Proofs and Refutations

Non-Classically (and Game Theoretically)

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November 2 - 4, 2022, Imre Lakatos Centenary Conference
London School of Economics and Political Science

In a Nutshell

Proofs and Refutations is an example of a game

with inconsistencies and strategies.

Overview

1. Methods of Discovery with Inconsistencies: Hintikka and Lakatos

2. *Proofs and Refutations*: A Paraconsistent Game

Methods of Discovery with Inconsistencies: Hintikka and Lakatos

Hintikka and Lakatos

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- Both offer a heuristic methodology for knowledge growth.

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In this talk, I will argue that both Hintikkan and Lakatosian methodologies are largely based on game theoretical and strategic reasoning, but with inconsistencies.

First, Hintikka's Theory of Interrogative Inquiry

In an interrogative inquiry, the Inquirer is given a starting theory T and a question. He then tries to answer the question based on T by posing questions to nature or an oracle.

The Inquirer is also allowed to draw conclusions by using T and the answers he has already received, supposedly by using the derivation rules of classical logic.

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What if the Inquirer receives irrelevant or uncertain answers?

Reference

CB, *Towards Paraconsistent Inquiry*, The Australasian Journal of Logic, vol. 13, no. 2, pp. 21-40, 2016.

Bracketing

> "An important aspect of this general applicability of the interrogative model is its ability to handle uncertain answers - that is, answers that may be false. The model can be extended to this case simply by al- lowing the inquirer to tentatively disregard ("bracket") answers that are dubious. (...) Equally obviously, further inquiry might lead the inquirer to reinstate ("unbracket") a previously bracketed answer. This means thinking of interrogative inquiry as a self-corrective process."

This suggests that we need an "answer-selection procedure".

Reference

J. Hintikka, *Socratic Epistemology*, p. 3, Cambridge UP, 2007.

More on *Bracketing*

> "In a typical application of interrogative inquiry –for instance in the cross-examination of a witness in a court of law– the inquirer cannot simply accept all answers at their face value. They can be false. Hence we must have rules allowing the rejection or, as I will call it, the "bracketing of an answer", and rules governing such bracketing.
>
> But this seems totally unrealistic. How can we possibly hope to formulate realistic rules for the rejection or acceptance of any answers –any data– that an inquirer might ever receive?"

Reference

J. Hintikka, *Socratic Epistemology*, p. 223, Cambridge UP, 2007.

Bracketing: From epistemology to game theory

Epistemically, there seems to be a major problem in bracketing, as Hintikka pointed out.

In an inquiry, how can the Inquirer know which answers to ignore? How can he know what to reject or accept? This epistemic problem empties the notion of bracketing.

A game theoretical response can be given to eliminate this problem. It can be suggested that the inquirer simply chooses the assumptions and responses that help him *win* the game of inquiry. If the Inquirer can win the game with a particular set of assumptions, then he can adopt those assumptions for a win. If he cannot, then he simply selects another set of assumptions and answers, and keeps playing.

Bracketing: From classical to non-classical logic

In Hintikkan inquiry, the Inquirer needs to reason with all the uncertain, *contradictory*, *ambiguous* responses he receives.

In order to make it work, the Inquirer needs to have an inconsistency-friendly logical system.

In Hintikkan inquiry, there are some propositions that follow from a contradiction, and there are some that do *not*.

Reference

CB, *Towards Paraconsistent Inquiry*, The Australasian Journal of Logic, vol. 13, no. 2, pp. 21-40, 2016.

Lessons from Hintikka

The process of interrogative inquiry requires

1. game theoretical strategising
   – to choose what move to make when bracketing is required,
2. a non-classical logical model
   – to work with the inconsistencies.

Lakatosian *Bracketing*

We have similar notions in Lakatosian method of proofs and refutations.

1. Dealing with *monsters* require game theoretical strategising,
2. Managing *proofs that do not prove* require non-classical logical tools

Therefore, *Proofs and Refutations* requires a formalism that can express strategic reasoning with non-classical logical methods – even if the goal is to maintain the consistency of the system.

This requires an inconsistency-friendly, paraconsistent model.

What is Paraconsistency?

A logic is paraconsistent if contradictions do not *explode*.

That is

$$F, \neg F \nvdash G$$

for some formula F, G.

The proof theoretical definition above can be supplemented with a semantic one.

Dialetheism is the view that some statements are both true and false.

Proofs and Refutations: A Paraconsistent Game

Proofs and Refutations, schematically i

1. Primitive conjecture.
2. Proof (a rough thought experiment or argument, decomposing the primitive conjecture into subconjectures and lemmas).
3. Global counterexamples.
4. Proof re-examined. The guilty lemma is spotted. The guilty lemma may have previously remained hidden or may have been misidentified.
5. Proofs of the other theorems are examined to see if the newly found lemma occurs in them.

