Modal Topologies

From classical to non-classical logics

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Topological semantics

provides a richer semantics for

classical as well as non-classical logics.

Overview

A Bit of History

1930s and 40s

A first connection between closure algebras and modal logic was made in a paper in 1938 by Tsao-Chen.

Reference

T. Tsao-Chen, “Algebraic Postulates and a Geometric Interpretation for the Lewis Calculus of Strict Implication”, Bulletin of the American Mathematical Society, vol. 44, pp. 737-744, 1938.

1938] STRICT IMPLICATION 737

ALGEBRAIC POSTULATES AND A GEOMETRIC INTERPRETATION FOR THE LEWIS CALCULUS OF STRICT IMPLICATION

TANG TSAO-CHEN

1. **Two further postulates for a Boolean ring with a unit element.** If addition, subtraction, and multiplication are properly defined in logic, it may be shown* that the postulates for these operations are identical with those in a ring, in which every element is idempotent, satisfying the postulate $xx=x$. Such a ring is called a Boolean ring. The postulates for a Boolean ring with a unit element are therefore the following:

A. *Addition is always possible, commutative, and associative.*
B. *Multiplication is always possible, associative, and both left- and right-distributive with respect to addition.*
C. *Subtraction is always possible.*
D. $xx=x$.
E. *There exists an element* 1 *such that* $x1=x$ *for every element* x *in the ring.*

Here we shall introduce a new operation, represented by x^∞, which satisfies the following two further postulates:

F_1. *For every element* x *there exists an element* x^∞ *such that* $x^\infty x=x^\infty$.
F_2. *For any two elements* x *and* y *we have* $(xy)^\infty=x^\infty y^\infty$.

The postulates A-F_2, obtained above, may be called the algebraic postulates for the Lewis calculus of strict implication.

2. **A geometric meaning of the symbol x^∞.** A geometric meaning† may be attached to x^∞ as follows: Let x be a point set in the euclidean

* See M. H. Stone. *The theory of representations for Boolean algebras*, Transactions of this Society, vol. 40 (1936), pp. 37–53.

† Another geometric meaning of x^∞ may be obtained by assuming 1^∞ to be any one fixed point or any set of fixed points (finite or infinite in number and continuous or discontinuous in character) and setting $x^\infty=x1^\infty$. If we assume that 1^∞ is a fixed point, we have then the following property:

G. x^∞ *is two-valued, that is,* $x^\infty=1^\infty$ or 0^∞,

which is independent of the postulates A-F_2. This sub-Boolean algebra with the postulates A-G does not become the ordinary two-valued Boolean algebra, unless we assume further that x is two-valued.

1930s and 40s

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3. **A geometric interpretation of Lewis' possibility functions.** When p is a proposition, Lewis introduces an undefined idea $\Diamond p$, which is read "p is possible" and may be called the possibility function of p. Now, if a class x be given, $\Diamond x$ must have a corresponding meaning which is obtained from the following definition:

DEFINITION. $\Diamond x = 1-(1-x)^\infty$.

By means of the definition of $\sim x$, we obtain $\sim x = 1-x$; and we have the following theorems:

THEOREM 1. $\Diamond x$ *is the closure of* x.

THEOREM 2. $\sim\Diamond x = (1-x)^\infty$ *is the exterior of* x.

THEOREM 3. $\Diamond\sim x = 1-x^\infty$ *is the closure of the complement of* x.

THEOREM 4. $\sim\Diamond\sim x = x^\infty$ *is the interior of* x.

On the basis of the postulates A-F we may then prove abstractly the following theorems:

THEOREM 5. $0^\infty = 0$ *and* $\sim\Diamond\sim 0. = .0$; *that is, the interior of the null class is a null class.*

THEOREM 6. $\sim\Diamond\sim x . \sim\Diamond x : = .0$, *and* $x^\infty(\sim x)^\infty = 0$; *that is, if* x *is a class, then the interior and the exterior of* x *have no point in common.*

DEFINITION. $x^f = 1-x^\infty-(\sim x)^\infty$, *that is,* x^f *is the frontier of* x.

