

Some Non-Classical Methods in Epistemic Logic and Games

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Outlook of the Presentation

- ▶ Dynamic Epistemic Logic in Topological Semantics
- ▶ Paraconsistency in Topological Semantics
- ▶ The Brandenburger - Keisler Paradox and Paraconsistency
- ▶ A Dynamic Strategy Logic

Introduction I

Public announcement logic is a well-known example of dynamic epistemic logics (Plaza, 1989; Gerbrandy, 1999; van Ditmarsch *et al.*, 2007).

The contribution of public announcement logic (PAL) to the field of knowledge representation is mostly due to its succinctness and clarity in reflecting the simple intuition as to how epistemic updates work in some situation. But, PAL is *not* more expressiveness than the basic epistemic logic (Kooi, 2007).

Public announcement logic has many applications in the fields of formal approaches to social interaction, dynamic logics, knowledge representation and updates (Balbiani *et al.*, 2008; Baltag & Moss, 2004; van Benthem, 2006; van Benthem *et al.*, 2005).

Introduction II

Virtually almost all applications of PAL make use of Kripke models for knowledge representation.

However, as it is very well known, Kripke models are not the only representational tool for modal and epistemic logics.

Topological Semantics for PAL I

The way PAL updates the epistemic states of the knower is by “state-elimination”. A truthful announcement φ is made, and consequently, the agents updates their epistemic states by eliminating the possible states where φ is false. Kripkean semantics for PAL is well-known.

Let $\mathcal{T} = \langle T, \tau, \nu \rangle$ be a topological model and φ be a public announcement. We now need to obtain the topological model \mathcal{T}_φ which is the updated model after the announcement. We denote the extension of a formula φ in model \mathcal{M} by $(\varphi)^\mathcal{M}$, so $(\varphi)^\mathcal{M} = \{w : \mathcal{M}, w \models \varphi\}$.

Define $\mathcal{T}_\varphi = \langle T_\varphi, \tau_\varphi, \nu_\varphi \rangle$ where $T_\varphi = T \cap (\varphi)$, $\tau_\varphi = \{O \cap T_\varphi : O \in \tau\}$ and $\nu_\varphi = \nu \cap T_\varphi$.

Clearly, τ_φ is a topology.

Topological Semantics for PAL II

Now, we can give a semantics for the public announcements in topological models.

$$\mathcal{T}, s \models [\varphi]\psi \text{ iff } \mathcal{T}, s \models \varphi \text{ implies } \mathcal{T}_\varphi, s \models \psi$$

Language of topological PAL is the language of propositional logic with the modal operator I (after interior).

Topological Semantics for PAL III

Therefore, the reduction axioms for PAL in topological spaces are given as follows.

1. $[\varphi]p \leftrightarrow (\varphi \rightarrow p)$
2. $[\varphi]\neg\psi \leftrightarrow (\varphi \rightarrow \neg[\varphi]\psi)$
3. $[\varphi](\psi \wedge \chi) \leftrightarrow ([\varphi]\psi \wedge [\varphi]\chi)$
4. $[\varphi]I\psi \leftrightarrow (\varphi \rightarrow I[\varphi]\psi)$

Theorem

PAL in topological spaces is complete (and decidable) with respect to the given semantics.

Topological Semantics for PAL - Multi-Agent I

Let $\mathcal{T} = \langle T, \tau \rangle$ and $\mathcal{T}' = \langle T', \tau' \rangle$ be two topological spaces. Let $X \subseteq T \times T'$. We call X horizontally open (*h-open*) if for any $(x, y) \in X$, there is a $U \in \tau$ such that $x \in U$, and $U \times \{y\} \subseteq X$. In a similar fashion, we call X vertically open (*v-open*) if or any $(x, y) \in X$, there is a $U' \in \tau'$ such that $y \in U'$, and $\{x\} \times U' \subseteq X$. Now, given two topological spaces $\mathcal{T} = \langle T, \tau \rangle$ and $\mathcal{T}' = \langle T', \tau' \rangle$, let us associate two modal operators I and I' respectively to these models. Then, we can obtain a product topology on a language with those two modalities. The product model will be of the form $\langle T \times T', \tau, \tau' \rangle$ on a language with two modalities I and I' .

Topological Semantics for PAL - Multi-Agent II

The semantics of those modalities are given as such.

$$(x, y) \models I\varphi \quad \text{iff} \quad \exists U \in \tau, x \in U \text{ and } \forall u \in U, (u, y) \models \varphi$$

$$(x, y) \models I'\varphi \quad \text{iff} \quad \exists U' \in \tau', y \in U' \text{ and } \forall u' \in U', (x, u') \models \varphi$$

It has been shown that the fusion logic $S4 \oplus S4$ is complete with respect to products of arbitrary topological spaces (van Benthem & Sarenac, 2004).