Proofs and Refutations, schematically ii

6. Hitherto accepted consequences of the original and now refuted conjecture are checked.
7. Counterexamples are turned into new examples, and new fields of inquiry open up.

Reference

David Corfield, *Assaying Lakatos's History and Philosophy of Science*, "Studies in History and Philosophy of Science", vol. 28, no. 1, pp. 99–121, 1997.

Proofs and Refutations: A Game with Inconsistencies

Corfield's Lakatosian algorithm allows us to make a lot of *searches* and gives us some room to control the parameters.

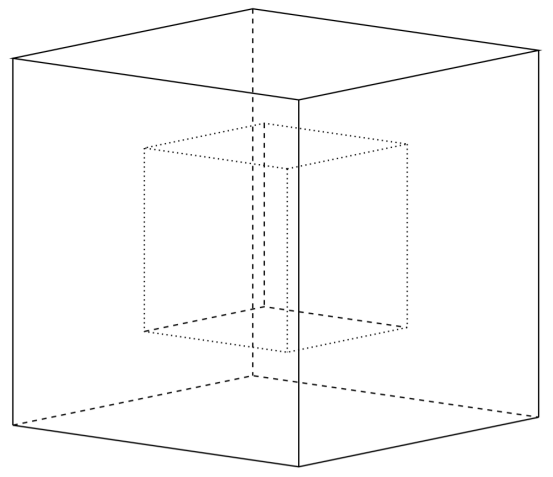
Searching for counterexamples, re-examining proofs and the methods that are developed to turn counter-examples into examples are all strategic moves.

This is a game with inconsistencies – a game with "proofs that do not prove".

Lakatosian Inconsistent Games: An Example

Proofs and Refutations offers a "rationally reconstructed" view of the history of Euler's Conjecture $V - E + F = 2$, call it χ.

At some stage of the dialogue of inquiry, we have both an example and a counter-example. A cube is an example for which the Euler's Conjecture holds. A *hollow cube* –an object bounded by a pair of nested cubes, one of which is inside, but does not touch the external one– is a counter-example.



Hollow-cube,
for which $V - E + F = 4$.

Notice
$V = 16, E = 24, F = 12$.

Lakatosian Inconsistent Games: An Example

Therefore, we have both χ and $\neg\chi$, supported by cube and hollow-cube, respectively, in our theory.

Having a contradiction suggests that there may be some available strategic moves.

In this case, one of the moves is **ReDefine**(Polyhedron).

(Let us use the same notation for the proposition which says that "redefining polyhedron" is the move to be made.)

Arguably, **ReExamineProof**, **IncorporateLemma**, **DeclareExceptions** are some of the other available moves.

Lakatosian Inconsistent Games: An Example

Now, we have

$$\chi \wedge \neg\chi \rightarrow \textbf{ReDefine}(\textit{Polyhedron})$$

for some conditional operator $\rightarrow$.

However, the very same contradiction $\chi \wedge \neg\chi$ does not entail everything, including the move **AcceptProof**. Because the existence of counter-examples suggest that the proof needs to be re-examined.

This is what renders *Proofs and Refutations* a paraconsistent and strategic methodology.

Game Theory of *Proofs and Refutations* i

An immediate question is how game theoretical and heuristic ideas match in *Proofs and Refutations*.

- **Strategies** Lakatos offers many strategies to deal with inconsistencies and proofs that do not prove: revising the lemma, redefining concepts, *monsters*, learning from proofs that do not prove, etc.
  - *Proofs that do not prove* are strategies that *knowingly* produce a loss.
  - *Re-examining proofs* is strategy pruning.
  - *Revising proofs* is strategy revision.

Game Theory of *Proofs and Refutations* ii

- **Moves** Each of these strategies have a set of available moves: re-examine the lemmas, redefine the concepts to exclude the counter-example, turn the counter-example into an example, etc.
- **Equilibria** Nash equilibrium means that the players are not better off by changing only their own strategy in a non-cooperative game.

If seen as a non-cooperative game, the dialogue suggests that players reach an equilibrium while maintaining a balance between strategies: they do not keep redefining concepts nor revising the proof.

- ***Homo Economicus* vs *Homo Heuristics*** *Proofs and Refutations* suggests that we can move from the rational decision maker to the heuristics decision maker in games.

An Intuitionistic Take

Game theoretical approach to Lakatosian heuristics helps us to re-establish the intuitionistic connection between

Truth *vs* **Proofs** *vs* **Strategies**

As such, we can then allow

- dialetheic truth,
- proofs that do not prove,
- (winning) strategies that do not only bring a win.

Reference

CB, *Game Theoretical Semantics for Some Non-Classical Logics*, Journal of Applied Non-Classical Logics, vol. 26, no. 3, pp. 208-39, 2016.

From *Homo Economicus* to *Homo Heuristics*

A broader philosophical goal of this project is to take a step forward from *Homo Economicus* to *Homo Heuristics* –from rational man to discoverer man– as new foundations of game theoretical and strategic reasoning.