THEOREM 7. $x^f = (\sim x)^f$, *that is,* x *and* $\sim x$ *have the same frontier.*

THEOREM 8. $x^f + x^\infty + (\sim x)^\infty = 1$, $x^f + (\sim\Diamond\sim x) + (\sim\Diamond x) = 1$; *that is, the frontier, the interior, and the exterior of* x *form the whole plane.*

THEOREM 9. $\Diamond x . = . x^f + x^\infty$ *and* $\Diamond x . = . x^f + (\sim\Diamond\sim x)$; *that is,* $\Diamond x$ *is the sum of the frontier and the interior of* x.

THEOREM 10. $\Diamond\sim x . = . x^f + (\sim x)^\infty$ *and* $\Diamond\sim x . = . x^f + \sim\Diamond x$; *that is,* $\Diamond\sim x$ *is the sum of the frontier and the exterior of* x.

THEOREM 11. $x(\sim x)^\infty = 0$ *and* $x . \sim\Diamond x : = .0$; *that is, the class* x *and its exterior have no point in common.*

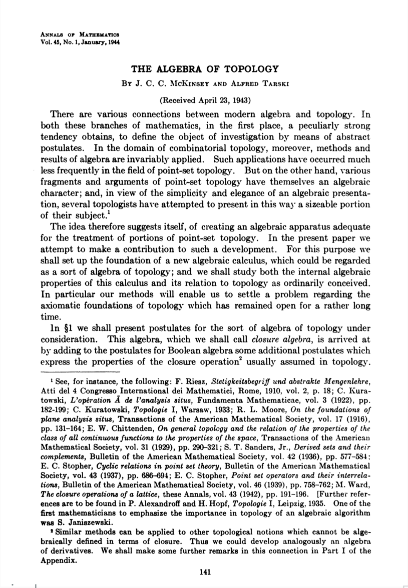
1930s and 40s

Tarski and McKinsey's paper is often considered a landmark.

In that paper they identified the algebraic and modal logical qualities of the topological closure operator.

Reference

J. C. C. McKinsey and A. Tarski, "The Algebra of Topology", The Annals of Mathematics, vol. 45, pp. 141-191, 1944.



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THE ALGEBRA OF TOPOLOGY

BY J. C. C. MCKINSEY AND ALFRED TARSKI

(Received April 23, 1943)

There are various connections between modern algebra and topology. In both these branches of mathematics, in the first place, a peculiarly strong tendency obtains, to define the object of investigation by means of abstract postulates. In the domain of combinatorial topology, moreover, methods and results of algebra are invariably applied. Such applications have occurred much less frequently in the field of point-set topology. But on the other hand, various fragments and arguments of point-set topology have themselves an algebraic character; and, in view of the simplicity and elegance of an algebraic presentation, several topologists have attempted to present in this way a sizeable portion of their subject.[1]

The idea therefore suggests itself, of creating an algebraic apparatus adequate for the treatment of portions of point-set topology. In the present paper we attempt to make a contribution to such a development. For this purpose we shall set up the foundation of a new algebraic calculus, which could be regarded as a sort of algebra of topology; and we shall study both the internal algebraic properties of this calculus and its relation to topology as ordinarily conceived. In particular our methods will enable us to settle a problem regarding the axiomatic foundations of topology which has remained open for a rather long time.

In §1 we shall present postulates for the sort of algebra of topology under consideration. This algebra, which we shall call *closure algebra*, is arrived at by adding to the postulates for Boolean algebra some additional postulates which express the properties of the closure operation[2] usually assumed in topology.

[1] See, for instance, the following: F. Riesz, *Stetigkeitsbegriff und abstrakte Mengenlehre*, Atti del 4 Congresso International dei Mathematici, Rome, 1910, vol. 2, p. 18; C. Kuratowski, *L'opération $\bar{A}$ de l'analysis situs*, Fundamenta Mathematicae, vol. 3 (1922), pp. 182-199; C. Kuratowski, *Topologie* I, Warsaw, 1933; R. L. Moore, *On the foundations of plane analysis situs*, Transactions of the American Mathematical Society, vol. 17 (1916), pp. 131–164; E. W. Chittenden, *On general topology and the relation of the properties of the class of all continuous functions to the properties of the space*, Transactions of the American Mathematical Society, vol. 31 (1929), pp. 290–321; S. T. Sanders, Jr., *Derived sets and their complements*, Bulletin of the American Mathematical Society, vol. 42 (1936), pp. 577–584; E. C. Stopher, *Cyclic relations in point set theory*, Bulletin of the American Mathematical Society, vol. 43 (1937), pp. 686–694; E. C. Stopher, *Point set operators and their interrelations*, Bulletin of the American Mathematical Society, vol. 46 (1939), pp. 758–762; M. Ward, *The closure operations of a lattice*, these Annals, vol. 43 (1942), pp. 191–196. [Further references are to be found in P. Alexandroff and H. Hopf, *Topologie* I, Leipzig, 1935. One of the first mathematicians to emphasize the importance in topology of an algebraic algorithm was S. Janiszewski.