The language of multi-agent topological PAL is as follows. We specify it for two-agents for simplicity, but it can easily be generalized to n -agents.

$$p \mid \neg\varphi \mid \varphi \wedge \psi \mid K_1\varphi \mid K_2\varphi \mid [\varphi]\psi$$

Topological Semantics for PAL - Multi-Agent III

For given two topological models $\mathcal{T} = \langle T, \tau, \nu \rangle$ and $\mathcal{T}' = \langle T', \tau', \nu' \rangle$, the product topological model $M = \langle T \times T', \tau, \tau', \nu \rangle$ has the following semantics.

$$\begin{aligned}
 M, (x, y) &\models K_1\varphi && \text{iff} && \exists U \in \tau, x \in U \text{ and } \forall u \in U, (u, y) \models \varphi \\
 M, (x, y) &\models K_2\varphi && \text{iff} && \exists U' \in \tau', y \in U' \text{ and } \forall u' \in U', (x, u') \models \varphi \\
 M, (x, y) &\models [\varphi]\psi && \text{iff} && M, (x, y) \models \varphi \text{ implies } M_\varphi, (x, y) \models \psi
 \end{aligned}$$

where $M_\varphi = \langle T_\varphi \times T'_\varphi, \tau_\varphi, \tau'_\varphi, \nu_\varphi \rangle$ is the updated model. We define all T_φ , T'_φ , τ_φ , τ'_φ , and ν_φ as before.

Topological Semantics for PAL - Multi-Agent IV

Therefore, the following axioms axiomatize the product topological PAL together with the axioms of $S4 \oplus S4$.

1. $[\varphi]p \leftrightarrow (\varphi \rightarrow p)$
2. $[\varphi]\neg\psi \leftrightarrow (\varphi \rightarrow \neg[\varphi]\psi)$
3. $[\varphi](\psi \wedge \chi) \leftrightarrow ([\varphi]\psi \wedge [\varphi]\chi)$
4. $[\varphi]K_i\psi \leftrightarrow (\varphi \rightarrow K_i[\varphi]\psi)$

Theorem

Product topological PAL is complete and decidable with respect to the given axiomatization.

Subset Space Semantics for PAL I

Let us recall a classical example for subset space logic (Moss & Parikh, 1992; Dabrowski *et al.*, 1996). Assume that a policeman is measuring the speed of the passing cars where the speed limit is 50 mph. Suppose that the measurement device he uses has an error range of 5 mph. In one instance, he measures the speed of a car, and reads that the speed lies in the interval $[45, 55]$.

Yet, the policeman does not exactly know the speed v . Therefore, he is at the neighborhood situation $(v, [45, 55])$. Then, let us assume that he hears an announcement, say, a message he receives via the police radio, saying that the speed of that particular car is faster than or equal to 48 mph.

Subset Space Semantics for PAL II

In other words, he learns that $v \in [48, \infty]$. Then, the policeman updates his situation to $(v, [48, 55])$ since the announcements are assumed to be truthful. Therefore, public announcement limits his possibilities leading to an improvement, and update in knowledge. The language of the subset space PAL is given as follows:

$$p \mid \perp \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Box\varphi \mid K\varphi \mid [\varphi]\psi$$

In SSL PAL, we have an additional axiom: $[\varphi]\Box\psi \leftrightarrow (\varphi \rightarrow \Box[\varphi]\psi)$

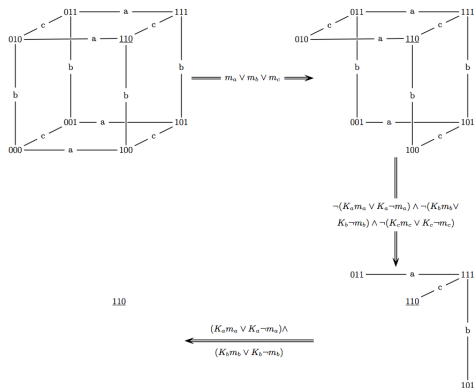
Theorem

SSL PAL is complete and decidable with respect to the given axiomatization.

Announcement Stabilization I

Muddy Children presents an interesting case for PAL (Fagin *et al.*, 1995). In that game, the model gets updated after each children says that she does not know if she had mud on her forehead. The model keeps updated until the announcement is negated (van Benthem, 2007).

Announcement Stabilization II



Announcement Stabilization III

For a model M and a formula φ , we define the announcement limit $\lim_{\varphi} M$ as the first model which is reached by successive announcements of φ that no longer changes after the last announcement is made. Announcement limits exist in both finite and infinite models (van Benthem & Gheerbrant, 2010).

In topological models, the stabilization of the fixed-point definition version of common knowledge may occur later than ordinal stage ω . However, it stabilizes in $\leq \omega$ steps in Kripke models (van Benthem & Sarenac, 2004).

Announcement Stabilization IV

Theorem

For some formula φ and some topological model M , it may take more than ω stage to reach the limit model $\lim_{\varphi} M$.

