Correction: Some non-classical approaches to the Brandenburger - Keisler Paradox

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In my paper (Başkent, 2015), there appears to be a problem. The problem regards the algebraic and category theoretical properties of co-Heyting (Brouwerian) algebras. Proposition 3.5 must read as follows.

Proposition 3.5 Co-Heyting algebras are co-Cartesian closed categories.

It is well-known that Heyting algebras are cartesian closed (Awodey, 2006). Therefore, their dual are co-cartesian closed: they have initial elements, co-products and co-exponentials.

In the case of co-Heyting algebras, the initial element is $0$ by containment. For every $x$, we have $0 \subseteq x$.

The co-product in a co-Heyting algebra is the join $\lor$ with respect to containment. Because, by definition, for all $x$, $y$, $x \lor y$ is the unique element such that $x \leq x \lor y$ and $y \leq x \lor y$. And, for all $z$ with $x \leq z$ and $y \leq z$, we have $x \lor y \leq z$. This shows that $\lor$ is a co-product, by definition.

The co-exponential object $x_y$ is a bit tricky. We define $x_y$ as $\neg x \land y$. It’s easy to see that the dual of this is $\neg x \lor y$ and it is exponential object in Heyting algebra. Nevertheless, let us show that $\neg x \land y$ is indeed the co-exponential object in co-Heyting algebras.

For two objects $x$, $y$, their co-exponential is an object $x_y$ together with a co-evaluation map $\text{coev} : y \mapsto x_y \lor x$ such that for any object $z$ and a map $f : y \mapsto z \lor x$, there is a unique map $\hat{f} : x_y \mapsto z$ such that the following diagram commutes:

$$
\begin{array}{c}
y \\
\downarrow f \\
z \lor x
\end{array}
\quad \begin{array}{c}
x_y \lor x \\
\downarrow \text{coev} \\
\end{array}
\quad \begin{array}{c}
f \lor \text{id}_x \\
\end{array}
$$

Now, we can claim that co-exponential $x_y$ is $\neg x \land y$. First of all, $\text{coev}$ arrow is $y \leq (\neg x \land y) \lor x$. This always holds, because

$$
y \leq (\neg x \land y) \lor x = (\neg x \land y) \land (y \lor x) = 1 \land (y \lor x) = y \lor x.
$$

We also need to show that $y \leq z \lor x$ iff $\neg x \land y \leq z$, reading off from the commutativity diagram above.
From left-to-right direction, let \( z \lor x \leq y \). Then,
\[
\neg x \land y \leq \neg x \land (z \lor x) = (\neg x \land z) \lor (\neg x \land x) = \neg x \land z \leq z,
\]
which produces \( \neg x \land y \leq z \).

From right-to-left direction, suppose \( \neg x \land y \leq z \). Then, reading from right to left,
\[
y \leq y \lor x = (\neg x \lor x) \land (y \lor x) = (\neg x \lor y) \lor x \leq z \lor x,
\]
which produces \( y \leq z \lor x \). Therefore, \( \neg x \land y \) is the co-exponential object \( x_y \).
This completes the proof.

Similarly, the closed sets in a topology is an example of a co-CCC, not a CCC. In that case, argumentation in the paper is correct: the co-product (not the product) is the union of closed sets \( C_1 \) and \( C_2 \), and the co-exponent (not the exponent) is \( \text{Clo}(C_1 \cap C_2) \).

Even if these corrections change the reasoning for Theorem 3.7, they don’t affect the main result: Co-Heyting algebras still admit fixed-points due to Lawvere’s Theorem because they have products (that is \( \land \)) and co-products (that is \( \lor \)).

**Theorem 3.7**  Co-Heyting algebras admit fixed-points. Therefore, there exists a co-Heyting algebraic model with a satisfiable BK sentence.

First, as demonstrated in [Abramsky & Zvesper, 2015] Proposition 4.3), Lawvere theorem works in any category with finite products, which includes co-Heyting algebras. Moreover, since aforementioned paper also establishes that the BK argument reduces to Lawvere’s theorem, our Theorem 3.7 follows immediately.

More precisely, in co-Heyting algebras, the valuation at the boundary \( \partial \) produces a fixed-point regardless of the truth value.

We define \( \partial(p) = p \land \neg p \) where \( \neg \) is the co-Heyting (paraconsistent) negation. The operator \( \neg \) is unary, thus has to admit fixed-points. Thus, for all \( x \in \partial(p) \), we have \( x = \neg x \).

For the BK paradox, simply take \( p \) as the BK sentence.

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**References**