[2] Similar methods can be applied to other topological notions which cannot be algebraically defined in terms of closure. Thus we could develop analogously an algebra of derivatives. We shall make some further remarks in this connection in Part I of the Appendix.

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Classical Logic of Topologies

Predating Kripke

Topological and algebraic semantics predates Kripke semantics by a couple of decades.

Kripke semantics is more popular because

- it is simpler,
- it is used by philosophers and computer scientists more often,
- it does not require an advanced mathematical background.

Yet, none of these reasons gives us an *a priori* reason to resort to Kripke semantics.

A lot is lost in translation!

Basics: Topologies

Open Set Definition

The structure (S, σ) is called a topological space if it satisfies the following conditions.

1. $S \in \sigma$ and $\emptyset \in \sigma$,
2. σ is closed under arbitrary unions and finite intersections.

Alternatively,

Closed Set Definition

The structure (T, τ) is called a topological space if it satisfies the following conditions.

1. $T \in \tau$ and $\emptyset \in \tau$,
2. τ is closed under finite unions and arbitrary intersections.

Basics: Opens and Closeds

¶ Collections σ and τ are called topologies.

¶ The elements of σ are called open sets whereas the elements of τ are called closed sets.

¶ A set is open if its complement in the same topology is closed and vice versa.

Basics: Logic

¶ Let us consider the language of propositional modal logic with the modality $\Diamond$, and define the dual modality $\Box$ in the usual sense.

¶ In topological semantics, the modal operator $\Box$ for necessitation corresponds to the topological interior operator $Int(\cdot)$ where $Int(O)$ is the largest open set contained in set O.

¶ Furthermore, one can dually associate the topological closure operator $Clo(\cdot)$ with the possibility modal operator $\Diamond$ where the closure $Clo(O)$ of a given set O is the smallest closed set that contains O.

Basics: Logic

¶ In the classical setting, modalities necessarily produce topological entities such as open or closed sets.

¶ However, the extension of Booleans may or may not be topological entities. For example, negation of an open set is not necessarily an open set. Therefore, the negation operator may not produce an open or closed set.

Basics: Derivative

¶ A point $s \in S$ is called a limit point of $A \subseteq S$ if for each open neighborhood U of s, the set $S \cap (U - \{s\})$ is nonempty.

¶ The set of limit points of A is called the derivative of A and is denoted by $d(A)$.

¶ Then $Clo(A) = A \cup d(A)$.

Therefore, $s \in Clo(A)$ if and only if $U \cap A$ is nonempty for each open neighborhood U of s.

Basics: Co-derivative

¶ Let $t(A) = S - d(S - A)$. We call $t(A)$ the co-derivative of A.

¶ Then, $s \in t(A)$ if and only if there exists an open neighborhood U of s such that $U \subseteq A \cup \{s\}$.

For $A \subseteq S$, we have the following for the derivative and co-derivative operators.

- $d(A \cup B) = d(A) \cup d(B)$
- $t(A \cap B) = t(A) \cap t(B)$
- $A \subseteq A \cup d(A)$
- $A \cap t(A) \subseteq A$

Basics: Derivative and Co-derivative

For $A \subseteq S$, we have the following for the derivative and co-derivative operators.

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- $t(A \cap B) = t(A) \cap t(B)$
- $A \subseteq A \cup d(A)$
- $A \cap t(A) \subseteq A$

Note the error in the "*Modal Logic of Space*" paper!

Some Results: S4

The logic S4 is defined by the KT4 axioms and the rules of modus ponens and necessitation which translate into the following axioms.

- $\Box\top$ — the whole space is open
- $(\Box p \wedge \Box q) \leftrightarrow \Box(p \wedge q)$ — the open sets are closed under finite intersections
- $\Box p \rightarrow \Box\Box p$ — the interior operator is idempotent
- $\Box p \rightarrow p$ — the interior of any set is contained in the set

Reference

Johan van Benthem & Guram Bezhanishvili, "Modal Logics of Space" in M. Aiello & Pratt-Hartmann & van Benthem (eds), *Handbook of Spatial Logics*, Springer, 2007.

