili et al., 2005; van Benthem et al., 2006; van Benthem & Bezhanishvili, 2007). Moreover, recently, the topological properties of paraconsistent systems have also been investigated (Başkent, 2011; Mortensen, 2000).

The proof theory of the topological models is as expected: we utilize modus ponens and necessitation. The modal logic S4 is long known to be complete with respect to the given semantics.

### 1.2.2 Subset Space Logic

Subset space logic (SSL, henceforth) was presented in early 90s as a bimodal logic to formalize reasoning about sets and points with an underlying motivation from epistemic logic (Moss & Parikh, 1992). One of the modal operators of SSL is intended to quantify over the sets ($\square$) whereas the other modal operator was intended to quantify in the current set ($K$). The underlying motivation for the introduction of these two modalities is to be able to speak about the notion of closeness. In this context, the K operator is intended to be the knowledge operator (for one agent only, as SSL is originally presented for single-agent), and the $\square$ modality is intended for the effort modality. Effort can correspond to various things: computation, observation, approximation - the procedures that can result in knowledge increase.

The language of subset space logic $L_S$ has a countable set $P$ of proposition letters, a truth constant $\top$, the usual Boolean operators $\neg$ and $\land$, and two modal
operators $K$ and $\Box$. A subset space model is a triple $S = \langle S, \sigma, v \rangle$ where $S$ is a non-empty set, $\sigma \subseteq \wp(S)$ is a collection of subsets (not necessarily a topology), $v : P \rightarrow \wp(S)$ is a valuation function. Semantics of SSL, then is given inductively as follows.

$$s, U \models p \iff s \in v(p)$$
$$s, U \models \varphi \land \psi \iff s, U \models \varphi \text{ and } s, U \models \psi$$
$$s, U \models \neg \varphi \iff t, U \models \varphi \text{ for all } t \in U$$
$$s, U \models \Box \varphi \iff s, V \models \varphi \text{ for all } V \in \sigma \text{ such that } s \in V \subseteq U$$

The duals of $\Box$ and $K$ are $\Diamond$ and $L$ respectively and defined as usual. The tuple $(s, U)$ is called a neighborhood situation if $U$ is a neighborhood of $s$, i.e. if $s \in U \in \sigma$. The axioms of SSL reflect the fact that the $K$ modality is S5-like whereas the $\Box$ modality is S4-like. Moreover, we will need an additional axiom to state the interaction between those two modalities: $K \Box \varphi \rightarrow \Box K \varphi$.

The rules of inference are as expected: modus ponens and necessitation for both modalities. Therefore, subset space logic is complete and decidable (Moss \\& Parikh, 1992; Georgatos, 1994; Georgatos, 1997).

Note that SSL is originally proposed as a single-agent system. There have been some attempts in the literature to suggest a multi-agent version of it, but to the best of our knowledge, as of today, there is no intuitive and clear presentation of multi-agent version of SSL.

1.2.3 Public Announcement Logic

Public announcement logic is a way to represent changes in knowledge. The way PAL updates the epistemic states of the knower is by “state-elimination”. A truthful announcement $\varphi$ is made, and consequently, the agents updates their epistemic states by eliminating the possible states where $\varphi$ is false (Balbiani et al., 2007; Balbiani et al., 2008; van Ditmarsch et al., 2007).

Public announcement logic is typically interpreted on multi-modal (or multi-agent) Kripke structures (Plaza, 1989). Notationwise, the formula $[\varphi] \psi$ is intended to mean that after the public announcement of $\varphi$, $\psi$ holds. As usual, $K_i$ is the epistemic modality for the agent $i$. Likewise, $R_i$ is the epistemic accessibility relation for the agent $i$. The language of PAL will be that of multi-agent (multi-modal) epistemic logic with an additional public announcement operator $[\ast]$ where $\ast$ can be replaced with any well-formed formula in the language of basic epistemic logic. To see the semantics of PAL, take a model $M = \langle W, \{R_i\}_{i \in I}, V \rangle$ where $i$ denotes the agents and varies over a finite set $I$. For atomic propositions, negations and conjunction the semantics is as usual. For modal operators, we have the following semantics.

$$M, w \models K_i \varphi \iff M, v \models \varphi \text{ for each } v \text{ such that } (w, v) \in R_i$$
$$M, w \models [\varphi] \psi \iff M, w \models \varphi \text{ implies } M|\varphi, w \models \psi$$

Here, the updated model $M|\varphi = \langle W', \{R'_i\}_{i \in I}, V' \rangle$ is defined by restricting $M$ to those states where $\varphi$ holds. Hence, $W' = W \cap (\varphi)^M$; $R'_i = R_i \cap (W' \times W')$, 6
and finally $V'(p) = V(p) \cap W'$. The axiomatization of PAL is the axiomatization of $S5_n$ with the following axioms.

1. $[\varphi]p \iff (\varphi \rightarrow p)$
2. $[\varphi]\neg \psi \iff (\varphi \rightarrow \neg[\varphi]\psi)$
3. $[\varphi](\psi \land \chi) \iff ([\varphi]\psi \land [\varphi]\chi)$
4. $[\varphi]K_i \psi \iff (\varphi \rightarrow K_i[\varphi]\psi)$

The additional rule of inference which we will need for announcement modality is called the announcement generalization and is described as expected: From $\vdash \psi$, derive $\vdash [\varphi]\psi$.

PAL is complete and decidable. The completeness proof is quite straightforward. Once the soundness of the given axiomatization is proved, then it means that every complex formula in the language of PAL can be reduced to a formula in the basic language of (multi-agent) epistemic logic. Since $S5_n$, epistemic logic is long known to be complete, we obtain the completeness of PAL.

Notice again that in this section, we have defined PAL in Kripke structures by following the literature. In the next section, we will see how PAL is defined in geometrical models. We will start with SSL and proceed to topological models with further observations.

### 1.3 Subset Space PAL

In SSL, we depend on neighborhoods instead of the epistemic accessibility relations. Therefore, if we want to adopt public announcement logic to the context of subset space logic, we first need to focus on the fact that the public announcements shrink the observation sets for each agent. Here, we model how a single agent in SSL update his model after a public announcement.

Let us set a piece of notation. For a formula $\varphi$, let us $(\varphi)^S$ be the extension of $\varphi$. In SSL, $(\varphi)^S = \{ (s,U) \in S \times \sigma : s \in U, (s,U) \models \varphi \}$ where $S = \langle S, \sigma, v \rangle$ is a subset model. Define $(\varphi)_1^S := \{ s : s, U \in (\varphi)^S \text{ for some } U \ni s \}$, and $(\varphi)_2^S := \{ U : s, U \in (\varphi)^S \text{ for some } s \in U \}$ be the projection of $(\varphi)^S$ to the second coordinate. We will drop the superscript when it is obvious.

Now, assume that we are in a subset space model $S = \langle S, \sigma, v \rangle$. Then, after public announcement $\varphi$, we will move to another subset space model $S_{\varphi} = \langle S|\varphi, \sigma_{\varphi}, v_{\varphi} \rangle$ where $S|\varphi = (\varphi)_1$, and $\sigma_{\varphi}$ is the reduced collection of subsets after the public announcement $\varphi$, and $v_{\varphi}$ is the reduct of $v$ on $S|\varphi$. The crucial point is to construct $\sigma_{\varphi}$. As we need to get rid of the refutative states, we eliminate the points which do not satisfy $\varphi$ for each observation set $U$ in $\sigma$. We will disregard the empty set as no neighborhood situations can be formed with empty set. Hence, $\sigma_{\varphi} = \{ U_{\varphi} : U_{\varphi} = U \land (\varphi)_2 \neq \emptyset \text{ for each } U \in \sigma \}$. In other words, we relativize the topologic $\sigma$ with respect to the announcement by keeping it in mind that the truth in SSL is defined with respect to tuples of points and sets.
But then, how would the neighborhood situations be affected by the public announcement? The corresponding semantics can be suggested as follows:

\[ s, U \models [\varphi] \psi \text{ iff } s, U \models \varphi \text{ implies } s, U_\varphi \models \psi \]

Intuitively, remembering the fact that the truth is defined with respect to tuples in SSL, we revise the subset component of the tuple by shrinking it. For example, consider that a policeman makes an observation and measures the speed of the passing cars with a device which has an error range of \( \pm 5 \text{ mph} \) (Taken from (?)). In one instance, he measures the speed of a car, and reads that the speed lies in the interval \([45, 55]\). Yet, the policeman does not exactly know the speed. Therefore, he is at the neighborhood situation \((v, [45, 55])\).

Then, let us assume that he hears an announcement, say, a message he receives via the police radio, saying that “the speed of that particular car is faster than or equal to 48 mph”. In other words, “\( v \in [48, \infty) \)”. Then, the policeman updates his situation to \((v, [48, 55])\) since the announcements are assumed to be truthful.

However, if there is an announcement such as “no car is slower than 52 mph”, then he cannot update his situation to \((50, [52, 55])\) as \( 50 \notin [52, 55] \). Therefore, the latter announcement is not truthful in his situation.

Before checking whether this semantics satisfies the axioms of public announcement logic, let us give the language and semantics of the topologic PAL. The language of the topologic public announcement logic interpreted in subset spaces is given as follows:

\[ p, \bot, \neg, \varphi \land \psi, \square \varphi, K \varphi, [\varphi] \psi \]

Now, let us consider the soundness of the following axioms of basic PAL that we discussed earlier.

1. \( [\varphi]p \leftrightarrow (\varphi \rightarrow p) \)
2. \( [\varphi] \neg \psi \leftrightarrow (\varphi \rightarrow [\varphi] \neg \psi) \)
3. \( [\varphi](\psi \land \chi) \leftrightarrow ([\varphi] \psi \land [\varphi] \chi) \)
4. \( [\varphi]K \psi \leftrightarrow (\varphi \rightarrow K [\varphi] \psi) \)

**Theorem 2.** The axioms of the basic PAL are sound in subset space logic.

**Proof.** As the atomic propositions do not depend on the neighborhood, the first axiom is satisfied by the subset space semantics of public announcement modality. To see this, assume \( s, U \models [\varphi]p \). So, by the semantics \( s, U \models \varphi \) implies \( s, U_\varphi \models p \). So, \( s \in v(p) \). For any set \( V \) where \( s \in V \), we have \( s, V \models p \). Hence, \( s, U \models \varphi \) implies \( s, U \models p \), that is \( s, U \models \varphi \rightarrow p \). Conversely, assume \( s, U \models \varphi \rightarrow p \). So, \( s, U \models \varphi \) implies \( s \in v(p) \). As \( s, U \models \varphi \), \( s \) will lie in \( U_\varphi \), thus \( (s, U_\varphi) \) will be a neighborhood situation. Thus, \( s, U_\varphi \models p \). Then, we conclude \( s, U \models [\varphi]p \).

The axioms for negation and conjunction are also straightforward formula manipulations and hence skipped.
The important reduction axiom is the knowledge announcement axiom. Assume, \( s, U \models [\varphi]K\psi \). Suppose further that \( s, U \models \varphi \). Then we have the following.

\[
\begin{align*}
  s, U \models [\varphi]K\psi & \iff s, U_\varphi \models K\psi \\
  & \iff \text{for each } t_\varphi \in U_\varphi, \text{ we have } t_\varphi, U_\varphi \models \psi \\
  & \iff \text{for each } t \in U, t, U \models \varphi \\
  & \quad \text{implies } t, U \models [\varphi]\psi \\
  & \iff s, U \models K(\varphi \to [\varphi]\psi) \\
  & \iff s, U \models K[\varphi]\psi
\end{align*}
\]

Hence, the above axioms are sound for the subset space semantics of public announcement logic.

Now, recall that SSL has an indispensable modal operator \( \square \). One can wonder whether we can have a reduction axiom for it as well. We start by considering the statement \([\varphi]\square\psi \leftrightarrow (\varphi \to \square[\varphi]\psi)\). Assume, \( s, U \models [\varphi]\square\psi \). Suppose further that \( s, U \models \varphi \). Then, we deduce the following.

\[
\begin{align*}
  s, U \models [\varphi]\square\psi & \iff s, U_\varphi \models \square\psi \\
  & \iff \text{for each } V_\varphi \subseteq U_\varphi, \text{ we have } s, V_\varphi \models \psi \\
  & \iff \text{for each } V \subseteq U, s, V \models \varphi \\
  & \quad \text{implies } s, V \models [\varphi]\psi \\
  & \iff s, U \models \square(\varphi \to [\varphi]\psi) \\
  & \iff s, U \models \square[\varphi]\psi
\end{align*}
\]

Now, it is easy to see that the following axiomatize the SSL-PAL together with the axiomatization of SSL:

1. \([\varphi]p \leftrightarrow (\varphi \to p)\)
2. \([\varphi]\neg\psi \leftrightarrow (\varphi \to \neg[\varphi]\psi)\)
3. \([\varphi](\psi \land \chi) \leftrightarrow ([\varphi]\psi \land [\varphi]\chi)\)
4. \([\varphi]K\psi \leftrightarrow (\varphi \to K[\varphi]\psi)\)
5. \([\varphi]\square\psi \leftrightarrow (\varphi \to \square[\varphi]\psi)\)

Referring to the above discussions, the completeness of topologic PAL follows easily.

**Theorem 3.** Topologic PAL is complete with respect to the axiom system given above.

**Proof.** By reduction axioms we can reduce each formula in the language of topologic PAL to a formula in the language of SSL as we have shown above. As SSL is complete, so is PAL in subset space models.

By the same idea, we can import the decidability result.

**Theorem 4.** PAL in subset space models is decidable.
1.4 Topological PAL

1.4.1 Single Agent Topological PAL

We can use similar ideas to give an account of PAL in topological spaces. Let $T = \langle T, \tau, v \rangle$ be a topological model and $\varphi$ be a public announcement. We now need to obtain the topological model $T_\varphi$ which is the updated model after the announcement. Recall that we denote the extension of a formula $\varphi$ in model $M$ by $(\varphi)^M$, so $(\varphi)^M = \{ w : M, w \models \varphi \}$.

Define $T_\varphi = \langle T_\varphi, \tau_\varphi, v_\varphi \rangle$ where $T_\varphi = T \cap (\varphi)$, $\tau_\varphi = \{ O \cap T_\varphi : O \in \tau \}$ and $v_\varphi = v \cap T_\varphi$. We now need to verify that $\tau_\varphi$ is indeed a topology. However, note that $\tau_\varphi$ need not be a subtopology of $\tau$.

The intuition as to how PAL works in topological spaces is similar to how it works in Kripke spaces. A truthful announcement is made by an external agent, and the agent(s) update their topology by reducing their topological model relative to the announcement. Now, we need to check whether the newly obtained model is a topology or not.