Announcement Stabilization V

Proof sketch

First, note that it was shown that in multi-agent topological models, stabilization of common knowledge with fixed-point definition may occur later than ω stage. Also note that it was also shown that if the limit model is not empty, the announcement has become common knowledge. Therefore, combining these two observations, we conclude that in some topological models with non-empty limit models, the number of stages for the announcement to be common knowledge may take more than ω steps.

Backward Induction I

Consider the backward induction solution where players trace back their moves to develop a winning strategy. Notice that the Aumann's backward induction solution assumes common knowledge of rationality (Aumann, 1995; Halpern, 2001) (Although according to Halpern, Stalnaker proved otherwise (Halpern, 2001; Stalnaker, 1998; Stalnaker, 1994; Stalnaker, 1996).

Granted, there can be several philosophical and epistemic issues about the centipede game and its relationship with rationality, but we will pursue this direction here (Artemov, 2009; Fitting, 2011; Parikh *et al.*, 2012).

Backward Induction II

However, this issue can also be approached from a dynamic epistemic perspective.

Recently, it has been shown that in any game tree model M taken as a PAL model, $\lim_{\text{rational}} M$ is the actual subtree computed by the backward induction procedure where the proposition rational means that “at the current node, no player has chosen a strictly dominated move in the past coming here” (van Benthem & Gheerbrant, 2010). Therefore, the announcement of node-rationality produces the same result as the backward induction procedure.

Backward Induction III

However, there seems to be a problem in topological models. The admissibility of limit models can take more than ω steps in topological models as we have showed earlier. Therefore, the BI procedure can take ω steps or more.

Theorem

In topological models of games, under the assumption of rationality, the backward induction procedure can take more than ω steps.

Backward Induction IV

Proof sketch

Notice that each tree can easily be converted to a topology by taking the upward closed sets as opens. By the previous discussion, we know that backward induction solution can be attained by obtaining the limit models by publicly announcing the proposition rationality.

Therefore, by an earlier theorem, stabilization can take more than ω steps. Thus, the corresponding backward induction scheme can also take more than ω steps.

Basics

The well-studied notion of deductive explosion describes the situation where every formula can be deduced from an inconsistent set of formulae, i.e. for all φ and ψ , we have $\{\varphi, \neg\varphi\} \vdash \psi$, where \vdash denotes logical consequence relation. In this respect, both “classical” and intuitionistic logics are known to be explosive. Paraconsistent logic, on the other hand, is the umbrella term for logical systems where the logical consequence relation \vdash is not explosive (Priest, 2002).

What is Paraconsistency?

Semantics I

We stipulate that extension of *any* propositional variable will be a closed set (Goodman, 1981; Mortensen, 2000). In that setting, conjunction and disjunction works fine for finite intersections and unions. Nevertheless, negation can be difficult as the complement of a closed set may not be a closed set, thus may not be the extension of a formula in the language. For this reason, we use the symbol \sim that returns the closed complement of a given set.

We can make a similar observation about the boundary points $\partial(\cdot)$ in σ . Now, take $x \in \partial((\varphi))$ where (φ) is a closed set in topology σ . By the above definition, since we have $x \in \partial((\varphi))$, we obtain $x \in (\varphi)$ as (φ) is closed.

Yet, $\partial((\varphi))$ is also included in $(\sim \varphi)$ which we have defined as a closed set. Thus, by the same reasoning, we conclude $x \in (\sim \varphi)$.

Semantics II

Thus, $x \in (\varphi \wedge \sim \varphi)$ yielding that $x \models \varphi \wedge \sim \varphi$. Therefore, in σ , any theory that includes the boundary points will be inconsistent. In this respect, the model $\langle S, \sigma, V \rangle$ with the negation symbol \sim will be called a paraconsistent topological model where V is a valuation function.

Continuity I

A recent research program that considers topological modal logics with continuous functions were discussed in an early work of Artemov *et al.*, and later by Kremer and Mints (Artemov *et al.*, 1997; Kremer & Mints, 2005). In these aforementioned works, they associated continuous functions with temporal modal operator and discussed the orbits of such functions.

Take two closed set topologies σ and σ' on a given set S and a homeomorphism $f : \langle S, \sigma \rangle \rightarrow \langle S, \sigma' \rangle$. We have a simple way to associate the respective valuations between two models M and M' which respectively depend on σ and σ' so that we can have a truth preservation result. Therefore, define $V'(p) = f(V(p))$. Then, we have $M \models \varphi$ iff $M' \models \varphi$.

Continuity II

Theorem

Let $M = \langle S, \sigma, V \rangle$ and $M' = \langle S, \sigma', V' \rangle$ be two paraconsistent topological models with a continuous f from $\langle S, \sigma \rangle$ to $\langle S, \sigma' \rangle$. Define $V'(p) = f(V(p))$. Then $M, w \models \varphi$ implies $M', w' \models \varphi$ for all φ where $w' = f(w)$.