Some Results: S4

The logic S4 is defined by the KT4 axioms and the rules of modus ponens and necessitation which translate into the following axioms.

Let (S, σ) be a topology and $A \subseteq S$.

- $\Box\top$ $\qquad$ $Int(S) = S$
- $(\Box p \wedge \Box q) \leftrightarrow \Box(p \wedge q)$ $\qquad$ $Int(A \cap B) = Int(A) \cap Int(B)$
- $\Box p \to \Box\Box p$ $\qquad$ $Int(A) \subseteq Int(Int(A))$
- $\Box p \to p$ $\qquad$ $Int(A) \subseteq A$

Some Results: S4 and Completeness

¶ In 1944 McKinsey and Tarski proved that S4 is complete for any dense-in-itself (that is no isolated points) metric separable space.

¶ Thus, S4 is also the logic of any Euclidean space $\mathbb{R}^n$ with the standard topology.

¶ Mints proved completeness of S4 for the Cantor space in 1998.

Some Results: S4 and Completeness

¶ S4 is the logic of the class of all topological spaces.

¶ S4 is the logic of the class of all finite topological spaces.

¶ S4 has the effective finite model property with respect to the class of topological spaces.

Some Results: Definability

A class of topological spaces K is topologically definable if there exists a set of modal formulas Γ such that for each topological space X, we have $X \in K$ iff $X \models \Gamma$.

Moroever,

- neither compactness nor connectedness is topologically definable,
- none of the separation axioms T_0, T_d, T_1, T_2, T_3 , $T_{3\frac{1}{2}}$, T_4 , T_5 and T_6 is topologically definable,

Reference

David Gabelaia, "Modal Definability in Topology", Masters Thesis, Institute for Logic, Language and Computation, University of Amsterdam, 2001.

Some Results: Definability

¶ The formula $p \to \Box p$ (or equivalently $\Diamond p \to p$) topologically defines the class of discrete spaces, as it renders every set as open.

¶ The formula $\Diamond\Box p \to \Box\Diamond p$ topologically defines the class of extremally disconnected spaces where the closure of each open is clopen.

Some Results: Derivative

¶ We can re-interpret $\Diamond$ as the derivative operator.

¶ The semantics of this re-interpretation is given as follows.

- $s \models \Diamond\varphi$ iff $\forall U \in \sigma.(s \in U \rightarrow \exists t \in U - \{s\}, t \models \varphi)$
- $s \models \Box\varphi$ iff $\exists U \in \sigma.(s \in U \wedge \forall t \in U - \{s\}, t \models \varphi)$

¶ Let wK4 denote the modal logic $K + (p \wedge \Box p) \rightarrow \Box\Box p$.

Some Results: Derivative

¶ wK4 is sound and complete with respect to the class of all weakly transitive frames where $(wRv \wedge vRu \wedge w \neq u) \rightarrow wRu)$.

¶ wK4 is sound and complete with respect to the class of all weakly transitive frames.

¶ wK4 has the finite model property.

¶ wK4 is complete with respect to finite rooted irreflexive wK4-frames where a frame is called rooted if there exists r (the root) such that rRw for every $w \neq r$.

Some Results: Derivative

¶ For a topological space (S, σ), define R_d on S by setting sR_dt iff $s \in d(t)$. Then, (S, R_d) is an irreflexive wK4-frame.

¶ For a nonempty set X, there is a 1-1 correspondence between

- Alexandroff topologies on X,
- Reflexive and transitive relations on X,
- Irreflexive and weakly transitive relations on X.

¶ Let us define a translation Θ between the formulas to establish a connection between S4 and wK4.

$$\begin{aligned}\Theta(p) &= p\\ \Theta(\neg\varphi) &= \neg\Theta(\varphi)\\ \Theta(\varphi\wedge\psi) &= \Theta(\varphi)\wedge\Theta(\psi)\\ \Theta(\Box\varphi) &= \Theta(\varphi)\wedge\Theta(\varphi)\wedge\Box\Theta(\varphi)\end{aligned}$$

¶ Let K be a class of topological spaces and $(S,\sigma)\in K$. Then, $(S,\sigma)\models\varphi$ iff $(S,\sigma)\models_d\Theta(\varphi)$, where $\models_d$ represents the derivative interpretation of modal logic.