**Proposition 1.** If $\tau$ is a topology, then $\tau_\varphi = \{ O \cap T_\varphi : O \in \tau \}$ is a topology as well.

Note that $\tau_\varphi$ is the induced topology, and hence we skip the proof.

Now, we can give a semantics for the public announcements in topological models.

$T, s \models [\varphi] \psi$ iff $T, s \models \varphi$ implies $T_\varphi, s \models \psi$

In a similar fashion, we can expect the reduction axioms to work in topological spaces. The reduction axioms for atoms and Booleans are quite straightforward. So, consider the reduction axiom for the interior modality $[\varphi]l\psi \leftrightarrow (\varphi \rightarrow l[\varphi]\psi)$.

Let $T, s \models [\varphi]l\psi$ which, by definition means $T, s \models \varphi$ implies $T_\varphi, s \models l\psi$. If we spell out the topological interior modality, we get $\exists U_\varphi \ni s \in \tau_\varphi$ s.t. $\forall t \in U_\varphi, t \models \psi$. By definition, since $U_\varphi \in \tau_\varphi$, it means that there is an open $U \in \tau$ such that $U_\varphi = U \cap (\varphi)$. Under the assumption that $T, s \models \varphi$, we observe that $\exists U \ni s \in \tau$ (as we just constructed it), such that after the announcement $\varphi$, the non-eliminated points in $U$ (namely, the ones in $U_\varphi$) will satisfy $\psi$. Thus, we get $T, s \models \varphi \rightarrow l[\varphi]\psi$.

The other direction is very similar and hence we leave it to the reader. Therefore, the reduction axioms for PAL in topological spaces are given as follows.

1. $[\varphi]p \leftrightarrow (\varphi \rightarrow p)$
2. $[\varphi]\neg \psi \leftrightarrow (\varphi \rightarrow \neg [\varphi] \psi)$
3. $[\varphi] (\psi \land \chi) \leftrightarrow ([\varphi] \psi \land [\varphi] \chi)$
4. $[\varphi]l\psi \leftrightarrow (\varphi \rightarrow l[\varphi] \psi)$
As a result, all the complex formulas involving the PAL operator can be reduced to a simpler one. This algorithm directly shows the completeness of PAL in topological spaces by reducing each formula in the language of topological PAL to the language of basic topological modal logic. Thus, we conjecture that the following result follows.

Conjecture 1. PAL in topological spaces is complete with respect to the axiomatization given.

By the same idea, we can import the decidability result.

Conjecture 2. PAL in topological models is decidable.

1.4.2 Multi-agent Topological PAL

It can be noted that the framework we have given was for a single agent. We can now discuss a multi-agent topological epistemic logic and how PAL works in that framework. Topological frameworks for multi-agent epistemic logics have been discussed already in the literature (van Benthem et al., 2006; van Benthem & Sarenac, 2004). Therefore, our treatment of the subject will be based on these works.

Let $T = \langle T, \tau \rangle$ and $T' = \langle T', \tau' \rangle$ be two topological spaces. Let $X \subseteq T \times T'$. We call $X$ horizontally open ($h$-open) if for any $(x, y) \in X$, there is a $U \in \tau$ such that $x \in U$, and $U \times \{y\} \subseteq X$. In a similar fashion, we call $X$ vertically open ($v$-open) if or any $(x, y) \in X$, there is a $U' \in \tau'$ such that $y \in U'$, and $\{x\} \times U' \subseteq X$.

Now, given two topological spaces $T = \langle T, \tau \rangle$ and $T' = \langle T', \tau' \rangle$, let us associate two modal operators $I$ and $I'$ respectively to these models. Then, we can obtain a product topology on a language with those two modalities. The product model will be of the form $\langle T \times T', \tau, \tau' \rangle$ on a language with two modalities $I$ and $I'$.

The semantics of those modalities are given as such.

$(x, y) \models I\varphi$ iff $\exists U \in \tau, x \in U$ and $\forall u \in U, (u, y) \models \varphi$

$(x, y) \models I'\varphi$ iff $\exists U' \in \tau', y \in U'$ and $\forall u' \in U', (x, u') \models \varphi$

It has been shown that the fusion logic $S4 \oplus S4$ is complete with respect to products of arbitrary topological spaces (van Benthem & Sarenac, 2004).

The language of multi-agent topological PAL is as follows. We specify it for two-agents for simplicity, but it can easily be generalized to $n$-agents.

$p \mid \neg \varphi \mid \varphi \land \varphi \mid K_1 \varphi \mid K_2 \varphi \mid [\varphi] \varphi$

For given two topological models $T = \langle T, \tau, v \rangle$ and $T' = \langle T', \tau', v' \rangle$, the product topological model $M = \langle T \times T', \tau, \tau', v \rangle$ has the following semantics.

$M, (x, y) \models K_1 \varphi$ iff $\exists U \in \tau, x \in U$ and $\forall u \in U, (u, y) \models \varphi$

$M, (x, y) \models K_2 \varphi$ iff $\exists U' \in \tau', y \in U'$ and $\forall u' \in U', (x, u') \models \varphi$

$M, (x, y) \models [\varphi] \psi$ iff $M, (x, y) \models \varphi$ implies $M, (x, y) \models \psi$
where $M_{\phi} = \langle T_{\phi} \times T'_{\phi}, \tau_{\phi}, \tau'_{\phi}, v_{\phi} \rangle$ is the updated model. We define all $T_{\phi}, T'_{\phi}, \tau_{\phi}, \tau'_{\phi},$ and $v_{\phi}$ as before.

Therefore, the following axioms axiomatize the product topological PAL together with the axioms of $S4 \oplus S4$.

1. $[\phi]p \leftrightarrow (\phi \rightarrow p)$
2. $[\phi]\neg \psi \leftrightarrow (\phi \rightarrow \neg[\phi]\psi)$
3. $[\phi](\psi \land \chi) \leftrightarrow ([\phi]\psi \land [\phi]\chi)$
4. $[\phi]K_i \psi \leftrightarrow (\phi \rightarrow K_i[\phi]\psi)$

**Conjecture 3.** Product topological PAL is complete and decidable with respect to the given axiomatization.

### 1.5 Applications

Now, we will briefly apply the previous discussions to some issues in PAL, foundational game theory and SSL. The purpose of such applications is to give the reader a sense how topological framework might affect the aforementioned issues, and in general how dynamic epistemic situations can be represented topologically.

#### 1.5.1 Announcement Stabilization

Muddy Children presents an interesting case for PAL (Fagin et al., 1995). In that game, the model gets updated after each children says that she does not know if she had mud on her forehead. The model keeps updated until the announcement is negated, and then becomes common knowledge (van Benthem, 2007). Therefore, after each update, we get smaller and smaller models up until the moment that the model gets stabilized in the sense that the same announcement does not update the model any longer.

As van Benthem pointed out, this is closely related to several issues (van Benthem, 2007). First, PAL behaves like a fixed-point operator where the fixed point is the model which is stabilized. Second, there seems to be a close relation between game theoretical strategy eliminations, and solution methods based on such approaches. Therefore, it is rather important to analyze announcement stabilization. Here, we will approach the issue from a topological angle.

For a model $M$ and a formula $\phi$, we define the announcement limit $\lim_{\phi} M$ as the first model which is reached by successive announcements of $\phi$ that no longer changes after the last announcement is made. Announcement limits exist in both finite and infinite models (van Benthem & Gheerbrant, 2010). For instance, for any model $M$, $\lim_{p} M = M|p$ for propositional variable $p$. Therefore, the limit model is the first updated model when the announcement is a ground Boolean formula. In muddy children, the announcement shrinks the model step by step, round by round (van Benthem, 2007). However, sometimes in dialogue games it may take too long to solve such puzzles until the model gets stabilized.
Figure 1: A model for muddy children played with 3 children \( a, b, c \) taken from (van Ditmarsch et al., 2007). The state \( n_an_bn_cn_c \) for \( n_a, n_b, n_c \in \{0, 1\} \) represent that child \( i \) has mud on her forehead iff \( n_i = 1 \) for \( i \in \{a, b, c\} \). The proposition \( m_i \) means that the child \( i \in \{a, b, c\} \) has mud on her forehead. The current state is underlined.

as shown by Parikh (Parikh, 1991). Similarly, even Zermelo considered similar approaches to understand as to how long it takes for the game to stabilize (Schwalbe & Walker, 2001).

Similar to the discussions of the aforementioned authors, we now analyze how the models stabilize in topological PAL. We know that topological models do present some differences in epistemic logical structures. For instance, in topological models, the stabilization of the fixed-point definition\(^1\) version of common knowledge may occur later than ordinal stage \( \omega \). However, it stabilizes in \( \leq \omega \) steps in Kripke models (van Benthem & Sarenac, 2004).

We also know that there are two possibilities for the limit models. Either it is empty or nonempty. If it is empty, it means that the negation of the an-

\(^1\)Formula \( \varphi \) is common knowledge among two-agents 1 and 2 \( C_{1,2}\varphi \) is represented with the (largest) fixed-point definition as follows: \( C_{1,2}\varphi := \nu p.\varphi \land K_{1}p \land K_{2}p \) where \( K_i \), for \( i = 1, 2 \) is the familiar knowledge operator (Barwise, 1988).
nouncement has become common knowledge, thus the announcement refuted itself. On the other hand, if the limit model is not empty, it means that the announcement has become common knowledge (van Benthem & Gheerbrant, 2010).

**Conjecture 4.** For some formula $\varphi$ and some topological model $M$, it may take more than $\omega$ stage to reach the limit model $\lim_{\varphi} M$.

**Proof.** (Sketch) Note that it was shown that in multi-agent topological models, stabilization of common knowledge with fixed-point definition may occur later than $\omega$ stage. However, in Kripke models it occurs before $\omega$ stage (van Benthem & Sarenac, 2004).

Also note that it was shown that if the limit model is not empty, the announcement has become common knowledge (van Benthem & Gheerbrant, 2010).

Therefore, combining these two observations, we conclude that in some topological models with non-empty limit models, the number of stage for the announcement to be common knowledge may take more than $\omega$ steps.

Even if the stabilization takes longer, we can still obtain stable models by taking intersections at the limit ordinals as a general rule (van Benthem & Gheerbrant, 2010). Therefore, we guarantee that the update procedure will terminate. Thus, the following result is now self-evident.

**Conjecture 5.** Limit models exist in topological models.

### 1.5.2 Backward Induction

The fact that limit models can be attained in more than $\omega$ steps can create some problems in games. Consider the backward induction solution where players trace back their moves to develop a winning strategy. Notice that the Aumann’s backward induction solution assumes common knowledge of rationality (Aumann, 1995; Halpern, 2001). Granted, there can be several philosophical and epistemic issues about the centipede game and its relationship with rationality, but we will not pursue this direction here (Artemov, 2009a; Artemov, 2009b).

This issue can also be approached from a dynamic epistemic perspective. Recently, it has been shown that in any game tree model $M$ taken as a PAL model, $\lim_{\text{rational}} M$ is the actual subtree computed by the backward induction procedure where the proposition rational means that “at the current node, no player has chosen a strictly dominated move in the past coming here” (van Benthem & Gheerbrant, 2010). Therefore, the announcement of node-rationality produces the same result as the backward induction procedure. Each backward step in the backward induction procedure can then be obtained by the public announcement of node rationality. This result is quite impressive in the sense that it establishes a closer connection between communication and rationality, and furthermore leads to several more intriguing discussions about rationality.

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2Although according to Halpern, Stalnaker proved otherwise (Halpern, 2001; Stalnaker, 1998; Stalnaker, 1994; Stalnaker, 1996).
In this work, we refrain ourselves from pursuing this line of thought for the time being.

However, there seems to be a problem in topological models. The admissibility of limit models can take more than $\omega$ steps in topological models as we have conjectured earlier. Therefore, the BI procedure can take $\omega$ steps or more.

**Conjecture 6.** In topological models of games, under the assumption of rationality, the backward induction procedure can take more than $\omega$ steps.

This is indeed a problem about the attainability in infinite games: how can a player continue playing the game when she hit the limit ordinal $\omega$-th step in the backward induction procedure? In order not to diverge from our current focus, we leave this question open.

Notice that a similar result was obtained by Parikh in the context of finite and infinite dialog games (Parikh, 1991). He showed that in a game where players make consecutive announcements, one can obtain a model for transfinite ordinals.

### 1.5.3 Persistence

A similar notion has already been defined in SSL. Define persistent formula in a model $M$ as the formula $\varphi$ whose truth is independent from the subsets in $M$ (Dabrowski et al., 1996). In other words, $\varphi$ is persistent if for all states $s$ and subsets $V \subseteq U$, we have $s, U \models \varphi$ implies $s, V \models \varphi$. Clearly, Boolean formulas are persistent in every model.

Intuitively, persistent formulas are those formulas whose truth does not depend on the observational, computational etc. factors. For instance, the formula $\lozenge$ is not persistent.

Now, we can apply this idea to the context of PAL. The punch-line is the fact that in PAL, we obtain a subset of the current observation set we are occupying. Therefore, persistent formulas do not get affected from announcements, and remain true.

**Conjecture 7.** Let $M$ be a model and $\varphi$ be persistent in $M$. Then, for any formula $\chi$ and neighborhood situation $(s, U)$, if $s, U \models \varphi$, then $s, U \models [\chi] \varphi$. In other words, true persistent formulas are immune to the public announcements.

**Proof.** (Sketch) Proof follows directly from the definitions and the fact that after the public announcement of $\chi$, we always have $U_\chi \subseteq U$. \hfill $\Box$
2 Paraconsistent Topologies and Homotopies

Let us now combine our geometric motivation with a non-classical logical system. In this section, we will discuss the connection between inconsistent logical systems and topological models.

2.1 What is Paraconsistency?

The well-studied notion of deductive explosion describes the situation where every formula can be deduced from an inconsistent set of formulae, i.e. for all $\varphi$ and $\psi$, we have $\{\varphi, \neg\varphi\} \vdash \psi$, where $\vdash$ denotes logical consequence relation. In this respect, both “classical” and intuitionistic logics are known to be explosive. Paraconsistent logic, on the other hand, is the umbrella term for logical systems where the logical consequence relation $\vdash$ is not explosive (Priest, 2002). Variety of philosophical and logical objections can be raised against paraconsistency, and almost all of these objections can be defended in a rigorous fashion. We will not here be concerned about the philosophical implications of it, yet we refer the reader to the following for a comprehensive defense of paraconsistency with a variety of well-structured applications chosen from mathematics and philosophy (Priest, 2006, Chapters 2 and 3). In this chapter, we present some further applications of paraconsistency in basic modal logic with topological semantics.