Theorem

Let $M = \langle S, \sigma, V \rangle$ and $M' = \langle S, \sigma', V' \rangle$ be two paraconsistent topological models with an open f from $\langle S, \sigma \rangle$ to $\langle S, \sigma' \rangle$. Define $V'(p) = f(V(p))$. Then $M', w' \models \varphi$ implies $M, w \models \varphi$ for all φ where $w' = f(w)$.

Basics I

Let S and S' be two topological spaces with continuous functions $f, f' : S \rightarrow S'$. A homotopy between f and f' is a continuous function $H : S \times [0, 1] \rightarrow S'$ such that if $s \in S$, then $H(s, 0) = f(s)$ and $H(s, 1) = g(s)$.

Given a model $M = \langle S, \sigma, V \rangle$, we call the family of models $\{M_t = \langle S_t \subseteq S, \sigma_t, V_t \rangle\}_{t \in [0, 1]}$ generated by M and homotopic functions homotopic models. In the generation, we put $V_t = f_t(V)$.

Basics II

Theorem

Given two topological paraconsistent models $M = \langle S, \sigma, V \rangle$ and $M' = \langle S', \sigma', V' \rangle$ with two continuous functions $f, f' : S \rightarrow S'$ both of which respect the valuation: $V' = f(V) = f'(V)$. If there is a homotopy H between f and f' , then M and M' satisfy the same modal formulae.

Therefore, we can “compare” different updates, and construct the connection between them.

An Epistemic Concept: Robust Knowledge/Belief I

Consider two believers Ann and Bob where Ann is an ordinary believer while Bob is a religious cleric of the religion that Ann is following. Therefore, they believe in the same religion and the same rules of the religion.

However, we feel that Bob believes it *more* than Ann even though they believe in exactly the same propositions. In other words, there is still a difference between their belief. Then, what is this difference?

An Epistemic Concept: Robust Knowledge/Belief II

The reason for this is the fact that the *extent* of Bob's knowledge is wider than that of Ann's.

In this context, we can ask the following two questions.

1. How much wider is Bob's belief?
2. How is Ann's belief transformed to Bob's?

These two questions are meaningful. Even if their language cannot tell us which one has wider knowledge, ontologically, we know that Bob has *more* knowledge in some sense even if they agree on every proposition. Clearly, the reason for that is the fact that Bob considers more possible worlds for a given proposition which makes his belief more robust than Ann's¹.

An Epistemic Concept: Robust Knowledge/Belief III

From a dynamic epistemic angle, homeomorphisms and homotopies can explain this transformation from Ann's beliefs to Bob's belief with respect to their models.

The parameter mentioned in the definition of homotopies can easily be considered as a temporal parameter. It help us to give a step by step account of the transformation between Ann's and Bob's belief.

¹We borrowed the term “robust” from Artemov.

The Paradox I

The Brandenburg-Keisler paradox ('BK paradox', henceforth) is a two-person self-referential paradox in epistemic game theory (Brandenburger & Keisler, 2006).

The following configuration of beliefs is impossible:

Paradox

Ann believes that Bob assumes that Ann believes that Bob's assumption is wrong.

The paradox appears if you ask whether "Ann believes that Bob's assumption is wrong".

The Paradox II

Suppose that answer to the above question is 'yes'. Then according to Ann, Bobs assumption is wrong. But, according to Ann, Bobs assumption is Ann believes that Bobs assumption is wrong. However, since the answer to the above question is 'yes', Ann believes that this assumption is correct. So Ann does not believe that Bobs assumption is wrong. Therefore, the answer to the above question must be 'no'. Thus, it is not the case that Ann believes that Bobs assumption is wrong. Hence Ann believes Bobs assumption is correct. That is, it is correct that Ann believes that Bobs assumption is wrong. So, the answer must have been yes. This is a contradiction (Pacuit, 2007). Notice that this is essentially a 2-person version of Russell's Paradox.

Model I

Brandenburger and Keisler use belief sets to represent the players' beliefs.

The model (U^a, U^b, R^a, R^b) that they consider is called a *belief structure* where $R^a \subseteq U^a \times U^b$ and $R^b \subseteq U^b \times U^a$.

The expression $R^a(x, y)$ represents that in state x , Ann believes that the state y is possible for Bob, and similarly for $R^b(y, x)$. We will put $R^a(x) = \{y : R^a(x, y)\}$, and similarly for $R^b(y)$.

At a state x , we say Ann believes $P \subseteq U^b$ if $R^a(x) \subseteq P$.

Model II

A modal logical semantics for the interactive belief structures can be given as well.