Non-Classical Logic of Topologies

The Jump towards Non-Classicity

Notice that the extensions of modal formulas are guaranteed to have a topological extensions in classical logics. Topological semantics for modal logic therefore works with both topological and non-topological sets.

We can take one step further, and suggest that extension of any propositional variable will be an open set.

Reference

G. Mints, "A short introduction to intuitionistic logic", Kluwer, 2000.

Reference

C. Mortensen, "Topological Separation Principles and Logical Theories", *Synthese*, Vol. 125, No. 1/2, pp. 169- 178, 2000.

Basics

In this case the extension of any propositional variable p will be an open set. This works well with finite conjunctions and disjunctions as they all are going to be open.

What about the negation then?

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What about the negation then?

We will interpret negation as the “open complement”: the largest open set contained in the set theoretic complement. That is the interior of the complement.

This procedure produces an intuitionistic logic with a Heyting algebra.

Basics

We can *dualise* this.

We can interpret negation as the "closed complement": the smallest closed set containing in the set theoretic complement. That is the closure of the complement.

This procedure produces a paraconsistent logic with a co-Heyting or Brouwerian algebra.

Basics

Let p be true at set K. If we impose that K is a closed set and the negation operator works as a closed complement to obtain a paraconsistent logic, then $\neg p$ will be true at the $Clo(\overline{K})$ — closure of the set theoretical complement of K.

Interestingly, $K \cap Clo(\overline{K}) \neq \emptyset$. This is equal to the boundary of K, denoted as ∂K.

In the intuitionistic case ∂K may be empty as K and $Int(\overline{K})$ may not intersect, when K is open.

In the paraconsistent case, $K \cap Clo(\overline{K}) = \partial K$.

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In the paraconsistent case, $K \cap Clo(\overline{K}) = \partial K$.

Question: For what sets X, ∂X is always empty?

The Philosophy of Non-Classical Topologies

How to interpret opens and closeds within logic is an important problem.

Just because it is possible to identify them with closure algebras does not immediately entail that they are the true objects of intuitionistic or paraconsistent logics.

Some Results

A formula φ is called "connected" in a model M, if for any two formulas α_1 and α_2 with nonempty closed (or dually, open) extensions in M, if $\varphi \equiv \alpha_1 \vee \alpha_2$, then $\alpha_1 \wedge \alpha_2$ will have a non-empty extension. We will call a theory connected if it is generated by a set of connected formulas.

Reference

CB, "Some topological properties of paraconsistent models", *Synthese*, vol. 190, pp. 4023–4040, 2013.

Some Results

- A paraconsistent topological model with no connected formulas cannot have true contradictions.
- A paraconsistent topological model with totally disconnected topology cannot be inconsistent.
- Every connected formula is satisfiable in some connected (classical) topological space.

Reference

CB, "Some topological properties of paraconsistent models", *Synthese*, vol. 190, pp. 4023–4040, 2013.

Some Results

- Every connected theory in a paraconsistent topological logic is inconsistent. Moreover, every connected theory in a paracomplete topological logic is incomplete.
- In a paraconsistent topological model, the only subtheory that is not inconsistent is the empty theory.

Reference

CB, "Some topological properties of paraconsistent models", *Synthese*, vol. 190, pp. 4023–4040, 2013.

Some Results

I gave topological models for some other non-classical and paraconsistent logics.

Reference

CB, "Public Announcement Logic in Geometric Frameworks", *Fundamenta Informaticae*, vol. 118, no. 3, pp. 207-223, 2012.

Reference

CB, "Topological Semantics for da Costa Paraconsistent Logics C_ω and C_ω^*", in *New Directions in Paraconsistent Logic*, Edited by J.-Y. Beziau, M. Chakraborty and S. Dutta, pp. 427-444, Springer, 2016.

Conclusion

Future Work

Topological models are rich.

It remains a major project to identify the topological qualities of paraconsistent models *in detail*.

What are the topologies of reflexive-insensitive logics?

How can we develop a logic for derivative operators or regular sets?

In a Nutshell

Topological semantics

provides a richer semantics for

classical as well as non-classical logics.

Thank you!

Talk slides are available at my website

CanBaskent.net/Logic