The use of topological semantics for paraconsistent logic is not new. To the best of our knowledge, the earliest work discussing the connection between inconsistency and topology goes back to Goodman (Goodman, 1981). In his paper, Goodman discussed “pseudo-complements” in a lattice theoretical setting and called the topological system he obtains “anti-intuitionistic logic”. In a recent work, Priest discussed the dual of the intuitionistic negation operator and considered that operator in topological framework (Priest, 2009). Similarly, Mortensen discussed topological separation principles from a paraconsistent and paracomplete point of view and investigated the theories in such spaces (Mortensen, 2000). Similar approaches from modal perspective was discussed by Béziau, too (Béziau, 2005).

2.2 Semantics

Recall that in topological models, modal formulas produce open (or dually, closed) sets due to the semantics of the modal operators in such models. We can take one step further and suggest that extension of any propositional variable will be an open set (Mortensen, 2000). In that setting, conjunction and disjunction works fine for finite intersections and unions. Nevertheless, negation can be difficult as the complement of an open set is not generally an open set, thus may not be the extension of a formula in the language. For this reason, we
will need to use a new negation symbol $\sim$ that returns the open complement (interior of the complement) of a given set.

A similar idea can also be applied to closed sets where we assume that the extension of any propositional variable will be a closed set. In order to avoid a similar problem with negation, we stipulate yet another negation operator which returns the closed complement (closure of the complement) of a given set. In this setting, we will use the symbol $\sim$ that returns the closed complement of a given set.

So far, we have set up a lot of notational conventions. Let us make them clear once more. Whenever we refer to a topology of open sets with the negation $\dot{\sim}$, we will call it $\tau$ and, whenever we refer to a topology of closed sets with the negation $\sim$, we will call it $\sigma$. We reserve the classical negation symbol $\neg$ for the set theoretical complement.

Now, let us consider the boundary $\partial(\cdot)$ of a set $X$ where $\partial(X)$ is defined as $\partial(X) := \text{Clo}(X) - \text{Int}(X)$. Consider now, for a given formula $\varphi$, the boundary of its extension $\partial((\varphi))$ in $\tau$. Let $x \in \partial((\varphi))$. Since $\varphi$ is open, $x \not\in \varphi$. Similarly, $x \not\in (\sim \varphi)$ as the open complement is also open by definition, and does not include the boundary. Thus, neither $\varphi$ nor $\sim \varphi$ is true at $x$. The point $x$ was selected arbitrarily from the boundary, thus, in $\tau$, any theory that includes the theory of the propositions that are true at the boundary is incomplete.

We can make a similar observation about the boundary points in $\sigma$. Now, take $x \in \partial((\varphi))$ where $\varphi$ is a closed set in $\sigma$. By the above definition, since we have $x \in \partial((\varphi))$, we obtain $x \in \varphi$ as $\varphi$ is closed. Yet, $\partial((\varphi))$ is also included in $(\sim \varphi)$ which we have defined as a closed set. Thus, by the same reasoning, we conclude $x \in (\sim \varphi)$. Thus, $x \in (\varphi \land \sim \varphi)$ yielding that $x \models \varphi \land \sim \varphi$. Therefore, in $\sigma$, any theory that includes the boundary points will be inconsistent. In this respect, the model $M = \langle S, \tau, V \rangle$ with the negation symbol $\dot{\sim}$ will be called a paracomplete topological model, and similarly, the model $M' = \langle S, \sigma, V \rangle$ with the negation symbol $\sim$ will be called a paraconsistent topological model where $V$ is a valuation function.

So far, we have recalled how paracomplete and paraconsistent logics can be obtained in a topological setting. However, an immediate observation yields that since extensions of all formulae in $\sigma$ (respectively in $\tau$) are closed (respectively, open), the topologies which are obtained in both paraconsistent and paracomplete logics are discrete. This observation may trivialize the matter as, for instance, discrete spaces with the same cardinality are homeomorphic.

\textbf{Conjecture 8.} Let $M_1$ and $M_2$ be paraconsistent and paracomplete topological models respectively. If $|M_1| = |M_2|$, then there is a homeomorphism from the paraconsistent topological model to the paracomplete one, and vice versa.

\subsection*{2.3 Topological Properties and Paraconsistency}

In this section, we investigate the relation between some basic topological properties and paraconsistency. Mostly, we will consider the closed set topology $\sigma$ with its negation operator $\sim$ as it is the natural candidate for paraconsistent
topological models. Our work can be seen as an extension of Mortensen’s earlier work (Mortensen, 2000). Here we extend his approach to some other topological properties and discuss the behavior of such spaces under some special functions.

### 2.3.1 Connectedness

In the above section, we observed that boundary points play a central role in paraconsistent theories defined in topological spaces. There is also a close connection between boundary and connectedness which motivates this section.

A topological space is called *connected* if it is not the union of two disjoint non-empty open (closed) sets. Formally, a set $X$ is then called connected if for two non-empty open (respectively closed) subsets $A, B$, if $X = A \cup B$; then $A \cap B \neq \emptyset$. Moreover, in any connected topological space, the only subsets with empty boundary are the space itself and the empty set (Bourbaki, 1966). Finally, connectedness is *not* definable in the basic modal language (Cate et al., 2009).

**Definition 5.** A formula $\varphi$ in the language of propositional modal logic is called connected if for any two formulae $\alpha_1$ and $\alpha_2$ with non-empty extensions, if $\varphi \equiv \alpha_1 \lor \alpha_2$, then we have $(\alpha_1 \land \alpha_2) \neq \emptyset$. We will call a theory $T$ connected, if it is generated by a set of connected formulae.

Based on this definition, we observe the following.

**Conjecture 9.** Every connected formula is satisfiable in some connected (classical) topological space.

Connected theories may be inconsistent or incomplete.

**Conjecture 10.** Every non-empty connected theory in closed set topology $\sigma$ is inconsistent. Moreover, every connected theory in open set topology $\tau$ is incomplete.

The converse direction is a bit more interesting. Do connected spaces satisfy only connected formulae?

**Conjecture 11.** Let $X$ be a connected topological space of closed sets. Then, the only subtheories that are not inconsistent are the trivial ones (i.e. empty theory and $X$ itself).

### 2.3.2 Continuity

A recent research program that considers topological modal logics with continuous functions were discussed in an early work of Artemov et al., and later by Kremer and Mints (Artemov et al., 1997; Kremer & Mints, 2005). In these aforementioned works, they associated continuous functions with temporal modal operator and discussed the orbits of such functions. In that setting, $\Box p =$
$f^{-1}(p)$ where $\circlearrowright$ is the temporal next time operator and $f$ is a continuous function. This framework allows us to discuss the orbits of continuous functions and express continuity modally.

An immediate theorem, which was stated and proved in variety of different work, would also work for paraconsistent logics. Now, let us take two closed set topologies $\sigma$ and $\sigma'$ on a given set $S$ and a homeomorphism $f : \langle S, \sigma \rangle \rightarrow \langle S, \sigma' \rangle$. We have a simple way to associate the respective valuations between two models $M$ and $M'$ which respectively depend on $\sigma$ and $\sigma'$ so that we can have a truth preservation result. Therefore, define $V'(p) = f(V(p))$.

**Conjecture 12.** Let $M = \langle S, \sigma, V \rangle$ and $M' = \langle S, \sigma', V' \rangle$ be two paraconsistent topological models with a homeomorphism $f$ from $\langle S, \sigma \rangle$ to $\langle S, \sigma' \rangle$. Define $V'(p) = f(V(p))$. Let $w \in S, w' = f(w)$, then for all $\varphi$, we have $M, w \models \varphi$ iff $M', w' \models \varphi$.

**Proof.** (Sketch) By induction on the complexity of the formulae.

Notice that the above theorem also works in paracomplete topological models, and we leave the details to the reader.

Assuming that $f$ is a homeomorphism may seem a bit strong. We can then separate it into two. One direction of the biconditional can be satisfied by continuity whereas the other direction is satisfied by the openness of $f$.

**Corollary 1.** Let $M = \langle S, \sigma, V \rangle$ and $M' = \langle S, \sigma', V' \rangle$ be two paraconsistent topological models with a continuous $f$ from $\langle S, \sigma \rangle$ to $\langle S, \sigma' \rangle$. Define $V'(p) = f(V(p))$. Then $M, w \models \varphi$ implies $M', w' \models \varphi$ for all $\varphi$ where $w' = f(w)$.

**Corollary 2.** Let $M = \langle S, \sigma, V \rangle$ and $M' = \langle S, \sigma', V' \rangle$ be two paraconsistent topological models with an open $f$ from $\langle S, \sigma \rangle$ to $\langle S, \sigma' \rangle$. Define $V'(p) = f(V(p))$. Then $M', w' \models \varphi$ implies $M, w \models \varphi$ for all $\varphi$ where $w' = f(w)$.

Proofs of both corollaries depend on the fact that $\text{Clo}$ operator commutes with continuous functions in one direction, and it commutes with open functions in the other direction. Furthermore, similar corollaries can be given for paracomplete frameworks as the $\text{Int}$ operator also commutes in one direction under similar assumptions, and we leave it to the reader as well.

Furthermore, any topological operator that commutes with continuous, open and homeomorphic functions will reflect the same idea and preserve the truth\footnote{Thanks to Chris Mortensen for pointing this out.}.

Therefore, these results can easily be generalized.

### 2.4 Homotopies

#### 2.4.1 Paraconsistent Case

We can now take one step further to discuss homotopies in paraconsistent topological modal models. A homotopy is a description of how two continuous functions from a topological space to another can be deformed to each other.
Definition 6. Let $S$ and $S'$ be two topological spaces with continuous functions $f, f' : S \rightarrow S'$. A homotopy between $f$ and $f'$ is a continuous function $H : S \times [0, 1] \rightarrow S'$ such that if $s \in S$, then $H(s, 0) = f(s)$ and $H(s, 1) = g(s)$.

An alternative definition can also be given: a homotopy between $f$ and $f'$ is a family of continuous functions $H_t : S \rightarrow S'$ such that for $t \in [0, 1]$ we have $H_0 = f$ and $H_1 = g$ and the map $t \mapsto H_t$ is continuous from $[0, 1]$ to the space of all continuous functions from $S$ to $S'$. It is easy to show that homotopy relation is an equivalence relation. Thus, if $f$ and $f'$ are homotopic, we denote it with $f \approx f'$.

Definition 7. Given a model $M = \langle S, \sigma, V \rangle$, we call the family of models $\{ M_t = \langle S_t \subseteq S, \sigma_t, V_t \rangle \}_{t \in [0, 1]}$ generated by $M$ and homotopic functions homotopic models. In the generation, we put $V_t = f_t(V)$.

Conjecture 13. Given two topological paraconsistent models $M = \langle S, \sigma, V \rangle$ and $M' = \langle S', \sigma', V' \rangle$ with two continuous functions $f, f' : S \rightarrow S'$ both of which respect the valuation: $V' = f(V) = f'(V)$. If there is a homotopy $H$ between $f$ and $f'$, then $M$ and $M'$ satisfy the same modal formulae.

Notice that we have discussed truth in the image sets that are obtained under $f, f', f_t, \ldots$. Nevertheless, the converse can also be true, once the continuous functions have continuous inverses: this is exactly what is guaranteed by homeomorphisms. The corresponding notion at the level of homotopies is an isotopy. An isotopy is a continuous transformation between homeomorphic functions. Thus, we have the following.

Conjecture 14. Given two topological paraconsistent models $M = \langle S, \sigma, V \rangle$ and $M' = \langle S', \sigma', V' \rangle$ with two homeomorphism $f, f' : S \rightarrow S'$ both of which respect the valuation: $V' = f(V) = f'(V)$. If there is an isotopy $H$ between $f$ and $f'$, then, for all $\varphi$, we have

$$M_t \models \varphi \iff M \models \varphi \iff M' \models \varphi$$
What makes the non-classical case easy is the fact that the extension of each formula is an open or a closed set. Furthermore, a similar theorem can be stated for paracomplete cases with a similar proof.

**Conjecture 15.** Given two topological paracomplete models \( M = \langle S, \sigma, V \rangle \) and \( M' = \langle S', \sigma', V' \rangle \) with two continuous functions \( f, f' : S \to S' \) both of which respect the valuation: \( V' = f(V) = f'(V) \). If there is a homotopy \( H \) between \( f \) and \( f' \), then \( M \) and \( M' \) satisfy the same formulae.

### 2.4.2 Classical Case

Let us now observe if we can have similar results in classical modal logics. Let \( N = \langle T, \eta, V \rangle \) and \( N' = \langle T', \eta', V' \rangle \) be classical topological modal models. Define a homotopy \( H : T \times [0, 1] \to T' \). Therefore, as before, for each \( t \in [0, 1] \), we obtain models \( N_t = \langle T_t, \eta_t, V_t \rangle \).

**Conjecture 16.** Given two classical topological modal models \( N = \langle T, \eta, V \rangle \) and \( N' = \langle T', \eta', V' \rangle \) with two continuous functions \( f, f' : T \to T' \) both of which respect the valuation: \( V' = f(V) = f'(V) \). If there is a homotopy \( H \) between \( f \) and \( f' \), then \( N \) and \( N' \) satisfy the same formula.

### 2.5 A Modal Logical Application

Consider the following two bisimilar Kripke models \( M \) and \( M' \). Assume that \( w, w' \) and \( u, u', u'' \) and \( v, v', v'' \) do satisfy the same propositional letters. Then it is easy to see that \( w \) and \( w' \) are bisimilar, and therefore satisfy the same model formulae. As it is well-known, modal logic cannot distinguish bisimilar models. However, the difference between bisimilar models may raise some philosophical and mathematical issues.

![Figure 3: Two Bisimilar Models](image)

We can still pose a conceptual question about the relation between \( M \) and \( M' \). Even if these two models are bisimilar, they are clearly different models. Moreover, it is plausible to contract \( M' \) to \( M \) in a validity preserving fashion.
Therefore, we may need to transform one model to its bisimilar copy. Furthermore, given a model, we may need to measure the level of change from the fixed model to another model which is bisimilar to the given one.

Especially, in epistemic logic, such concerns do make sense. Given an epistemic situation, we can model it up to bisimulation. In other words, from an agent’s perspective whose language is propositional modal logic, bisimilar models are indistinguishable. But, from a (modal) model theoretical perspective, they are distinguishable. Therefore, there can be several ways to model the given epistemic situation. We will now define how these models are related and different from each other by using the constructions we have presented earlier.

Before proceeding further, let us give the definition of topological bisimulations (Aiello & van Benthem, 2002).