We use two modalities \Box and \heartsuit for the belief and assumption operators respectively with the following semantics.

$$\begin{aligned}
 x \models \Box^{ab}\varphi & \text{ iff } \forall y \in U^b.R^a(x, y) \text{ **implies** } y \models \varphi \\
 x \models \heartsuit^{ab}\varphi & \text{ iff } \forall y \in U^b.R^a(x, y) \text{ **iff** } y \models \varphi
 \end{aligned}$$

Model III

A belief structure (U^a, U^b, R^a, R^b) is called *assumption complete* with respect to a set of predicates Π on U^a and U^b if for every predicate $P \in \Pi$ on U^b , there is a state $x \in U^a$ such that x assumes P , and for every predicate $Q \in \Pi$ on U^a , there is a state $y \in U^b$ such that y assumes Q .

We will use special propositions \mathbf{U}^a and \mathbf{U}^b with the following meaning: $w \models \mathbf{U}^a$ if $w \in U^a$, and similarly for \mathbf{U}^b . Namely, \mathbf{U}^a is true at each state for player Ann, and \mathbf{U}^b for player Bob.

Model IV

Brandenburger and Keisler showed that no belief model is complete for its first-order language. Therefore, “not every description of belief can be represented” with belief structures (Brandenburger & Keisler, 2006).

The incompleteness of the belief structures is due to the *holes* in the model. A model, then, has a hole at φ if either $\mathbf{U}^b \wedge \varphi$ is satisfiable but $\heartsuit^{ab}\varphi$ is not, or $\mathbf{U}^a \wedge \varphi$ is satisfiable but $\heartsuit^{ba}\varphi$ is not. A big hole is then defined by using the belief modality \square instead of the assumption modality \heartsuit .

Two Lemmas I

In the original paper, the authors make use of two lemmas before identifying the holes in the system.

First, let us define a special propositional symbol **D** with the following valuation

$$D = \{w \in W : (\forall z \in W)[P(w, z) \rightarrow \neg P(z, w)]\}.$$

Lemma

1. If $\heartsuit^{ab}\mathbf{U}^b$ is satisfiable, then $\Box^{ab}\Box^{ba}\Box^{ab}\heartsuit^{ba}\mathbf{U}^a \rightarrow \mathbf{D}$ is valid.
2. $\neg\Box^{ab}\heartsuit^{ba}(\mathbf{U}^a \wedge \mathbf{D})$ is valid.

Two Lemmas II

Theorem (Modal Version)

There is either a hole at \mathbf{U}^a , a hole at \mathbf{U}^b , a big hole at one of the formulas

$$\heartsuit^{ba}\mathbf{U}^a, \quad \Box^{ab}\heartsuit^{ba}\mathbf{U}^a, \quad \Box^{ba}\Box^{ab}\heartsuit^{ba}\mathbf{U}^a$$

a hole at the formula $\mathbf{U}^a \wedge \mathbf{D}$, or a big hole at the formula $\heartsuit^{ba}(\mathbf{U}^a \wedge \mathbf{D})$. Thus, there is no complete interactive frame for the set of all modal formulas built from \mathbf{U}^a , \mathbf{U}^b , and \mathbf{D} .

Concept I

Non-well-founded set theory is a theory of sets where the axiom of foundation is replaced by the *anti-foundation axiom* which is due to Mirimanoff (Mirimanoff, 1917).

Then, decades later, it was formulated by Aczel within graph theory, and this motivates our approach here (Aczel, 1988). In non-well-founded (NWF, henceforth) set theory, we can have true statements such as ' $x \in x$ ', and such statements present interesting properties in game theory. NWF theories are natural candidates to represent circularity (Barwise & Moss, 1996).

Games with Non-well-founded Type Spaces I

To the best of our knowledge, the idea of using non-well-founded sets as Harsanyi type spaces was first suggested by Lismont (Lismont, 1992), and extended later by Heifetz (Heifetz, 1996). Heifetz motivated his approach by “making the types an explicit part of the states’ structure”, and hence obtained a circularity that enabled him to use non-well-founded sets.

The way he motivated his approach, which is related to our perspective here, is by arguing that NWF type spaces can be used “once states of nature and types would be longer be associated with states of the world, but constitute *their very definition*.” [ibid, (his emphasis)].

Games with Non-well-founded Type Spaces II

There can be argued, at this stage that circularity in a game theoretical model is not desirable. However, considering the fact that Harsanyi type spaces represent uncertainty, NWF models indeed become good candidates to formalize uncertainty. Here is Heifetz on the very same issue.

Games with Non-well-founded Type Spaces III

*Nevertheless, one may continue to argue that a state of the world should indeed be a circular, self-referential object: A state represents a situation of human uncertainty, in which a player considers what other players may think in other situations, and in particular about what they may think there about the current situation. According to such a view, one would seek a formulation where states of the world are indeed self-referring mathematical entities.
(Heifetz, 1996, p. 204).*

Games with Non-well-founded Type Spaces IV

On the other hand, NWF set theory is not immune to the problems that the classical set theory suffers from.

For example, note that Russell's paradox is not solved in NWF setting, and moreover the subset relation stays the same in NWF theory (Moss, 2009).

Therefore, we may not expect the BK paradox to disappear in NWF setting. Yet, NWF set theory will give us many other tools in game theory.