A Prisoners' Dilemma for *Proofs and Refutations* i

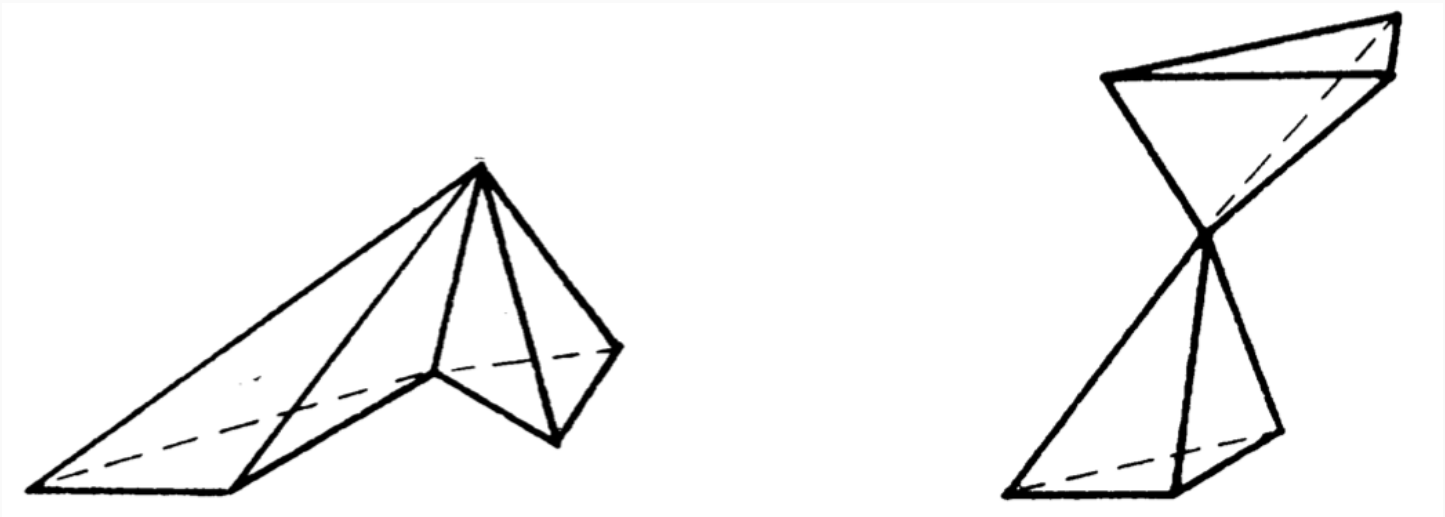
Let us put it all together.

Consider *Proofs and Refutations* when pupils were redefining polyhedron as the method of *monster-barring*.

> GAMMA: *A polyhedron is a solid whose surface consists of polygonal faces.*
>
> DELTA: *A polyhedron is a surface consisting of a system of polygons.*
>
> DELTA: [referring to the hollow-cube] *A woman with a child in her womb is not a counterexample to the thesis that human beings have one head.*

A Prisoners' Dilemma for *Proofs and Refutations* ii

The following counterexample is offered by ALPHA, after which DELTA revises her definition.



Following, DELTA offers a counter-example which agrees with her definition but refutes the conjecture. And the dialogue continues.

A Prisoners' Dilemma for *Proofs and Refutations* iii

What we observe here is that both players either cooperate and revise their definition, or they may choose not to revise and insist on their definition. The risk is that if they do not agree on the definition, they may not reach the (mathematical) truth.

Let us consider the following toy matrix for this game.

	Gamma cooperates	Gamma defects
Delta cooperates	Delta gets 5 Gamma gets 5	Delta gets 2 Gamma gets 3
Delta defects	Delta gets 3 Gamma gets 2	Delta gets 1 Gamma gets 1

A Prisoners' Dilemma for *Proofs and Refutations* iv

Recall that the equilibria for the prisoners' dilemma is when both cooperate (in this case).

This is a simple and quick explanation that once met a counter-example, both players need to cooperate and revise their definitions.

Some other explanations can be given using the *matching pennies* and *battle of sexes* cases.

Conclusion

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Common themes between Hintikkan and Lakatosian heuristics encourage us for the applicability of game theoretical reasoning to understand and contrast both methodologies better.

It also serves paraconsistency: a direct relation to philosophy of mathematical practice, an exciting case for logical pluralism.

A lot to be done: a logical and game theoretical explanation for *proofs that do not prove*, a logic of rigour, ...

Dialectic, Method of Proofs and Refutations and Dialetheism

The relationship between Hegelian dialectic, Lakatosian heuristics and logical dialetheism / pluralism remains largely unexplored from game theoretical and strategic reasoning perspectives.

I hope the current work would motivate some further work in these directions.

In a Nutshell

Proofs and Refutations is a game with

inconsistencies and non-classical strategies.

Thank you!

Talk slides are available at my website

CanBaskent.net/Logic