**Definition 8.** Let $M = \langle S, \sigma, v \rangle$ and $M' = \langle S', \sigma', v' \rangle$ be two topological models. A topo-bisimulation is a nonempty relation $\leftrightarrow \subseteq S \times S'$ such that if $s \leftrightarrow s'$, then we have the following:

1. **BASE CONDITION**
   
   $s \in v(p)$ if and only if $s' \in v'(p)$ for any propositional variable $p$.

2. **FORTH CONDITION**
   
   $s \in U \in \sigma$ implies that there exists $U' \in \sigma'$ such that $s' \in U'$ and for all $t' \in U'$ there exists $t \in U$ with $t \leftrightarrow t'$.

3. **BACK CONDITION**
   
   $s' \in U' \in \sigma'$ implies that there exists $U \in \sigma$ such that $s \in U$ and for all $t \in U$ there exists $t' \in U'$ with $t \leftrightarrow t'$.

We can take one step further and define a homoeomorphism that respect bisimulations. Let $M = \langle S, \sigma, v \rangle$ and $M' = \langle S', \sigma', v' \rangle$ be two topo-bisimilar models. If there is a homeomorphism $f$ from $\langle S, \sigma \rangle$ to $\langle S', \sigma' \rangle$ that respect the valuation, we call $M$ and $M$ homeo-topo-bisimilar models. We give the precise definition as follows.

**Definition 9.** Let $M = \langle S, \sigma, v \rangle$ and $M' = \langle S', \sigma', v' \rangle$ be two topological models. A homeo-topo-bisimulation is a nonempty relation $\Rightarrow f \subseteq S \times S'$ based on a homeomorphism $f$ from $S$ into $S'$ such that if $s \Rightarrow f s'$, then we have the following:

1. **BASE CONDITION**
   
   $s \in v(p)$ if and only if $s' \in v'(p)$ for any propositional variable $p$.

2. **FORTH CONDITION**
   
   $s \in U \in \sigma$ implies that there exists $f(U) \in \sigma'$ such that $s' \in f(U)$ and for all $t' \in f(U)$ there exists $t \in U$ with $t \Rightarrow f t'$.

3. **BACK CONDITION**
   
   $s' \in f(U) \in \sigma'$ implies that there exists $U \in \sigma$ such that $s \in U$ and for all $t \in U$ there exists $t' \in f(U)$ with $t \Rightarrow f t'$.
Based on this definition, we immediately observe the following.

**Conjecture 17.** Homeo-topo-bisimulation preserve the truth.

Now, we can discuss the homotopy of homeo-topo-bisimilar models. What we aim is the following. Given a topological model (either classical, intuitionistic or paraconsistent), we will construct two homeomorphic image of it respecting homeo-topo-bisimulation where these two homeomorphisms are homotopic. Then, by using homotopy, we will measure the level of change of the intermediate homeomorphic models with respect to these two functions.

Let $M$ be a given topological model. Construct $M_f$ and $M_g$ as the homeomorphic image of $M$ respecting the valuation where $f$ and $g$ are homeomorphism. For simplicity, assume that $M \xrightarrow{f} M_f$ and $M \xrightarrow{g} M_g$. Now, if $f$ and $g$ are homotopic, then we have functions $h_x$ for $x$ continuous on $[0,1]$ with $h_0 = f$ and $h_1 = g$.

Therefore, given $x \in [0,1]$ the model $M_x$ will be obtained by applying $h_x$ to $M$ respecting the valuation. Hence, $M_0 = M_f$ and $M_1 = M_g$. Therefore, given $M$, the distance of any homeo-topo-bisimilar model $M_x$ to $M$ will be $x$, and it will be the measure of non-modal change in the model. In other words, even if $M \xrightarrow{h(x)} M_x$, we will say $M$ and $M_x$ are $x$-different than each other.

The procedure we described offers a well-defined method of indexing the homeo-topo-bisimilar models. But, indexing is not random. It is continuous on the closed unit interval.

### 2.6 An Epistemic Application

Consider two believers Ann and Bob where Ann is an ordinary believer while Bob is a religious cleric of the religion that Ann is following. Therefore, they believe in the same religion and the same rules of the religion. For the sake of our example, let us assume that the religion in question is really simple and there is nothing that Ann does not know about it. In other words, even if Bob is a clergyman, Ann believes in the religion as much as Bob does. However, we feel that Bob believes it more than Ann even though they believe in exactly the same propositions. In other words, there is still a difference between their belief. Then, what is this difference? The reason for this is the fact that the extent of Bob’s knowledge is wider than that of Ann’s. Namely, Bob believes better. What does this mean?

In this context, we can ask the following two questions.

1. How much wider is Bob’s belief?
2. How is Ann’s belief transformed to Bob’s?

These two questions are meaningful. Even if their language cannot tell us which one has wider knowledge, ontologically, we know that Bob has more knowledge in some sense even if they agree on every proposition. Clearly, the reason for that is the fact that Bob considers more possible worlds for a given
proposition which makes his belief more robust than Ann’s. Let us set up a
piece of notation first. By \([w]\), we denote the set of accessible states from \(w\), i.e. 
\([w] = \{v : wRv\} \).

**Definition 10.** Given two agents \(i\) and \(j\) with their respective bisimilar models 
\(M\) and \(N\). We say \(i\)'s knowledge more robust than \(j\)'s at \(w\) if 
\((\varphi)^N \cap [w] \subseteq (\varphi)^M \cap [w]\) for all formula \(\varphi\) in the language.

Moreover, from a dynamic epistemic angle, homeomorphisms and homoto-
phies can explain this transformation from Ann’s beliefs to Bob’s belief with respect to their models.

The timestamp subscript in the definition of homotopies can easily be con-
sidered as a temporal parameter. In our simple example, this reading of the homotopy index is especially helpful. It help us to give a step by step account of the transformation between Ann’s and Bob’s belief.

Therefore, focusing on the transformations reveals more information on the ontology of the agents when the epistemics of the agents are already known. Robust knowledge/belief captures this notion by focusing on the ontology of the model and connects the epistemics with ontology.

\(^5\)We borrowed the term “robust” from Artemov.
3 A Paradox in Game Theory

Can we use such non-classical methods and even more to analyze a well-known paradox in epistemic game theory? With some help from category theory, we obtain some intriguing results using paraconsistency and non-well-founded set theory.

3.1 Introduction

The Brandenburg-Keisler paradox ('BK paradox', henceforth) is a two-person self-referential paradox in epistemic game theory (Brandenburger & Keisler, 2006). Due to its considerable impact on the various branches of game theory and logic, it has gained increasing interest in the literature.

In short, for players Ann and Bob, the BK paradox arises when we consider the following statement “Ann believes that Bob assumes that Ann believes that Bob’s assumption is wrong” and ask the question if “Ann believes that Bob’s assumption is wrong”.

There can be considered two main reasons why the Brandenburger-Keisler argument turns out to be a paradox. First, the limitations of set theory presents some restrictions on the mathematical model which is used to describe self-referantiality and circularity in the formal language. Second, Boolean logic comes with its own Aristotelian meta-logical assumptions about consistency. Namely, Aristotelian principle about consistency, principium contradictionis, maintains that contradictions are impossible. In this paper, we will consider some alternatives to such assumptions, and investigate their impact on the BK paradox.

The BK paradox is based on the ZF(C) set theory. The ZFC set theory comes with its own restrictions one of which is the axiom of foundation. It can be deduced from this axiom that no set can be an element of itself. In non-well-founded set theory, on the other hand, the axiom of foundation is replaced by the anti-foundation axiom. In non-well-founded set theory, among many other things, it is possible to have sets to be their own member (Mirimanoff, 1917; Aczel, 1988). Therefore, we claim that switching to non-well-founded set theory suggests a new approach to the paradox, and game theory in general. The power of non-well-founded set theory comes from its genuine methods to deal with circularity (Barwise & Moss, 1996; Moss, 2009).

Second, what makes the BK paradox a paradox is the principium contradictionis. Paraconsistent logics challenge this assumption (Priest, 1998; Priest, 2006). Therefore, we will also investigate the BK paradox in paraconsistent systems. This line of research, as we shall see, is rather fruitful. The reason for this is the following. The BK paradox is essentially a self-referential paradox, and for any other paradox of the same kind, it can be analyzed from a category theoretical or algebraic point of view (Yanofsky, 2003; Abramsky & Zvesper, 2010). Moreover, paraconsistent logics also present an algebraic and category theoretical structure. In this work, we will make the connection between self-referantiality and paraconsistency clearer and see whether we can solve the paradox if we embrace a paraconsistent framework.
What is the significance of adopting non-classical frameworks then? There are many situations where circularity and inconsistency are integral parts of the game. A game when some players can reset the game can be thought of a situation where the phenomenon of circularity appears. Moreover, inconsistencies occur in games quite often as well. Situations where information sets of some players become inconsistent after receiving some information in a dialogue without a consequent belief revision are such examples where inconsistencies occur (Lebbink et al., 2004; Rahman & Carnielli, 2000).

The Brandenburger - Keisler paradox was presented in its final form in a relatively recent paper in 2006 (Brandenburger & Keisler, 2006). A general framework for self-referential paradoxes was discussed earlier by Yanofsky in 2003 (Yanofsky, 2003). In his paper, Yanofsky used Lawvere’s category theoretical arguments in well-known mathematical arguments such as Cantor’s diagonalization, Russell’s paradox, and Gödel’s Incompleteness arguments. Lawvere, on the other hand, discussed self-referential paradoxes in cartesian closed categories in his early paper which appeared in 1969 (Lawvere, 1969). Abramsky and Zvesper used Lawvere’s arguments to analyze the BK paradox in a category theoretical framework (Abramsky & Zvesper, 2010).

Pacuit approached the paradox from a rather modal logical perspective and presented a detailed investigation of the paradox in neighborhood models and in hybrid systems (Pacuit, 2007). Neighborhood models are used to represent modal logics weaker than K, and can be considered as weak versions of topological semantics (Chellas, 1980). This argument later was extended to assumption-incompleteness in modal logics (Zvesper & Pacuit, 2010).

Paraconsistent games in the form of dialogical games were largely discussed by Rahman and his co-authors (Rahman & Carnielli, 2000). Co-Heyting algebras, on the other hand, have gained interest due to their use in “region based theories of space” within the field of mereotopology (Stell & Worboys, 1997). Mereotopology discusses the qualitative topological relations between the wholes, parts, contacts and boundaries and so on.

Now, we can discuss the paradox briefly. The BK paradox can be considered as a game theoretical two-person version of Russell’s paradox where players interact in a self-referential fashion. Let us call the players Ann and Bob with associated type spaces $U^a$ and $U^b$ respectively. Now, consider the following statement which we call the BK sentence:

\[ \text{Ann believes that Bob assumes that Ann believes that Bob’s assumption is wrong.} \]

A Russell-like paradox arises if one asks the question whether \textit{Ann believes that Bob’s assumption is wrong}. In both cases, we get a contradiction, hence the paradox. Because, if the answer to the question is “Yes”, then it turns out that it is not the case that Ann believes that Bob’s assumption is wrong, which contradicts with the “Yes” answer. Similarly, if the answer to that question is “No”, we get a contradiction as well. Thus, the BK sentence is impossible.

Brandenburger and Keisler use belief sets to represent the players’ beliefs. The model $(U^a, R^a, R^b)$ that they consider is called a belief structure where
$R^a \subseteq U^a \times U^b$ and $R^b \subseteq U^b \times U^a$. The expression $R^a(x, y)$ represents that in state $x$, Ann believes that the state $y$ is possible for Bob, and similarly for $R^b(y, x)$. We will put $R^a(x) = \{ y : R^a(x, y) \}$, and similarly for $R^b(y)$. At a state $x$, we say Ann believes $P \subseteq U^b$ if $R^a(x) \subseteq P$. Now, a modal logical semantics for the interactive belief structures can be given. We use two different modalities $\Box$ and $\Diamond$ which stand for the belief and assumption operators respectively with the following semantics.

\[
\begin{align*}
x & = \Box^{ab} \varphi \iff \forall y \in U^b, R^a(x, y) \implies y \models \varphi \\
x & = \Diamond^{ab} \varphi \iff \forall y \in U^b, R^a(x, y) \iff y \models \varphi
\end{align*}
\]

A belief structure $(U^a, U^b, R^a, R^b)$ is called assumption complete with respect to a set of predicates $\Pi$ on $U^a$ and $U^b$ if for every predicate $P \in \Pi$ on $U^b$, there is a state $x \in U^a$ such that $x$ assumes $P$, and for every predicate $Q \in \Pi$ on $U^a$, there is a state $y \in U^b$ such that $y$ assumes $Q$. We will use special propositions $U^a$ and $U^b$ with the following meaning: $w \models U^a$ if $w \in U^a$, and similarly for $U^b$. Namely, $U^a$ is true at each state for player Ann, and $U^b$ for player Bob.

Brandenburger and Keisler showed that no belief model is complete for its first-order language. Therefore, “not every description of belief can be represented” with belief structures (Brandenburger & Keisler, 2006). The incompleteness of the belief structures is due to the holes in the model. A model, then, has a hole at $\varphi$ if either $U^b \land \varphi$ is satisfiable but $\Diamond^{ab} \varphi$ is not, or $U^a \land \varphi$ is satisfiable but $\Box^{ba} \varphi$ is not. A big hole is then defined by using the belief modality $\Box$ instead of the assumption modality $\Diamond$.

In the original paper, the authors make use of two lemmas before identifying the holes in the system. These lemmas are important for us as we will challenge them in the next section. First, let us define a special propositional symbol $D$ with the following valuation $D = \{ w \in W : (\forall z \in W)[P(w, z) \rightarrow \neg P(z, w)] \}$.

**Lemma 1** ((Brandenburger & Keisler, 2006)).

1. If $\Diamond^{ab} U^b$ is satisfiable, then $\Box^{ab} \Box^{ba} \Box^{ab} U^a \rightarrow D$ is valid.
2. $\neg \Box^{ab} \Diamond^{ba} (U^a \land D)$ is valid.

Based on these lemmas, the authors observe that there are no complete belief models. Here, we give the theorem in two forms.

**Theorem 11** ((Brandenburger & Keisler, 2006)).

- **First-Order Version**: Every belief model $M$ has either a hole at $U^a$, a hole at $U^b$, a big hole at one of the formulas

  (i) $\forall x. P^b(y, x)$

  (ii) $x$ believes $\forall x. P^b(y, x)$

  (iii) $y$ believes $[x$ believes $\forall x. P^b(y, x)]$

  a hole at the formula
or a big hole at the formula

Thus, there is no belief model which is complete for a language \( \mathcal{L} \) which contains the tautologically true formulas and formulas (i)-(v).