Definitions

What we call a non-well-founded model is a tuple $M = (W, V)$ where W is a non-empty non-well-founded set (*hyperset*, for short), and V is a valuation. We will use the symbol \models^+ to represent the semantical consequence relation in a NWF model based on (Gerbrandy, 1999).

$$\begin{aligned}
 M, w \models^+ \Box^{ij} \varphi \quad \text{iff} \quad & M, w \models^+ \mathbf{U}^i \wedge \\
 & \forall v \in w(M, v \models^+ \mathbf{U}^j \rightarrow M, v \models^+ \varphi) \\
 M, w \models^+ \heartsuit^{ij} \varphi \quad \text{iff} \quad & M, w \models^+ \mathbf{U}^i \wedge \\
 & \forall v \in w(M, v \models^+ \mathbf{U}^j \leftrightarrow M, v \models^+ \varphi)
 \end{aligned}$$

Lemmas I

Define $D^+ = \{w \in W : \forall v \in W. (v \in w \rightarrow w \notin v)\}$.

We define the propositional variable \mathbf{D}^+ as the propositional variable with the valuation set D^+ .

Theorem

There exists a NWF belief structure in which if $\heartsuit^{ab}\mathbf{U}^b$ is satisfiable, then the formula $\Box^{ab}\Box^{ba}\Box^{ab}\heartsuit^{ba}\mathbf{U}^a \wedge \neg\mathbf{D}^+$ is also satisfiable.

Theorem

The formula $\Box^{ab}\heartsuit^{ba}(\mathbf{U}^a \wedge \mathbf{D}^+)$ is satisfiable in some NWF belief structures.

Lemmas II

Yet, we have to be careful here.

This argument does *not* establish that NWF belief models are complete. It establishes the fact that they do not have the same holes as the standard belief models. We will get back to this issue later on, and give an answer from category theoretical point of view.

Self-Reference I

Recently, a category theoretical approach has been presented (Abramsky & Zvesper, 2010).

They focus on the fixed points and extend their analysis to category theory.

Lawvere's Theorem says that if $g : X \rightarrow V^X$ is surjective, then every function $f : V \rightarrow V$ has a fixed point (Lawvere, 1969).

BK paradox occurs if f plays the role of a Boolean negation.

There is an important restriction:

- ▶ X should be cartesian closed (actually, should only admit exponents)

Usually people consider the category of sets **Set**.

Co-Heyting: definitions

Let L be a bounded distributive lattice. If there is defined a binary operation $\Rightarrow: L \times L \rightarrow L$ such that for all $x, y, z \in L$,

$$x \leq (y \Rightarrow z) \text{ iff } (x \wedge y) \leq z,$$

then we call (L, \Rightarrow) a Heyting algebra.

Dually, if we have a binary operation $\setminus: L \times L \rightarrow L$ such that

$$(y \setminus z) \leq x \text{ iff } y \leq (x \vee z),$$

then we call (L, \setminus) a co-Heyting algebra.

We call \Rightarrow implication, \setminus subtraction.

Co-Heyting: definitions

In Boolean algebras, Heyting and co-Heyting algebras give two different operations. We interpret $x \Rightarrow y$ as $\neg x \vee y$, and $x \setminus y$ as $x \wedge \neg y$.

In other words, a co-Heyting algebra is a generalization of a Boolean algebra that allows a generalization in which *principium contradictionis* is relaxed.

Closed set topologies are co-Heyting algebras. The topological paraconsistent negation \sim is defined as $\sim\varphi \equiv \mathbf{1} \setminus \varphi$ where $\mathbf{1}$ is the top element of the lattice.

Paraconsistent BK Paradox

Therefore, even if we have paraconsistent framework. we will have fixed points.

How:

- ▶ Take a co-Heyting algebra - which is a natural candidate for paraconsistency.
- ▶ Observe that it admits exponents: $x^y \equiv x \wedge \neg y$.
- ▶ Thus, Lawvere's Theorem applies.
- ▶ It will still have fixed points: instead of the Boolean negation, take co-Heyting negation as the unary operator.

Topological BK Paradox

Therefore, by following this idea, paraconsistent topological frameworks can be given. We claim that the BK sentence is satisfiable in some paraconsistent topological framework where we use topologies and topological products instead of accessibility relations.

NWF Categories

Category of hypersets is also CCC.

Thus, Lawvere theorem applies.

Therefore, we will have “different” fixed points, thus BK sentences in the NWF setting.

Motivation I

In game theory, *strategy* for a player is defined as “a set of rules that describe exactly how (...) [a] player should choose, depending on how the [other] players have chosen at earlier moves” (Hodkinson *et al.*, 2000). Nevertheless, this definition of strategies is static, and presumably is constructed *before* the game is actually played.

While people play games, they observe, learn, recollect and update their strategies *during* the game as well as adopting deontological strategies and goals before the game. Players update and revise their strategies, for instance, when their opponent makes an *unexpected* or *irrational* move.

Motivation II

For instance, assume that you play a video game by using a gamepad or a keyboard, and in the middle of the game, one of the buttons on the gamepad brakes. Hence, from that moment on, you will not be able to make some moves in the game that are controlled by that button on the gamepad. This is most certainly is not part of your strategy. Therefore, you will need to revise your strategy in such a way that some moves will be excluded from your strategy from then on. However, for your opponent, that is not the case as she can still make all the moves available to her.

Strategy Logic I

Consider a game played between two players given by the set $N = \{1, 2\}$ and a single admissible set of moves Σ for both (Ramanujam & Simon, 2008a; Ramanujam & Simon, 2008b). Let $\mathbf{T} = (S, \Rightarrow, s_0)$ be a tree rooted at s_0 , on the set of vertices S . A partial function $\Rightarrow: S \times \Sigma \rightarrow S$ specifies the labeled edges of such a tree where labels represent the moves at the states. The extensive form game tree, then, is a pair $T = (\mathbf{T}, \lambda)$ where \mathbf{T} is a tree as defined before, and $\lambda: S \rightarrow N$ specifies whose turn it is at each state. A strategy μ^i for a player $i \in N$ is a function $\mu^i: S^i \rightarrow \Sigma$ where $S^i = \{s \in S : \lambda(s) = i\}$.

Strategy Logic II

For player i and strategy μ^i , the strategy tree

$T_\mu = (S_\mu, \Rightarrow_\mu, s_0, \lambda_\mu)$ is the least subtree of T satisfying the following two conditions:

1. $s_0 \in S_\mu$;
2. For any $s \in S_\mu$, if $\lambda(s) = i$, then there exists a unique $s' \in S_\mu$ and action a such that $s \xRightarrow{a}_\mu s'$. Otherwise, if $\lambda(s) \neq i$, then for all s' with $s \xRightarrow{a}_\mu s'$ for some a , we have $s \xRightarrow{a}_\mu s'$.

Strategy Logic III

The most basic constructions in SL are the strategy specifications. First, for a given countable set X , a set of formulas $BF(X)$ is defined as follows, for $a \in \Sigma$:

$$BF(X) := x \in X \mid \neg\varphi \mid \varphi \wedge \varphi \mid \langle a \rangle \varphi$$

Let P^i be a countable set of atomic observables for player i , with $P = P^1 \cup P^2$. The syntax of strategy specifications is given as follows for $\varphi \in BF(P^i)$:

$$Strat^i(P^i) := [\varphi \rightarrow a]^i \mid \sigma_1 + \sigma_2 \mid \sigma_1 \cdot \sigma_2$$

Strategy Logic IV

The specification $[\varphi \rightarrow a]^i$ at player i 's position stands for “play a whenever φ holds”. The specification $\sigma_1 + \sigma_2$ means that the strategy of the player conforms to the specification σ_1 or σ_2 and $\sigma_1 \cdot \sigma_2$ means that the strategy of the player conforms to the specifications σ_1 and σ_2 . By the abuse of the notation, we will use \leftrightarrow to denote the equivalence of specifications.

Let $M = (T, V)$ where $T = (S, \Rightarrow, s_0, \lambda)$ is an extensive form game tree as defined before, and V is a valuation function ($V : S \rightarrow 2^P$) for the set of propositional variables P . The truth of a formula $\varphi \in BF(P)$ is given as usual for the propositional, Boolean and modal formulas.

Strategy Logic V

The notion “strategy μ *conforms* to specification σ for player i at state s ” (notation $\mu, s \models_i \sigma$) is defined as follows, where **out** $_{\mu}(s)$ denotes the unique outgoing edge at s with respect to μ .

$$\mu, s \models_i [\varphi \rightarrow a]^i \quad \text{iff} \quad M, s \models \varphi \text{ implies } \mathbf{out}_{\mu}(s) = a$$

$$\mu, s \models_i \sigma_1 + \sigma_2 \quad \text{iff} \quad \mu, s \models_i \sigma_1 \text{ or } \mu, s \models_i \sigma_2$$

$$\mu, s \models_i \sigma_1 \cdot \sigma_2 \quad \text{iff} \quad \mu, s \models_i \sigma_1 \text{ and } \mu, s \models_i \sigma_2$$

Strategy Logic VI

Now, based on the strategy specifications, the syntax of the strategy logic SL is given as follows:

$$p \mid \neg\varphi \mid \varphi_1 \wedge \varphi_2 \mid \langle a \rangle \varphi \mid (\sigma)_i : a \mid \sigma \rightsquigarrow_i \psi$$

for propositional variable $p \in P$, action $a \in \Sigma$, strategy $\sigma \in \text{Strat}^i(P^i)$, and Boolean formula ψ over P^i . The intuitive reading of $(\sigma)_i : a$ is that at the current state the strategy specification σ for player i suggests that the move a can be played. The intuitive meaning of $\sigma \rightsquigarrow_i \psi$ is that following strategy σ player i can ensure ψ . The other Boolean connectives and modalities are defined as usual.