- Modal Version: There is either a hole at \( U^a \), a hole at \( U^b \), a big hole at one of the formulas

\[
\bigotimes^{ba} U^a, \quad \Box^{ab} \bigotimes^{ba} U^a, \quad \Box^{ba} \Box^{ab} \bigotimes^{ba} U^a
\]

a hole at the formula \( U^a \land D \), or a big hole at the formula \( \bigotimes^{ba} (U^a \land D) \).

Thus, there is no complete interactive frame for the set of all modal formulas built from \( U^a \), \( U^b \), and \( D \).

### 3.2 Non-well-founded Set Theoretical Approach

Non-well-founded set theory is a theory of sets where the axiom of foundation is replaced by the anti-foundation axiom which is due to Mirimanoff (Mirimanoff, 1917). Then, decades later, it was formulated by Aczel within graph theory, and this motivates our approach here (Aczel, 1988). In non-well-founded (NWF, henceforth) set theory, we can have true statements such as ‘\( x \in x \)’, and such statements present interesting properties in game theory. NWF theories are natural candidates to represent circularity (Barwise & Moss, 1996).

On the other hand, NWF set theory is not immune to the problems that the classical set theory suffers from. For example, note that Russell’s paradox is not solved in NWF setting, and moreover the subset relation stays the same in NWF theory (Moss, 2009). Therefore, we may not expect the BK paradox to disappear in NWF setting. Yet, NWF set theory will give us many other tools in game theory.

What we call a non-well-founded model is a tuple \( M = (W, V) \) where \( W \) is a non-empty non-well-founded set (hyperset, for short), and \( V \) is a valuation assigning propositional variables to the elements of \( W \). We also relax the condition that \( U^a \cap U^b = \emptyset \). Namely, some states may belong to both of the players, and whoever gets there may make a move there.

Now, we give the semantics of (basic) modal logic in non-well-founded setting (Gerbrandy, 1999). We will use the symbol \( \models^+ \) to represent the semantical consequence relation in a NWF model.

\[
M, w \models^+ \bigotimes \varphi \quad \text{iff} \quad \exists v \in w. \text{ such that } M, v \models^+ \varphi
\]
\[
M, w \models^+ \Box \varphi \quad \text{iff} \quad \forall v \in w. \; v \in w \text{ implies } M, v \models^+ \varphi
\]

Now, we can give a non-standard semantics for the belief and assumption modalities \( \Box^i \) and \( \bigotimes^i \) respectively for \( i, j \in \{a, b\} \).
\[ M, w \models □^b \varphi \quad \text{iff} \quad M, w \models □^a \left( □^b \varphi \land \forall v \in w(M, v \models □^b \varphi) \right) \]
\[ M, w \models □^b \varphi \quad \text{iff} \quad M, w \models □^a \left( □^b \varphi \land \forall v \in w(M, v \models □^b \varphi) \right) \]

Note that the following formulas discussed in the original paper are still valid as before.

\[ \Box^{ab} U^b \leftrightarrow U^a, \quad \Box^{ba} U^a \leftrightarrow U^b, \quad \Box^{ab} U^a \leftrightarrow \bot, \quad \Box^{ba} U^b \leftrightarrow \bot \]

Furthermore, the following formulas are still unsatisfiable as before.

\[ \Box^{ab} U^b \rightarrow U^b, \quad \Box^{ba} U^b \rightarrow \Box^{ba} U^b, \quad \Box^{ab} U^b \rightarrow \Box^{ab} U^b \]

Now, we need to redefine the diagonal set in NWF case. Recall that, in the standard case, diagonal set \( D \) is defined with respect to the accessibility relation \( P \) which we defined earlier. In NWF case, we will use membership relation for that purpose.

**Definition 12.** Define \( D^+ = \{ w \in W : \forall v \in W. (v \in w \rightarrow w \notin v) \} \).

We define the propositional variable \( D^+ \) as the propositional variable with the valuation set \( D^+ \). We call a set \( A \) transitive whenever \( a \in A \) and \( b \in a \), then \( b \in A \).

Now, we observe how the NWF models make a difference in the context of the BK paradox. Notice that BK argument relies on two lemmas which we have mentioned earlier in Lemma 1. Now, we present counter-models in NWF theory for them.

**Conjecture 18.** In a NWF belief structure, if \( \Diamond^{ab} U^b \) is satisfiable, then the formula
\[ \Box^{ab} \Diamond^{ba} U^a \land D^+ \]
is also satisfiable.

**Conjecture 19.** The formula \( \Box^{ab} \Diamond^{ba} (U^a \land D^+) \) is satisfiable in some NWF belief structures.

Therefore, the BK Lemma 1 is refuted in NWF belief models. Notice that Lemma 1 is central in Brandenburger and Keisler’s proof of the incompleteness of belief structures. Thus, we ask whether the failure of Lemma 1 would mean that there can be complete NWF belief structures.

Consider the following NWF counter-model \( M \). Let \( W = \{ w, u, v, t, y \} \) where \( U^a = \{ w, u \} \), and \( U^b = \{ v, t, y \} \). Put \( w = \{ v, t \}, v = \{ u, w \}, u = \{ t \}, y = \{ u \} \).

Then, \( M \) satisfies the formulas given in Theorem 11. First, \( M \) has no holes at \( U^a \) and \( U^b \) as the first is assumed at \( v \), and the latter is assumed at \( w \). Therefore, \( v \models \Diamond^{ba} U^a \). Moreover, it has no big holes, thus \( w \) believes \( \Diamond^{ba} U^a \) giving \( w \models \Diamond^{ba} U^a \). Similarly, \( v \) believes \( \Diamond^{ab} \Diamond^{ba} U^a \) yielding \( v \models \Diamond^{ba} \Diamond^{ba} U^a \).

The state \( u \) also satisfies \( D^+ \), and it is assumed by \( y \), thus \( y \) assumes \( D^+ (u) \). This counter-model shows that Theorem 11 does not hold in NWF belief structures. Yet, we have to be careful here. Our counter model does not establish the fact...
that NWF belief models are complete. It does establish the fact that they do not have the same holes as the standard belief models. We will get back to this question later on, and give an answer from a category theoretical point of view.

Games with NWF Belief Structures  In our analysis, we altered the set theory that the belief structures and belief sets use. Then, the immediate question is the following: What is the game theoretical meaning of it? Let us briefly elaborate on it.

Aczel’s Anti-Foundation Axiom states that for each connected rooted directed graph, there corresponds a unique set (Aczel, 1988). Since the set in question is the game theoretical belief set, it means that for every (labeled) directed graph we have NWF set that represent this game (up to the name and the order of the players). Hence, the following theorem.

Conjecture 20. For every (labeled) rooted directed connected graph, there corresponds to a unique two-player NWF belief structure up to the permutation of type spaces, and the order of players.

Therefore, we can use any connected graphs (i.e. not only trees, but also connected graphs with loops) to represent games in extensive form (up to the natural conditions in the theorem). This extension of game theory and similar issues have been raised by Harsanyi (Harsanyi, 1967). Universal type spaces do employ some circularity in definitions, and NWF theories are needed to give an explanation of such structures. Harsanyi mentions the following.

It seems to me that the basic reason why the theory of games with incomplete information has made so little progress so far lies in the fact that these games give rise, or at least appear to give rise, to an infinite regress in reciprocal expectations on the part of the players. (Harsanyi, 1967)

Moss, for example, takes this quote and considers universal type spaces from NWF set theory point of view (Moss, 2009). The following example illustrates Theorem 20.

Example 13. Consider the following labeled, connected directed graph.

The two-player NWF belief structure of this game is as follows. Put \( W = \{w, v, u\} \) where \( w = \{u, v\} \) and \( u = \{w\} \). Assume that \( U^a = \{w\} \), \( U^b = \{u, v\} \) (or any other combination of type spaces). Therefore, this graph corresponds to the game where Bob can reset the game if Alice plays \( L \) at \( w \).
We conclude that NWF belief sets allow us to consider a much larger collection of belief sets for games which was implied by Harsanyi much earlier. By construction, as the Example 13 demonstrates, NWF allows circularity in games. This is not surprising considering the motivation for NWF set theory.

3.3 Paraconsistent Approach

Paraconsistent logics can be captured by using several rather strong algebraic, topological and category theoretical structures. In this chapter, we will approach paraconsistency from such directions, and analyze the BK paradox within paraconsistent logics interpreted in such systems.

3.3.1 Algebraic and Category Theoretical Approach

A recent article on the BK paradox shows the general pattern of such paradoxical cases, and gives some positive results such as fixed-point theorems (Abramsky & Zvesper, 2010). In this section, we will generalize their fixed-point results to some other mathematical structures that can represent paraconsistent logics.

First, let us recall some facts about paraconsistency. Paraconsistency is the umbrella term for logical systems where *ex contradictione quodlibet* fails. Namely, in paraconsistent logics, for some $\varphi, \psi$, we have $\varphi, \neg \varphi \not\vdash \psi$.

Note that the semantical issue behind the failure of *ex contradictione quodlibet* in paraconsistent systems is the fact that in those logics, the intersection of a formula and its negation may not be the empty set.

There are a variety of different logical and algebraic structures to represent paraconsistent logics (Priest, 2002). Co-Heyting algebras are natural algebraic candidates to represent paraconsistency.

**Definition 14.** Let $L$ be a bounded distributive lattice. If there is a binary operation $\Rightarrow : L \times L \to L$ such that for all $x, y, z \in L$,

$$x \leq (y \Rightarrow z) \text{ iff } (x \land y) \leq z,$$

then we call $(L, \Rightarrow)$ a Heyting algebra.

Dually, if we have a binary operation $\setminus : L \times L \to L$ such that

$$(y \setminus z) \leq x \text{ iff } y \leq (x \lor z),$$

then we call $(L, \setminus)$ a co-Heyting algebra. We call $\Rightarrow$ implication, $\setminus$ subtraction.

An immediate example of a co-Heyting algebra is the closed subsets of a given topological space and subtopoi of a given topos (Lawvere, 1991; Mortensen, 2000; Başkent, 2011; Priest, 2002).

Both operations $\Rightarrow$ and $\setminus$ give rise to two different negations. The *intuitionistic negation* $\sim$ is defined as $\sim \varphi \equiv \varphi \to 0$ and *paraconsistent negation* $\sim$ is defined as $\sim \varphi \equiv 1 \setminus \varphi$ where 0 and 1 are the bottom and the top elements of the lattice respectively. Therefore, $\sim \varphi$ is the largest element disjoint from
\( \varphi \), and \( \sim \varphi \) is the smallest element whose join with \( \varphi \) gives the top element \( 1 \) (Reyes & Zolghaghi, 1996). In a Boolean algebra the intuitionistic and paraconsistent negations coincide, and give the usual Boolean negation where we interpret \( \varphi \Rightarrow \psi \) as \( \neg \varphi \lor \psi \), and \( \varphi \setminus \psi \) as \( \varphi \land \neg \psi \) with the usual Boolean negation \( \neg \). What makes closed topologies a paraconsistent structure is the fact that theories that are true at boundary points include formulas and their negation (Mortensen, 2000; Başkent, 2011). Because, a formula \( \varphi \) and its paraconsistent negation \( \sim \varphi \) intersect at the boundary of their extensions. We will discuss the paraconsistent negation in the following sections as well.

On the other hand, the algebraic structures we have mentioned can be approached from a category theory point of view. Before discussing Lawvere’s argument, we need to define weakly point surjective maps. An arrow \( f : A \times A \to B \) is called weakly point surjective if for every \( p : A \to B \), there is an \( x : 1 \to A \) such that for all \( y : 1 \to A \) where \( 1 \) is the terminal object, we have

\[
p \circ y = f \circ \langle x, y \rangle : 1 \to B
\]

In this case, we say, \( p \) is represented by \( x \). A category is called cartesian closed (CCC for short, henceforth), if it has a terminal object, and admits products and exponentiation. A set \( X \) is said to have the fixed-point property for a function \( f \), if there is an element \( x \in X \) such that \( f(x) = x \). Category theoretically, an object \( X \) is said to have the fixed-point property if and only if for every endomorphism \( f : X \to X \), there is \( x : 1 \to X \) with \( xf = x \) (Lawvere, 1969).

**Theorem 15** ((Lawvere, 1969)). In any cartesian closed category, if there exists an object \( A \) and a weakly point-surjective morphism \( g : A \to Y^A \), then \( Y \) has the fixed-point property for \( g \).

It was observed that CCC condition can be relaxed, and Lawvere’s Theorem works for categories that have only finite products (Abramsky & Zvesper, 2010)\(^6\). These authors showed how to reduce Lawvere’s Lemma to the BK paradox and, to reduce the BK paradox to Lawvere’s Lemma\(^7\). Now, our goal is to go one step further, and investigate some other cartesian closed categories which represent the non-classical frameworks that we have investigated in this paper. Therefore, by Lawvere and Abramsky & Zvesper results, we will be able to show the existence of fixed-points in our framework, which will give the BK paradox in those frameworks.

Now, we observe the category theoretical properties of co-Heyting algebras and the category of hypersets. Recall that the category of Heyting algebras is a CCC. A canonical example of a Heyting algebra is the set of opens of a topological space (Awodey, 2006). The objects of such a category will be the open sets. The unique morphisms in that category exists from \( O \) to \( O' \) if \( O \subseteq O' \). What about co-Heyting algebras?

---

\(^6\)This point was already made by Lawvere and Schanuel in *Conceptual Mathematics*. Thanks to Noson Yanofsky for pointing this out.

\(^7\)In order to be able to avoid the technicalities of categorical logic, we do not give the details of their construction and refer the reader to (Abramsky & Zvesper, 2010)
Proposition 2. Co-Heyting algebras are Cartesian closed categories.

Proof. We skip the standard proof.

Example 16. The co-Heyting algebra of the closed sets of a topology is a well-known example of a CCC. Given two objects $C_1, C_2$, we define the unique arrow from $C_1$ to $C_2$, if $C_1 \supseteq C_2$. The product is the union of $C_1$ and $C_2$ as the finite union of closed sets exists in a topology. The exponent $C_1^{C_2}$ is then defined as $\text{Clo}(C_1 \cap C_2)$ where $C_1$ is the complement of $C_1$.

Now, we have the following corollary for Theorem 15.

Corollary 3. In a co-Heyting algebra, if there is an object $A$ and a weakly point-surjective morphism $g : A \to Y^A$, then $Y$ has the fixed-point property.

This is interesting. In other words, even if we allow nontrivial inconsistencies and represent them as a co-Heyting algebra, we will still have fixed-points. This is our first step to establish the possibility of having the BK paradox in paraconsistent setting.

Corollary 4. In a co-Heyting algebraic model, there exists an impossible BK sentence.

The procedure is simple. Take a co-Heyting algebra (or co-Brouwer lattice) that represents a paraconsistent logic. Therefore, Corollary 3 applies. Together with Abramsky & Zvesper’s construction reducing Lawvere’s Lemma to the BK paradox, the result follows. Yet, we need to be careful as the paraconsistent (or co-Heyting) negation is not the same as the Boolean negation. Therefore, we obtain the BK sentence in a structure with different negation. On the other hand, the aforementioned theory does not mean that there cannot be some satisfiable BK sentences. We will elaborate on this later on.

What about the category of non-well-founded sets? Consider the category AFA of hypersets with total maps between them. Category AFA admits a final object $1 = \{\emptyset\}$. Moreover, it also admits exponentiation and products in the usual sense, making it a CCC.

Corollary 5. There exists an impossible BK sentence in all non-well-founded interactive belief structures.

This complements our earlier result. Namely, non-well founded belief structures with hypersets may have fixed-points, indeed different fixed-points.

3.3.2 Topological Approach

Now, we construct the BK argument in the paraconsistent topological setting, and call it paraconsistent topological belief structure. For the agents $a$ and $b$, we have a corresponding type space $A$ and $B$ and define closed set topologies $\tau_A$.

\footnote{Thanks to Florian Lengyel for pointing this out.}
and \( \tau_B \) on \( A \) and \( B \) respectively. Furthermore, to connect them in order to represent belief interaction for the players, we will have additional constructions \( t_A \subseteq A \times B \), and \( t_B \subseteq B \times A \). We will then call \( F = (A, B, \tau_A, \tau_B, t_A, t_B) \) paraconsistent topological belief structure. In this setting, the set \( A \) represent the possible epistemic states of the player \( a \) in which she holds beliefs about player \( b \), or \( b \)'s beliefs etc, and vice versa for the set \( B \) and the player \( a \). Moreover, the topological structure determines those beliefs. For instance, for player \( a \) at the state \( x \in A \), \( t_A \) returns a closed set in \( Y \in \tau_B \subseteq \wp(B) \). In this case, we write \( t_A(x, Y) \) which means that at state \( x \), player \( a \) believes that the states in \( Y \in \tau_B \) are possible for the player \( b \). Moreover, a state \( x \in A \) believes \( \varphi \subseteq B \) if \( \{ y : t_A(x, y) \} \subseteq \varphi \). Furthermore, a state \( x \in A \) assumes \( \varphi \) if \( \{ y : t_A(x, y) \} = \varphi \). Notice that in this definition, we identify logical formulas with their extensions.

The modal language we use will have two modalities representing the beliefs of each agent. Based on the previous modal logical structures, we will give a topological semantics for the BK argument in paraconsistent topological belief structures (Brandenburger & Keisler, 2006; Pacuit, 2007). Let us first give the formal language which we use.

\[
\varphi := p \mid \lnot \varphi \mid \varphi \land \varphi \mid \Box_a \mid \Box_b \mid \bigoplus_a \mid \bigoplus_b
\]

where \( \lnot \) is the topological negation symbol which we defined earlier, and \( \bigoplus_i \) are the belief and assumption operators for player \( i \) respectively.

We have discussed the semantics of the negation already. For \( x \in A, y \in B \), the semantics of the modalities are given as follows with a modal valuation attached to \( F \).

\[
\begin{align*}
x & \models \Box_a \varphi \quad \text{iff} \quad \exists Y \in \tau_B \text{ with } t_A(x, Y) \Rightarrow \forall y \in Y, y \models \varphi \\
x & \models \bigoplus_a \varphi \quad \text{iff} \quad \exists Y \in \tau_B \text{ with } t_A(x, Y) \iff \forall y \in Y, y \models \varphi \\
y & \models \Box_b \varphi \quad \text{iff} \quad \exists X \in \tau_A \text{ with } t_B(y, X) \Rightarrow \forall x \in X, x \models \varphi \\
y & \models \bigoplus_b \varphi \quad \text{iff} \quad \exists X \in \tau_A \text{ with } t_B(y, X) \iff \forall x \in X, x \models \varphi
\end{align*}
\]

We define conjunction, and \( \Diamond_a \) and \( \Diamond_b \) as usual.

Now, we have sufficient tools to represent the BK sentence in our paraconsistent topological belief structure with respect to a state \( x_0 \):

\[
x_0 \models \Box_a \bigoplus_b \varphi \land \Diamond_a \top
\]

Let us analyze this formula in our structure. Notice that the second conjunct guarantees that for the given \( x_0 \in A \), there exists a corresponding set \( Y \in \tau_B \) with \( t_A(x, Y) \). On the other hand, the first conjunct deserves closer attention:

\[
\begin{align*}
x_0 & \models \Box_a \bigoplus_b \varphi \quad \text{iff} \quad \exists Y \in \tau_B \text{ with } t_A(x_0, Y) \Rightarrow \forall y \in Y, y \models \bigoplus_b \varphi \\
& \quad \text{iff} \quad \exists Y \in \tau_B \text{ with } t_A(x_0, Y) \Rightarrow \\
& \quad \quad [\forall y \in Y, \exists X \in \tau_A \text{ with } t_B(y, X) \iff \forall x \in X, x \models \varphi]
\end{align*}
\]

Notice that in our framework, some special \( x \) can satisfy falsehood \( \perp \) to give \( x_0 \models \Box_a \bigoplus_b \perp \land \Diamond_a \top \) for some \( x_0 \). Let the extension of \( p \) be \( X_0 \). Pick \( x_0 \in \partial X_0 \) where \( \partial(\cdot) \) operator denotes the boundary of a set \( \partial(\cdot) = \text{Clo}(\cdot) - \text{Int}(\cdot) \). By the
assumptions of our framework $X_0$ is closed. Moreover, by simple topology $\partial X_0$ is closed as well. By the second conjunct of the formula in question, we know that some $Y \in \tau_B$ exists such that $t_A(x_0, Y)$. Now, for all $y$ in $Y$, we make an additional supposition and associate $y$ with $\partial X_0$ giving $t_B(y, \partial X_0)$. We know that for all $x \in \partial X_0$, we have $x \models p$ as $\partial X_0 \subseteq X_0$ where $X_0$ is the extension of $p$. Moreover, $x \models \neg p$ for all $x \in \partial X_0$ as $\partial X_0 \subseteq (\neg X_0)$, too. Thus, we conclude that $x_0 \models \square_a \Box_b \bot \land \Diamond_a \top$ for some carefully selected $x_0$.

In this construction, we have several suppositions. First, we picked the current state from the boundary of the extension of some proposition (ground or modal). Second, we associate the epistemic accessibility of the second player to the same boundary set. Namely, $a$’s beliefs about $b$ includes her current state.

Now, the BK paradox appears when one substitutes $\varphi$ with the following diagonal formula (whose extension is a closed set by definition of the closed set topology), hence breaking the aforementioned circularity:

$$D(x) = \forall y. [t_A(x, y) \rightarrow \neg t_B(y, x)]$$

The BK impossibility theorem asserts that, under the seriality condition, there is no such $x_0$ satisfying the following.

$$x_0 \models \square_a \Box_b D(x) \iff \exists Y \in \tau_B \text{ with } t_A(x_0, Y) \Rightarrow \left[ \forall y \in Y, \exists X \in \tau_A \text{ with } t_B(y, X) \Leftrightarrow \forall x \in X, x \models \forall y'. (t_A(x, y') \rightarrow \neg t_B(y', x)) \right]$$

Motivated by our earlier discussion, let us analyze the logical statement in question. Let $X_0$ satisfy the statement $t_A(x, y')$ for all $y' \in Y$ and $x \in X_0$ for some $Y$. Then, $\partial X_0 \subseteq X_0$ will satisfy the same formula. Similarly, let $\neg X_0$ satisfy $\neg t_B(y', x)$ for all $y' \in Y$ and $x \in X_0$. Then, by the similar argument, $\partial(\neg X_0)$ satisfy the same formula. Since $\partial(X_0) = \partial(\neg X_0)$, we observe that any $x_0 \in \partial X_0$ satisfy $t_A(x, y')$ and $\neg t_B(y', x)$ with the aforementioned quantification. Thus, such an $x_0$ satisfies $\square_a \Box_b D(x)$. Therefore, the states at the boundary of some closed set satisfy the BK sentence in paraconsistent topological belief structures.

**Conjecture 21.** The BK sentence is satisfiable in paraconsistent topological belief structures.

### 3.3.3 Product Topologies

Now, we will use product topologies to represent belief interaction among the players. The novelty of this approach is not only to economize on the notation and the model, but also to present a more natural way to connect such belief sets of respective agents. For our purposes here, we will only consider two-player games, and our results can easily be generalized to $n$-player. We make use of the constructions presented in some recent works (van Benthem et al., 2006; van Benthem & Sarenac, 2004).
Definition 17. Let $a, b$ be two players with corresponding type space $A$ and $B$. Let $\tau_A$ and $\tau_B$ be the (paraconsistent) closed set topologies of respective type spaces. The product topological paraconsistent belief structure for two agents is given as $(A \times B, \tau_A \times \tau_B)$.

In this framework, we will assume that the topologies are full on their sets - namely $\bigcup \tau_A = A$, and likewise for $B$. Therefore, if player $a$ believes proposition $P \subseteq B$ at state $x \in A$, we will stipulate that there is a closed set $X \in \tau_A$ such that $x \in X$ and a closed set $Y \in \tau_B$ with $Y \subseteq P$, all implying $X \times Y \in \tau_A \times \tau_B$. Player $a$ assumes $P$ if $Y = P$, and likewise for player $b$. Similar to the previous section, we will make use of paraconsistent topological structures with closed sets and paraconsistent negation.

Borrowing a standard definitions from topology, we say that given a set $S \subseteq A \times B$, we say that $S$ is horizontally closed if for any $(x, y) \in S$, there exists a closed set $X \subseteq \tau_A$ with $x \in X \in \tau_A$ and $X \times \{y\} \subseteq S$. Similarly, $S$ is vertically closed if for any $(x, y) \in S$, there exists a closed set $Y \subseteq \tau_B$, and $\{x\} \times Y \subseteq S$ (van Benthem et al., 2006; van Benthem & Sarenac, 2004). In this framework, we say player $a$ at $x \in A$ is said to believe a set $Y \subseteq B$ if $\{x\} \times Y$ is vertically closed.

Now, we can define assumption-complete structures in product topologies. For a given language $\mathcal{L}$ for our belief model, let $\mathcal{L}^a$ and $\mathcal{L}^b$ be the families of all subsets of $A$ and $B$ respectively. Then, we observe that by assumption-completeness, we require every non-empty set $Y \in \mathcal{L}^b$ is assumed by some $x \in A$, and similarly, every non-empty set $X \in \mathcal{L}^a$ is assumed by some $y \in B$.

We can now characterize assumption-complete paraconsistent topological belief models. Given type spaces $A$ and $B$, we construct the coarsest topologies on respective type spaces $\tau_A$ and $\tau_B$ where each subset of $A$ and $B$ are in $\tau_A$ and $\tau_B$. Therefore, it is easy to see that $A \times B$ is vertically and horizontally closed for any $S \subseteq A \times B$. Moreover, under these conditions, our belief structure in question is assumption-complete.

We can relax some these conditions. Assume that now $\tau_A$ and $\tau_B$ are not the coarsest topologies on $A$ and $B$ respectively. Therefore, we define, we weak assumption-completeness for a topological belief structure if every set $S \in A \times B$ is both horizontally and vertically closed. In other words, weak assumption-complete models focus only on the formula that are available in the given structure. There can be some formulas expressible in $\mathcal{L}$, but not available in $\mathcal{L}^a$ or in $\tau_A$ for some reasons. Epistemic game theory, indeed, is full of such cases where players may or may not be allowed to send some particular signals, and some information may be unavailable to some certain players. The following theorem follows directly from the definitions.

Conjecture 22. Let $M = (A \times B, \tau_A \times \tau_B)$ be a product topological paraconsistent belief model. If $M$ is horizontally and vertically closed, then it is weak assumption-complete.
4 Strategies

In this section, we consider strategies as the primitive object of our inquiry. Considering a formal framework that treats them as “first-class citizens”, we will give a dynamic logic that formalizes some types of strategy updates.

4.1 Introduction

In game theory, strategy for a player is defined as “a set of rules that describe exactly how (...) [a] player should choose, depending on how the [other] players have chosen at earlier moves” (Hodkinson et al., 2000). Nevertheless, this definition of strategies is static, and presumably is constructed before the game is actually played.

For example, consider chess. According to Zermelo’s well-known theorem, chess is determined. Then, why would you play chess if you know you will lose the game? Similarly, why would you even play if you know you will win. Clearly, if we have logical omniscience (which we don’t), then it is pointless for the player, who is going to lose, to even start playing the game as she knows the outcome already. If we are not logical omniscient, then chess is only a perfect information game for God, not for the players. Therefore, there seems to be a problem. The static notion of strategies falls short of analyzing perfect information games. Because, we, people, do not strategize as such even in perfect information games - largely because we are not logically omniscient, and we have limited computational power and bounded memory.

While people play games, they observe, learn, recollect and update their strategies during the game as well as adopting deontological strategies and goals before the game. Players update and revise their strategies, for instance, when their opponent makes an unexpected or irrational move. Similarly, external factors may sometimes force the players not to make some certain moves. For instance, assume that you play a video game by using a gamepad or a keyboard, and in the middle of the game, one of the buttons on the gamepad breaks. Hence, from that moment on, you will not be able to make some moves in the game that are controlled by that button on the gamepad. This is most certainly

\[
\begin{align*}
&s_0, P_1 \rightarrow s_1, P_2 \rightarrow s_2, P_1 \rightarrow s_3, P_2 \rightarrow s_4, P_1 \rightarrow s_5 (4, 6) \\
&d \rightarrow d \rightarrow d \rightarrow d \rightarrow d \rightarrow d \\
&s_6 (1, 0) \rightarrow s_7 (0, 2) \rightarrow s_8 (3, 1) \rightarrow s_9 (2, 4) \rightarrow s_{10} (5, 2)
\end{align*}
\]

Figure 4: Centipede game
not part of your strategy. Therefore, you will need to revise your strategy in such a way that some moves will be excluded from your strategy from then on. However, for your opponent, that is not the case as she can still make all the moves available to her.