$$M, s \models \langle a \rangle \varphi \quad \text{iff} \quad \exists s' \text{ such that } s \xrightarrow{a} s' \text{ and } M, s' \models \varphi$$

$$M, s \models (\sigma)_i : a \quad \text{iff} \quad a \in \sigma(s)$$

$$M, s \models \sigma \rightsquigarrow_i \psi \quad \text{iff} \quad \forall s' \text{ such that } s \Rightarrow_\sigma^* s' \text{ in } T_s|_\sigma, \\ \text{we have } M, s' \models \psi \wedge (\text{turn}_i \rightarrow \text{enabled}_\sigma)$$

Strategy Logic VII

where $\sigma(s)$ denotes the set of the enabled moves at state s in strategy σ , and \Rightarrow_σ^* denotes the reflexive transitive closure of \Rightarrow_σ . SL has been given a rather complex axiomatization and rules of inferences. However, it is complete for its semantics (Ramanujam & Simon, 2008a; Ramanujam & Simon, 2008b). Yet, it was not known that if SL was decidable or not. Similarly, the complexity of the satisfiability problem was not known.

Restricted Strategy Logic I

We denote the move restriction by $[\sigma!a]^i$ for a strategy specification σ and action a for player i . Informally, after the move restriction of σ by a , player i will not be able to make an a move. We incorporate restrictions to SL obtain Restricted Strategy Logic (RSL), and we incorporate these new dynamic operators at the level of strategy specifications.

In SL, recall that strategies are functions. Therefore, they only produce one move per state. However, our dynamic take in strategies cover more general cases where strategies can offer a set of moves to the player. Thus, in RSL, we define strategy μ^i as $\mu^i : S^i \rightarrow 2^\Sigma$. By $\mathbf{outr}_{\mu^i}(s)$ we will denote the set of moves

Restricted Strategy Logic II

returned by μ^i at s . Then, the extended syntax of strategy specifications for player i is given as follows.

$$\text{Strat}^i(P^i) := [\psi \rightarrow a]^i \mid \sigma + \sigma \mid \sigma \cdot \sigma \mid [\sigma!a]^i$$

Notice that the restrictions affect only the player who gets a move restriction. In other words, if a is prohibited to player i , it does not mean that some other player j cannot make an a move.

Once a move is restricted at a state, we will prune the strategy tree removing the prohibited move from that state on. Therefore, given $\mu^i : S^i \rightarrow 2^\Sigma$, we define the updated strategy relation $\mu^i!a : S^i \rightarrow 2^{\Sigma - \{a\}}$.

Restricted Strategy Logic III

We are now ready to define confirmation of restricted specifications to strategies. Note that we skip the cases for \cdot and $+$ as they are exactly the same.

$$\begin{aligned} \mu, s \models_i [\varphi \rightarrow a]^i & \quad \text{iff} \quad M, s \models \varphi \text{ implies } a \in \mathbf{outr}_\mu(s) \\ \mu^i, s \models_i [\sigma!a]^i & \quad \text{iff} \quad a \notin \mathbf{outr}_{\mu^i}(s) \text{ and } \mu!a, s \models_i \sigma \end{aligned}$$

where $\mu!a$ is the updated strategy tree.

RSL: Completeness and Complexity I

Here is the the syntax of RSL, which is the same as that of SL:

$$p \mid (\sigma)_i : a \mid \neg\varphi \mid \varphi_1 \wedge \varphi_2 \mid \langle a \rangle\varphi \mid \sigma \rightsquigarrow_i \psi$$

The semantics and the truth definitions of the formulas are defined as earlier with the exception of strategy specifications for restrictions. The axiom system of RSL consists of the axioms and rules of SL together with the following axioms for the added specification construct:

$$\blacktriangleright (\sigma!a)_i : c \leftrightarrow \mathbf{turn}_i \wedge \neg((\sigma)_i : a) \wedge (\sigma)_i : c$$

Theorem

$$(\sigma!a)_i : a \leftrightarrow \perp$$

RSL: Completeness and Complexity II

Theorem

The axiom system of RSL is complete with respect to the given semantics.

Theorem

The model checking problems for SL and RSL are in PSPACE.

RSL: Completeness and Complexity III

Proof Sketch

We have two different ways to prove it.

One is to translate RSL to CTL (computational tree logic) with a polynomial translation. The translation is rather immediate, and considers the two sorted-ness of the language of RSL.

Second is to use a direct method to show that the decision procedure uses PSPACE amount of space. This is very similar to the decidability proofs of most modal logics. Namely, scan the branches of the tree up to some depth determined by the formula, and then forget about that branch, and scan the next one.

Thanks!

Thanks for your attention!

Talk slides and the papers are available at:

`www.CanBaskent.net/Logic`

Thanks!

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