In some cases, assumptions about the game and the players may fail as well. For instance, consider the centipede game between two players \( P_1 \) and \( P_2 \). Under the assumption of common rationality, the usual backward induction scheme produces the solution that \( P_1 \) needs to make a \( d \) move at \( s_0 \). What happens then, if \( P_1 \) is prohibited from or prevented from making a \( d \) move at \( s_0 \) and onwards right after the beginning of the game (or similarly, if the key that is used to make a \( d \) move is broken)? It means that from a behavioral perspective, the assumption of common rationality is violated since the player did not follow the move which gives the highest pay-off, and \( P_2 \) may need to update her strategy during the game based on what she observed. There can be many reasons why \( P_1 \) may make such a move. The move can be prohibited, a taboo, or simply forbidden or restricted by nature/God, thus becomes unavailable.

Based on this short discussion, we underline that the game theory does not distinguish the following two sorts of strategies. First is the class of strategies constructed before the game - which we call deontological. Second is the class of strategies which are constructed as you go in the game-play. For instance, when you are on your way back home after work, being hungry, you plan to cook some pasta. You strategize how to cook it, and what to add, how to boil and so on so forth. Then, when you are home, you notice that you are out of pasta. This is certainly not part of your strategy, because you didn’t even know that you did not have any pasta. Therefore, you revise your strategy, and cook rice instead. Clearly, there are many epistemic issues here that deal with awareness and learning that we will not address at the present paper. Yet, our work falls in between these two different understandings of strategies, and aims at to draw attention to the differences.

4.2 Strategy logic

In this section, we give a short overview of the strategy logic as presented in (Ghosh et al., 2010) based on the framework of (Ramanujam & Simon, 2008). The focus is on games played between two players given by the set \( N = \{1,2\} \) and a single admissible set of moves \( \Sigma \) for both. Let \( T = (S, \Rightarrow, s_0) \) be a tree rooted at \( s_0 \), on the set of vertices \( S \). A partial function \( \Rightarrow : S \times \Sigma \rightarrow S \) specifies the labeled edges of such a tree where labels represent the moves at the states. The extensive form game tree, then, is a pair \( T = (T, \lambda) \) where \( T \) is a tree as defined before, and \( \lambda : S \rightarrow N \) specifies whose turn it is at each state. A strategy \( \mu_i \) for a player \( i \in N \) is a function \( \mu_i : S^i \rightarrow \Sigma \) where \( S^i = \{ s \in S : \lambda(s) = i \} \). For player \( i \) and strategy \( \mu_i \), the strategy tree \( T_\mu = (S_\mu, \Rightarrow_\mu, s_0, \lambda_\mu) \) is the least subtree of \( T \) satisfying the following two conditions:

1. \( s_0 \in S_\mu \);
2. For any \( s \in S_\mu \), if \( \lambda(s) = i \), then there exists a unique \( s' \in S_\mu \) and action \( a \) such that \( s \xrightarrow{\mu} s' \). Otherwise, if \( \lambda(s) \neq i \), then for all \( s' \) with \( s \xrightarrow{\mu} s' \) for some \( a \), we have \( s \xrightarrow{a} s' \).

In other words, in the strategy tree, the root is included, and for the states that belong to the strategizing player, a unique move is assigned to the player, and for the other player, all possible moves are considered. Notice that strategies return unique moves in SL. Nevertheless, we’d still have a tree even if the strategies are set-valued.

The most basic constructions in SL are the strategy specifications. First, for a given countable set \( X \), a set of formulas \( BF(X) \) is defined as follows, for \( a \in \Sigma \):

\[
BF(X) := x \in X \mid \neg \varphi \mid \varphi \land \varphi \mid \langle a \rangle \varphi
\]

Let \( P^i \) be a countable set of atomic observables for player \( i \), with \( P = P^1 \cup P^2 \). The syntax of strategy specifications is given as follows for \( \varphi \in BF(P^i) \):

\[
Strat^i(P^i) := [\varphi \rightarrow a]^i \mid \sigma_1 + \sigma_2 \mid \sigma_1 \cdot \sigma_2
\]

The specification \([\varphi \rightarrow a]^i \) at player \( i \)'s position stands for “play \( a \) whenever \( \varphi \) holds”. The specification \( \sigma_1 + \sigma_2 \) means that the strategy of the player conforms to the specification \( \sigma_1 \) or \( \sigma_2 \) and \( \sigma_1 \cdot \sigma_2 \) means that the strategy of the player conforms to the specifications \( \sigma_1 \) and \( \sigma_2 \). By the abuse of the notation, we will use \( \leftrightarrow \) to denote the equivalence of specifications: namely, those specifications return the same moves under the same preconditions.

Let \( M = (T, V) \) where \( T = (S, \Rightarrow, s_0, \lambda) \) is an extensive form game tree as defined before, and \( V \) is a valuation function \( (V : S \rightarrow 2^P) \) for the set of propositional variables \( P \). The truth of a formula \( \varphi \in BF(P) \) is given as usual for the propositional, Boolean and modal formulas. The notion “strategy \( \mu \) conforms to specification \( \sigma \) for player \( i \) at state \( s \)” (notation \( \mu, s \models_i \sigma \)) is defined as follows, where \( \text{out}_\mu(s) \) denotes the unique outgoing edge at \( s \) with respect to \( \mu \).

\[
\begin{align*}
\mu, s \models_i [\varphi \rightarrow a]^i & \quad \text{iff} \quad M, s \models \varphi \text{ implies } \text{out}_\mu(s) = a \text{ for } a \in \Sigma \\
\mu, s \models_i \sigma_1 + \sigma_2 & \quad \text{iff} \quad \mu, s \models_i \sigma_1 \text{ or } \mu, s \models_i \sigma_2 \\
\mu, s \models_i \sigma_1 \cdot \sigma_2 & \quad \text{iff} \quad \mu, s \models_i \sigma_1 \text{ and } \mu, s \models_i \sigma_2
\end{align*}
\]

Now, based on the strategy specifications, the syntax of the strategy logic SL is given as follows:

\[
p \mid \neg \varphi \mid \varphi_1 \land \varphi_2 \mid \langle a \rangle \varphi \mid (\sigma)_i : a \mid \sigma \models_i \psi
\]

for propositional variable \( p \in P \), action \( a \in \Sigma \), strategy \( \sigma \in Strat^i(P^i) \), and Boolean formula \( \psi \) over \( P^i \). The intuitive reading of \((\sigma)_i : a \) is that at the current state the strategy specification \( \sigma \) for player \( i \) suggests that the move \( a \) can be played. The intuitive meaning of \( \sigma \models_i \psi \) is that following strategy \( \sigma \) player \( i \) can ensure \( \psi \). The other Boolean connectives and modalities are defined as usual.
We now define the set of available moves \( \text{moves}(s) := \{ a \in \Sigma : \exists s' \in S \text{ with } s \xrightarrow{a} s' \} \). Then, based on \( \text{moves} \), we construct the set of enabled moves at state \( s \) in strategy \( \sigma \) is defined by induction as follows.

\[
\begin{align*}
\text{moves}(s) & = \{ a \in \Sigma : \exists s' \in S \text{ with } s \xrightarrow{a} s' \} \\
\{ \text{moves}(s) \} & = \{ \lambda(s) = i; M, s \models \psi, a \in \text{moves}(s) \} \\
\text{moves}(s) & = \emptyset : \lambda(s) = i; M, s \models \psi, a \notin \text{moves}(s) \\
\Sigma & = \text{otherwise}
\end{align*}
\]

\[
\sigma_1 + \sigma_2(s) = \sigma_1(s) \cup \sigma_2(s)
\]

\[
\sigma_1 : \sigma_2(s) = \sigma_1(s) \cap \sigma_2(s)
\]

The truth definition for the strategy formulas are as follows:

\[
\begin{align*}
M, s \models (a) \varphi & \iff \exists s' \text{ such that } s \xrightarrow{a} s' \text{ and } M, s' \models \varphi \\
M, s \models (\sigma)_i : a & \iff a \in \sigma(s) \\
M, s \models \sigma \xrightarrow{i}, \psi & \iff \forall s' \text{ such that } s \xrightarrow{a} s' \text{ in } T_s|\sigma, \\
we have \ M, s' \models \psi \land (\text{turn}_i \rightarrow \text{enabled}_\sigma)
\end{align*}
\]

where \( \sigma(s) \) denotes the set of the enabled moves at state \( s \) in strategy \( \sigma \), and \( \Rightarrow^*_\sigma \) denotes the reflexive transitive closure of \( \Rightarrow \). Furthermore, \( T_s \) is the tree that consists of the unique path from the root \( (s_0) \) to \( s \) and the subtree rooted at \( s \), and \( T_s|\sigma \) is the least subtree of \( T_s \) that contains a unique path from \( s_0 \) to \( s \) and from \( s \) onwards, for each player \( i \) node, all the moves enabled by \( \sigma \), and for each node of the opponent player, all possible moves. The proposition \( \text{turn}_i \), denotes that it is \( i \)’s turn to play. Finally, define \( \text{enabled}_\sigma = \bigvee_{a \in \Sigma} (a) \top \land (\sigma)_i : a \). The axioms of SL are given as follows:

- All the substitutional instances of the tautologies of propositional calculus
- \( [a](\varphi \rightarrow \psi) \rightarrow ([a] \varphi \rightarrow [a] \psi) \)
- \( [a] \varphi \rightarrow [a] \varphi \)
- \( [a] \top \rightarrow ([\psi \rightarrow [a] i : a]) \)
- \( \text{turn}_i \land (\psi \rightarrow [a] i : a) \rightarrow (a) \top \)
- \( \text{turn}_i \land ((\psi \rightarrow [a] i : a) : c \leftrightarrow \neg \psi) \text{ for all } a \neq c \)
- \( (\sigma + \sigma')_i : a \leftrightarrow (\sigma : a)_i \lor (\sigma' : a)_i \)
- \( (\sigma \cdot \sigma')_i : a \leftrightarrow (\sigma : a)_i \land (\sigma' : a)_i \)
- \( \sigma \xrightarrow{i} \psi \rightarrow [\psi \land \text{inv}_0^\sigma(a, \psi) \land \text{inv}_0^\sigma(\psi) \land \text{enabled}_\sigma] \)

Here, \( \text{inv}_0^\sigma(a, \psi) = (\text{turn}_i \land (\sigma)_i : a) \rightarrow [a](\sigma \xrightarrow{i} \psi) \) which expresses the fact that after an \( a \) move by \( i \) which conforms to \( \sigma \), the statement \( \sigma \xrightarrow{i} \psi \) continues to hold, and \( \text{inv}_0^\sigma(\psi) = \text{turn}_i \rightarrow \sigma(\sigma \xrightarrow{i} \psi) \) states that after any move of \( -i \), \( \sigma \xrightarrow{i} \psi \) continues to hold. Here, \( \bigcirc \varphi \equiv \bigvee_{a \in \Sigma} (a) \varphi \) and \( \bigcirc \varphi \equiv \neg \bigcirc \neg \varphi \).
Now, we discuss the inference rules that SL employs. Modus ponens is familiar as well as generalization for \([a]\) for each \(a \in \Sigma\). The induction rule is a bit more complex: From the formulas \(\varphi \land (\text{turn}_i \land (\sigma) : a) \rightarrow [a]\varphi, \varphi \land \text{turn}_{i-1} \rightarrow \circ \varphi, \varphi \rightarrow \psi \land \text{enabled}_i\), derive \(\varphi \rightarrow \sigma \rightarrow_i \psi\). The axiom system of SL is sound and complete with respect to the given semantics (Ramanujam & Simon, 2008).

4.3 Restricted strategy logic

4.3.1 Basics

Let us now extend SL to restricted strategy logic, henceforth RSL, by allowing move restrictions during the game. Recall that our motivation can be illustrated with the example of a gamepad/keyboard which gets broken during the game play disallowing the player to make some certain moves from that moment on.

We denote the move restriction by \([\sigma]a\) for a strategy specification \(\sigma\) and action \(a\) for player \(i\). Informally, after the move restriction of \(\sigma\) by \(a\), player \(i\) will not be able to make an \(a\) move. We incorporate restrictions in RSL at the level of strategy specifications. In SL, recall that strategies are functions. Therefore, they only produce one move per state. However, our dynamic take in strategies cover more general cases where strategies can offer a set of moves to the player. Thus, in RSL, we define strategy \(\mu_i\) as \(\mu_i : S \rightarrow 2^{\Sigma}\). By \(\text{out}_{\mu_i}(s)\) we will denote the set of moves returned by \(\mu_i\) at \(s\). Then, the extended syntax of strategy specifications for player \(i\) is given as follows.

\[
\text{Strat}_i(P_i) := [\psi \rightarrow a]^i_i | \sigma + \sigma | \sigma \cdot \sigma | [\sigma!a]^i_i
\]

Notice that the restrictions affect only the player who gets a move restriction. In other words, if \(a\) is prohibited to player \(i\), it does not mean that some other player \(j\) cannot make an \(a\) move. In other words, if my gamepad/keyboard is broken, it doesn’t mean that yours is broken as well. Thus, we do not allow constructs like \([\psi \rightarrow a]^{1|2}\).

Once a move is restricted at a state, we will prone the strategy tree removing the prohibited move from that state on. Therefore, given \(\mu_i : S^i \rightarrow 2^\Sigma\), we define the updated strategy relation \(\mu_i! : S^i \rightarrow 2^{\Sigma - \{a\}}\). We are now ready to define confirmation of restricted specifications to strategies. Note that we skip the cases for \(\cdot\) and \(+\) as they are exactly the same.

\[
\mu, s \models_i [\varphi \rightarrow a]^i_i \iff M, s \models \varphi \text{ implies } a \in \text{out}_{\mu}(s)
\]

\[
\mu_i, s \models_i [\sigma!a]^i_i \iff a \notin \text{out}_{\mu_i}(s) \text{ and } \mu!a, s \models_i \sigma
\]

In the sequel, we omit the superscript that indicates the agents, thus, we write \(S\) for \(S^i\), and \(\sigma!a\) for \([\sigma!a]^i_i\) when it is obvious. Given a strategy \(\mu\) and its strategy tree \(T_\mu = (S_\mu, \Rightarrow_\mu, s_0, \lambda_\mu)\), we define the restricted strategy structure \(T_{\mu!a}\) with respect to an action \(a\). Once we removed the restricted moves, the updated structure may not be a tree (it may be a forest). For this reason, we take \((\mu!a, s)\) as the connected component of \(T_{\mu!a}\) that includes \(s\). Therefore, for a fixed strategy, restrictions may yield different restricted strategy trees at different states.
This is perfectly fine for our intuition, because the state of the game where the restriction is made is important. In other words, it matters where may gamepad is broken during the game play. Now, for player $i$ and strategy $\mu$ and move $a$, the restricted strategy tree $T_{\mu a} = (S_{\mu a}, \Rightarrow_{\mu a}, s', \lambda_{\mu a})$ is the least subtree of $T$ satisfying the following two conditions:

1. $s' \in S_{\mu a}$;

2. For any $s \in S_{\mu a}$, if $\lambda(s) = i$, then there exists a unique $t \in S_{\mu a}$ and action $b \neq a \in \mu a(s)$ such that $s \Rightarrow_{\mu a} t$. Otherwise, if $\lambda(s) \neq i$, then for all $t$ with $s \Rightarrow_{b} t$ for $b \neq a$, we have $s \Rightarrow_{\mu a} t$.

Note that we introduce such conditions so that an RSL model can easily be considered as a submodel of a SL model with some additional assumptions. We can now make some observations.

Conjecture 23. For any strategy $\mu$, state $s$, specification $\sigma$, and formula $\psi$, we have $\mu, s \models [\psi \rightarrow a]!a$.

Conjecture 24. For strategy specifications $\sigma$ and $\sigma'$, and action $a$, we have $\langle \sigma \cdot \sigma' \rangle a \leftrightarrow (\sigma '=a \cdot (\sigma' a) \cdot (\sigma a) + (\sigma + \sigma') a \leftrightarrow (\sigma' a) + (\sigma'a)$.

Restrictions stabilize immediately. For $n \geq 1$, we use notation $\sigma^n a$ to denote $n \geq 1$-consecutive restrictions of $\sigma$ by move $a$. Similarly, we put $\mu^n a$ for the corresponding strategy tree $\mu$.

Conjecture 25. For arbitrary strategy specification $\sigma$, action $a$ and state $s$, we have $\sigma a!a \leftrightarrow \sigma a$. Moreover, we have $\sigma^n!a \leftrightarrow \sigma!a$.

Since restrictions are local and operate by elimination, the order of the restrictions does not matter.

Conjecture 26. For any actions $a, b$, we have $\langle \sigma!a \rangle b \leftrightarrow (\sigma!b) a$.

4.3.2 A Case Study: The Centipede Game

Let us consider the centipede game (see Figure 4) and see how RSL can formalize it when a restricted strategy specification can change the game after an unexpected move. Let us call the players $P_1$ and $P_2$. The set of actions in the centipede game is $\Sigma = \{d, a\}$ where $d, a$ mean that the player moves down or across, respectively. Utilities for individual players are indicated by a tuple $(x, y)$ where $x$ is the utility for $P_1$, and $y$ is the utility for $P_2$. For the sake of generality, we will not impose any further conditions on the strategies, such as rationality or $\max \min$ etc.

In a recent work, Artemov approached the centipede game from a rationality and epistemology based perspective (Artemov, 2009b). Now, similar to his approach, we will use symbols $r_1$ and $r_2$ to denote the propositions “$P_1$ is rational” and “$P_2$ is rational”, respectively. Let us now construct rational strategies $\mu$ and $\nu$ for $P_1$ and $P_2$ respectively following the backward induction scheme. At $s_4$,
The case that $P_1$ makes a $d$ move, if she is rational. Therefore, we have $\mu, s_4 \models [r_1 \to d]^{P_1}$.

However, at $s_3$, $P_2$ would be aware of $P_1$'s possible move at $s_4$ and the fact that $P_1$ is rational as well, thus makes a $d$ move if she herself is rational. So, we have $\nu, s_3 \models [(r_1 \land (a)(r_1 \lor (d)r_1)) \lor (d)(r_2 \land ((a)r_1 \lor (d)r_1))] \to d]^{P_2}$. Following the same strategy, we obtain the following.

$$\mu, s_2 \models [r_1 \land (a)(r_1 \lor (d)r_1) \lor (d)(r_2 \land ((a)r_1 \lor (d)r_1))] \to d]^{P_1}$$

$$\nu, s_1 \models [r_2 \land (a)(r_1 \lor (d)r_1) \lor (d)(r_2 \land ((a)r_1 \lor (d)r_1))] \to d]^{P_1}$$

$$\mu, s_0 \models [r_1 \land (a)(r_1 \lor (d)r_1) \lor (d)(r_2 \land ((a)r_1 \lor (d)r_1))] \to d]^{P_1}$$

Let $\diamond \varphi := (a)\varphi \lor (d)\varphi$. Then, we have the following statements.

$$\mu, s_4 \models [r_1 \to d]^{P_1}$$

$$\nu, s_3 \models [r_2 \land (a)\varphi \lor (d)\varphi \lor (d)\varphi \lor (d)\varphi \lor (d)\varphi] \to d]^{P_2}$$

$$\mu, s_2 \models [r_1 \land (a)\varphi \lor (d)\varphi \lor (d)\varphi \lor (d)\varphi \lor (d)\varphi] \to d]^{P_2}$$

$$\nu, s_1 \models [r_2 \land (a)\varphi \lor (d)\varphi \lor (d)\varphi \lor (d)\varphi \lor (d)\varphi] \to d]^{P_2}$$

$$\mu, s_0 \models [r_1 \land (a)\varphi \lor (d)\varphi \lor (d)\varphi \lor (d)\varphi \lor (d)\varphi] \to d]^{P_2}$$

Therefore, backward inductively, under the assumption of common rationality, we observe that $P_1$ should make a $d$ move at $s_0$. Furthermore, this can be generalized to many other games.

**Conjecture 27.** Assuming common rationality in RSL framework, backward induction scheme produces a unique solution in games with ordinal pay-offs.

Argument for the proof of the theorem is quite straight-forward. Even if the strategies may be set valued, and hence return a set of moves per state, the assumption of common rationality forces the player to choose the move which returns the highest pay-off. Since this fact is known among players, by induction, we can show that the solution is unique.

Let us follow the backward induction scheme again with the updated game tree given above. Notice that the specification $[r_1 \to d]^{P_1}$ does not conform with $\mu d$ (as $\mu d, s, s' \not\vDash [r_1 \to d]^{P_1}[d]^{P_1}$ for all $s$)\(^9\). Therefore, the move that rationality

\(^9\)We slightly abuse the formal language here. For a specification $\sigma$, we put $\mu, s \models \neg \sigma$ if it is not the case that $\mu, s \models \sigma$. 

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implies is not admissible for \( P1 \) from that point on. Thus, \( \mu d \) conforms to those specifications that implies an \( a \) move. Thus, \( \mu d, s_4 \models [T \rightarrow a]P1 \cdot [r_1 \rightarrow d]P1 \). Therefore, at \( s_3 \), being rational, \( P2 \) choses the move with the highest pay-off, and makes an \( a \) move. Thus, \( \nu, s_3 \models r_2 \land (\diamond \lnot [r_1 \rightarrow d]) \rightarrow a]P2 \). Similarly, at \( s_1 \), we have \( \nu, s_1 \models r_2 \land (\diamond \lnot [r_1 \rightarrow d] \land (r_2 \land \diamond \lnot [r_1 \rightarrow d])) \rightarrow a]P2 \). Finally, at the root, we have \( \mu d, s_0 \models [T \rightarrow a]P1 \). Thus, in the restricted centipede game, \( P1 \) makes an \( a \) move.

### 4.3.3 Axiomatization, Completeness, and Complexity

Before we discuss the axiomatization of RSL and its completeness with respect to the intended semantics, we first define the set of enabled moves for the new construct as \( \{[\sigma a]i(s) = \sigma(s) - \{a\} \} \) reflecting our intuition. We now give the syntax of RSL, which is the same as that of SL:

\[
p | (\sigma)i : a | \neg \varphi | \varphi_1 \land \varphi_2 | \langle a \rangle \varphi | (\sigma) \rightarrow_i \psi
\]

The semantics and the truth definitions of the formulas are defined as earlier with the exception of strategy specifications for restrictions (cf. Section 2). The axiom system of RSL consists of the axioms and rules of SL together with the following axioms for the added specification construct:

- \( (\sigma!a)i : c \leftrightarrow (\lnot \delta^σ_i(a)) \land (\sigma : c), c \neq a \)
- \( (\sigma!a)i : a \leftrightarrow \bot \)

Here, \( \delta^σ_i(a) = \text{turn}_i \land (\sigma)_i : a \), meaning it is \( i \)'s turn and the move \( a \) is enabled by strategy \( \sigma \) for the player \( i \). Now, observe that the satisfaction of \( ([\sigma!a]i)_i : c \) gives \( c \in ([\sigma!a])_i(s) \). For this reason, we have \( ([\sigma!a])_i : a \) false. The soundness of the given axioms are straightforward and hence skipped. The new specification we introduced does not bring along an extra derivation rule since the new operator is at the level of specifications, not the formulas.

**Conjecture 28.** The axiom system of RSL is complete with respect to the given semantics.

Now we discuss a reduction\(^{10}\), of first SL, and then of RSL to (multi-modal) Computational Tree Logic CTL*. We refer the readers who are not familiar with CTL* to (Emerson, 1990; Emerson & Halpern, 1986). First, we discuss how we construct a CTL* model based on a given SL/RSL model, then describe a translation from SL to CTL*. Let us take a strategy model \( M = (T, V) \) where \( T = (S, \Rightarrow, s_0, \lambda) \). Notice that the function \( \Rightarrow \) was defined from \( S \times \Sigma \) to \( S \). We can redefine it by *currying*. Given \( \Rightarrow : S \times \Sigma \rightarrow S \), we can define the transition \( a : S \rightarrow S \) for each move \( a \). Therefore, we can curry \( \Rightarrow \) to get \( \Rightarrow = \bigcup_{a \in \Sigma} \Rightarrow^a \). We can think of \( M \) as a pointed multi-modal CTL* tree model \( M^* = (S, s_0, \{a\}_{a \in \Sigma}, V) \) that has next time modalities for each action. In this case, corresponding to

\(^{10}\)Thanks to Sujata Ghosh for suggesting this reduction.
each binary relation $\mathcal{R}$, we will have a dynamic next-time modality $X_a$ which quantifies over the sub-path on the same branch (note that state formulas are also path formulas (Emerson & Halpern, 1986)). For the reflexive-transitive closure of $\mathcal{R}$, we will use $\Box$ for all accessible future times in the same branch. Before giving the translation of SL specifications and formulas into CTL*, let us introduce some special propositions. We label the states that are returned by a strategy $\mu$ with the proposition $\text{strategy}_\mu$, stipulating that $\text{strategy}_\mu$ holds only at those points. Notice that given two points in the domain of $\mu$, there is a unique path between these two (which satisfies $\text{strategy}_\mu$). Moreover, we use $\text{turn}_i$ as a proposition that denotes that it is $i$’s turn to play, i.e. $s \models \text{turn}_i$ iff $s \in S^i$. We give two translations. First, $\text{tr}$ translates strategy specifications to CTL* formulas while the second $\text{Tr}$ translates SL formulas to CTL* formulas. Given a strategy $\mu$, conformation to $\mu$ is translated as follows.

- $\text{tr}(\psi \rightarrow a^i) = \text{Tr}(\psi) \rightarrow \Box (\text{strategy}_\mu \land X_a \top)$
- $\text{tr}(\sigma_1 + \sigma_2) = \text{tr}(\sigma_1) \lor \text{tr}(\sigma_2)$
- $\text{tr}(\sigma_1 \cdot \sigma_2) = \text{tr}(\sigma_1) \land \text{tr}(\sigma_2)$

Notice that in the first case the translation makes use of the formula translation $\text{Tr}$ for formula $\psi$ defined below. Assuming the correctness of $\text{Tr}$ (which we will show next in Theorem 30), correctness of the translation $\text{tr}$ is straightforward.

**Conjecture 29.** Let $\mu$ be a strategy and $\sigma$ a strategy specification in SL, and let $\mu^*$ be the corresponding (sub)tree in a CTL* model. Then, $\mu, s \models \sigma$ iff $\mu^*, s \models \text{tr}(\sigma)$.

Here follows the translation $\text{Tr}$ of formulas from SL to CTL* skipping the Boolean cases. Note that the translation is very similar to the Kripke semantics for Propositional Dynamic Logic where for each action $a$, a relation $R_a$ and a modality associated with $R_a$ are introduced.

- $\text{Tr}((a) \varphi) = X_a \text{Tr}(\varphi)$
- $\text{Tr}(\langle c \rangle_i : \varphi) = X_c \top$ for $c \in \sigma(s)$
- $\text{Tr}(\langle c \rangle_i : \varphi) = \bot$ if $c \notin \sigma(s)$
- $\text{Tr}(\sigma \rightarrow_i \psi) = \Box (\text{strategy}_\sigma \land (\text{Tr}(\psi) \land (\text{turn}_i \rightarrow \text{enabled}_c)))$

Here, the atom $\text{enabled}_c$ is true at a state $s$ in CTL* if and only if for at least one $a \in \Sigma$ we have $X_a \top$ and $a \in \sigma(s)$. Finally, the CTL* correspondence of $\sigma(s)$ to the set of enabled moves at $s$ by the strategy specification $\sigma$ is defined exactly as before with one small arrangement in the definition of admissible moves at a given state $s$, namely $\text{moves}(s) = \{a : s \models_{\text{CTL}^*} X_a \top\}$. As an example, consider the translation of the proposition $\text{out}_\mu = a$. Recall that it means that $a$ is the unique outgoing edge according to strategy $\mu$ at the state where the formula is interpreted. Therefore, there is a branch that is followed by strategy $\mu$, and at
that branch, at the current state, $a$ is an admissible move. This corresponds to
the translation $E(\text{strategy}_\mu \land X_a \top)$. The following theorem summarizes our
efforts here.

**Conjecture 30.** Let $M$ be a SL model and let $M^*$ be its CTL* correspondent. Then,
$M, s \models \varphi \iff M^*, s \models \text{Tr}(\varphi)$ for any state $s \in S$.

Notice that the translation we suggest is model-dependent. For example,
the predicate $\text{strategy}_\mu$, depends on the strategy $\mu$, thus the model. For this
reason, the suggested translation is not entirely syntactic and does not give us
an immediate decidability result (using the fact that CTL* is decidable). How-
ever, model checking problem for CTL* is PSPACE-complete (Emerson, 1990).
Therefore, we have an upper bound for the complexity of the model checking
problem for SL. We next show that the model checking problem for both SL and
RSL is in PSPACE.

**Conjecture 31.** The model checking problem for SL is in PSPACE.

**Corollary 6.** The model checking problem for RSL is in PSPACE.
References


GHOSH, SUJATA, RAMANUJAM, RAMASWAMY, & SIMON, SUNIL. 2010. On Strategy Composition and Game Composition.


